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ON THE BLOCK ALGEBRAS HAVING ONLY ONE IRREDUCIBLE MODULE**

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Abstract

The following result is proved: Let B be a block ideal of group algebra kG over a splitting field k with characteristic p. Suppose that B has only one irreducible module L and abelian defect group D, then $B \simeq \mathrm{Mat}_m(kD)$, where $m = \mathrm{Dim}_k L$. This result generalizes Külshammer's theorem concerning the structure of block algebras with inertial index 1.

§1. Introduction

Let k be a field of characteristic p and G a finite group. Assume k is a splitting field of G and each subgroup of G. The blocks we discuss are all p-blocks. One of the basic results of the structure of block algebra B is that if the defect $d(B) = 0$, then $k(B) = l(B) = 1$ and $B \simeq \mathrm{Mat}_m(k)$, where $k(B)$ and $l(B)$ denote the numbers of the irreducible ordinary characters of B and the irreducible Brauer characters of B, respectively. In 1971, Brauer ([2] 6G) proved that if the defect group D of B is abelian and the inertial index of B is 1, then $k(B) = |D|$, $l(B) = 1$. In 1980, Külshammer[5] proved that if B satisfies the above condition then $B \simeq \mathrm{Mat}_m(kD)$. In 1988, Puig[6] generalized Kulshammer's result and proved that if B is a nilpotent block then $B \simeq \mathrm{Mat}_m(RD)$, where R is a complete discrete valuation ring with residual field k.

In this paper, we follow the approach of local methods of block theory developed in [1] and use some elementary results of representation theory to generalize Kulshammer's theorem.

§2. Preliminaries

First we introduce some concepts of local representation and the main results we will use. Let P be a p-subgroup of G. The Brauer map is a k-linear map $Br_P : kG \to kC_G(P)$ such that for each $x \in G$ if $x \in C_G(P)$, then $Br_P(x) = x$, otherwise $Br_P(x) = 0$. The following Theorem A shows the importance of the Brauer map, while Theorem B is another form of the well-known Brauer first main theorem.

Theorem A[1]. *Br_P is a k-algebraic homomorphism of $(kG)^P$ onto $kC_G(P)$. Here P acts on kG by conjugation and $(kG)^P$ denotes the subalgebra of the fixed points. Br_P is also $N_G(P)$-module homomorphism.*

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Theorem B[1]. *Br_P induces a one-to-one correspondence from the set of all centrally primitive idempotents with defect group P onto the set of all centrally primitive idempotents with defect group P of $kN_G(P)$.*

Suppose that b is the centrally primitive idempotent corresponding to block B. Let P be a p-subgroup of G and e a block idempotent of $kC_G(P)$. We will call (P,e) a Brauer B-pair of G, or simply, a B-pair, in case $eBr_P(b)=e$. If (Q,f) is another B-pair with $Q\lhd P$, P fixes f and $eBr_P(f)=e$, then we call $(Q,f)\lhd(P,e)$. For $(Q,f)\le(P,e)$ we mean $(Q,f)\lhd\lhd(P,e)$. If P is a defect group of block B of G, then B-pair (P,e) is called Sylow B-pair.

Theorem C[1]. *Let B be a block of G. Then*

(1) The Sylow B-pairs are all the maximal B-pairs of G.

(2) The Sylow B-pairs are conjugate to each other.

(3) If (D,e) is a Sylow B-pair, then D is a defect group of e in $DC_G(D)$.

The other terminologies and notations we use are all standard and follow [4] and [1].

§3. Proof of the Theorem

Theorem. *Let G be a finite group and k a splitting field of G and all subgroups of G with characteristic p. Let B be a block ideal of kG with abelian defect group D. Suppose that there is only one irreducible Brauer character in B and its dimension is m. Then $B\simeq \mathrm{Mat}_m(kD)$.*

Proof. First we prove two lemmas.

Lemma 3.1. *Let B be a block ideal with defect group D. Assume that there is only one irreducible kG-module L in B. Let $|D|=p^d$ and $\dim_k L=m$. Suppose $M=ikG$ is the projective cover of L, where i is the primitive idempotent of kG corresponding to M. Then $B\simeq \mathrm{Mat}_m(ikGi)$ and $\dim_k(ikGi)=p^d$.*

Proof of Lemma 3.1. Since B has only one irreducible module, $B\simeq M\oplus\cdots\cdots\oplus M$ and the multiplicity of M in B is m by [3, I.16.9]. Let $E=\mathrm{End}_{kG}(M)$. Then $E\simeq ikGi$ and $\mathrm{End}_{kG}(B)\simeq \mathrm{Mat}_m(E)$ by [3, I.8.2 and I.8.6]. Thus we have

$$B\simeq \mathrm{End}_B(B)\simeq \mathrm{End}_{kG}(B)\simeq \mathrm{Mat}_m(E).$$

We compute $\dim_k(ikGi)$. From the above isomorphism we have

$$\dim_k B=m^2\dim_k E=m\dim_k M$$

and so

$$\dim_k E=\frac{1}{m}\dim_k(M).$$

Since there is only one irreducible module in B, p^d is the unique Cartan invariant of B, and so

$$\dim_k(M)=p^d m.$$

Finally we have

$$\dim_k(ikGi)=\dim_k E=p^d.$$

Lemma 3.2. *Let D be a p-subgroup of G with $G=C_G(D)$. Let B be a block ideal of kG with defect group D. Let i be a primitive idempotent of B. Then we have k-algebraic isomorphism $ikGi\simeq kD$.*

Proof of Lemma 3.2. Set $M = ikG$. It is sufficient to prove $E = \mathrm{End}_{kG}(M) \simeq kD$. We have the following decomposition of direct sum of kD-module:

$$kG = x_1kD \oplus \cdots\cdots \oplus x_tkD,$$

where $\{x_1, \cdots\cdots, x_t\}$ is a transversal of D in G. Since M is a direct summand of kG as a kD-module, M is a projective kD-module. From a theorem of Kaplansky [4] M is a free kD-module, for kD is a local ring. Thus $\mathrm{End}_k(M)$ is also a free kD-module by [3, II. 2.7], and so

$$\mathrm{End}_k(M) \simeq kD \oplus \cdots\cdots \oplus kD.$$

But obviously

$$E = \mathrm{End}_{kG}(M) = (\mathrm{End}_k(M))^G \simeq kD \oplus \cdots\cdots \oplus kD.$$

From [3, V. 4.6], there is a unique irreducible module in B, so $\dim_k E = p^d$ by Lemma 3.1. Hence $E \simeq kD$.

Now we prove the theorem. Let b be the centrally primitive idempotent corresponding to block B. Let (D, e) be a Brauer B-pair of G. Put $f = \sum_i e^{y_i}$, where y_i runs over a cross set of $C_G(D)$ in $N_G(D)$. Then f is a centrally primitive idempotent of $kN_G(D)$. Since $eBr_D(b) = e$ and Br_D is a morphism of $N_G(D)$-module, $e^{y_i}Br_D(b) = e^{y_i}$ for each y_i. Then $fBr_D(b) = f$. By Theorem B $Br_D(b)$ is a centrally primitive idempotent with defect group D, and so $Br_D(b) = f$. Let $e_i = e^{y_i}$. Then each pair (D, e_i) is a Sylow B-pair of G. Therefore each block e_i of $kC_G(D)$ has a defect group D and there is only one irreducible Brauer character in e_i by [3, V.4.6]. Hence by Lemma 3.1 and Lemma 3.2

$$kC_G(D)e_i \simeq \mathrm{Mat}_n(jkC_G(D)j) \simeq \mathrm{Mat}_n(kD),$$

where j is a primitive idempotent of $kC_G(D)e_i$ and n the dimension of the unique irreducible Brauer character in e_i. Thus

$$ZkC_G(D)e_i \simeq Z(jkC_G(D)j) \simeq kD.$$

Here ZX denotes the center of a k-algebra X.

Since $Br_D(b) = f$, we have

$$Br_D(ZkGb) = ZkC_G(D)(\sum_i e_i) \simeq kD \oplus \cdots\cdots \oplus kD.$$

We derive a surjective k-algebraic homomorphism $ZkGb \to kD$, and so there is a surjective k-algebraic homomorphism $Z(ikGi) \to kD$ by Lemma 3.1, where i is the idempotent mentioned in Lemma 3.1. But $\dim_k(ikGi) = p^d$, so $ikGi \simeq kD$. Thus from Lemma 3.1 $B \simeq \mathrm{Mat}_m(kD)$ and the theorem is proved.

Example. There are some examples to show that the result in this paper is real generalization of Kulshammer's theorem, among of them we just mention that given by Broue and Puig.

Let Q be an extra-special q-group with q a prime distinct from p. Let $Z = Z(Q)$. Suppose that Q/Z acts faithfully on an abelian p-Group P. Let $G = P \rtimes Q$. Let ρ be a non-indentity irreducible character of Z and β be the block of kZ such that $\rho \in \beta$. Let e be the block idempotent of β. Since $Z = Z(G)$, $Z(kZ) = ZkG$. So e is also a block idempotent of kG and kQ corresponds to block B of kG and block b of kQ, respectively. For each irreducible

Brauer character $\varphi \in B$, we have $P \leq \text{Ker } \varphi$. So φ is also an irreducible Brauer character of $G/P \simeq Q$. Hence $l(B) = 1$, P is a defect group of B and the inertial index of B is

$$|G : PC_G(P)| = |G : PZ| = |Q : Z| \neq 1.$$

Therefore B does not suffice the condition of Kulshammer's theorem but B suffices the condition of the theorem in this paper.

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