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LIFE SPAN OF CLASSICAL SOLUTIONS TO $\Box u = |u|^p$ IN TWO SPACE DIMENSIONS

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Abstract

The author studies the life span of classical solutions to the following Cauchy problem

$$\Box u = |u|^p,$$

$$t = 0 : u = \varepsilon\varphi(x), u_t = \varepsilon\psi(x), \qquad x \in I\!R^2,$$

where φ, $\psi \in C_0^\infty(I\!R^2)$ and not both identically zero, $\Box = \partial_t^2 - \partial_1^2 - \partial_2^2$, $p \geq 2$ is a real number and $\varepsilon > 0$ is a small parameter, and obtains respectively upper and lower bounds of the same order of magnitude for the life span for $2 \leq p \leq p_0$, where p_0 is the positive root of the quadratic $\chi^2 - 3\chi - 2 = 0$.

§1. Introduction

In this paper, we consider the following Cauchy problem

$$\Box u = |u|^p, \tag{1.1}$$

$$t = 0 : u = \varepsilon\varphi(x), u_t = \varepsilon\psi(x), \qquad x \in I\!R^2 \tag{1.2}$$

where φ, $\psi \in C_0^\infty(I\!R^2)$ and not both identically zero, $\Box = \partial_t^2 - \partial_1^2 - \partial_2^2$, $p \geq 2$ is a real number and $\varepsilon > 0$ is a small parameter.

We are interested in estimating the life span T_ε of classical solutions of (1.1) (1.2). By definition, T_ε is the largest number such that classical solution of (1.1) (1.2) exists for $0 < t < T_\varepsilon$.

Let p_0 be the positive root of the quadratic

$$\chi^2 - 3\chi - 2 = 0. \tag{1.3}$$

R. T. Glassey [2] proved that when $p > p_0$ and ε is small enough, one has $T_\varepsilon = +\infty$, that is, global existence. When $p = 2$, H. Lindblad [1] has proved precise estimate for the life span T_ε; we restate his results as follows.

Theorem 1.1. *Let T_ε be the life span of classical solution to* (1.1) (1.2) *with* $p = 2$. *Then*

(I) *when* $\int g(x)dx \neq 0$,

$$\lim_{\varepsilon \to 0} a(\varepsilon)^{-1} T_\varepsilon = T \text{ exists with } 0 < T < +\infty,$$

where $a(\varepsilon)$ satisfies

$$\varepsilon^2 a^2(\varepsilon) \log(1 + a(\varepsilon)) = 1; \tag{1.4}$$

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(II) *when* $\int g(x)dx = 0$,

$$\lim_{\varepsilon\to 0} \varepsilon T_\varepsilon = T \quad \textit{exists with} \quad 0 < T < +\infty.$$

We in this paper will prove similar results for the case $2 < p < p_0$.

Theorem 1.2. *Let T_ε be the life span of classical solutions to* (1.1) (1.2) *with* $2 < p < p_0$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{k(p)} T_\varepsilon = T_p \quad \textit{exists with} \quad 0 < T_p < +\infty,$$

where

$$k(p) = 2p(p-1)/(2+3p-p^2). \tag{1.5}$$

We prove Theorem 1.2 by following the steps of H. Lindblad [1], but in order to go through with it, much more technical difficulties has to be overcome.

In case $p = p_0$, by applying the method the author proposed in [3], we shall be able to prove the following

Theorem 1.3. *Let T_ε be the life span of classical solutions to* (1.1) (1.2) *with* $p = p_0$. *Then there exists an* $\varepsilon_0 > 0$ *such that for any* ε *with* $0 < \varepsilon < \varepsilon_0$

$$\exp\{\kappa_1 \varepsilon^{-p_0(p_0-1)}\} \le T_\varepsilon \le \exp\{\kappa_2 \varepsilon^{-p_0(p_0-1)}\},$$

where κ_1 *and* κ_2 *are two positive constants indepent of* ε.

§2. Preliminaries

For the reason of convenience, we impose the initial data at time $t = 2M$

$$t = 2M : u = \varepsilon\varphi(x), u_t = \varepsilon\psi(x) \tag{2.1}$$

and suppose supp $\varphi, \psi \subset \{x \big| |x| < M\}$.

We extend any function $F(t,x)$ defined on $[2M, +\infty) \times \mathbb{R}^2$ to be zero on $[0, 2M) \times \mathbb{R}^2$.

We associate with u the linear solution u_L of

$$\Box u_L = 0 \tag{2.2}$$

with

$$t = 2M : u_L = \varphi, \partial_t u_L = \psi. \tag{2.3}$$

We denote by E the forward fundamental solution of $\Box$, that is,

$$(E * F)(t,x) = \int_0^t d\tau \int_{|x-y|\le t-\tau} F(\tau,y)[(t-\tau)^2 - |x-y|^2]^{-1/2} dy/2\pi. \tag{2.4}$$

For $v^0(t,x)$ such that $v^0 \in L^p_{\rm loc}$, $E * |v^0|^p \in L^\infty_{\rm loc}$, we define $J_p(v^0)$ as the solution of the following problem:

$$u = v^0 + E * |u|^p. \tag{2.5}$$

In what follows, c will always denote a positive constant independent of ε and its meaning may change from line to line.

We first give some basic estimates.

Lemma 2.1. *If $p > 1$, we have*

$$|a+b| \le 2^{p-1}(|a|^p + |b|^p), \tag{2.6}$$

$$\big||a|^p - |b|^p\big| \le \big||a| - |b|\big| p \max(|a|^p, |b|^p), \tag{2.7}$$

$$\big||a+b|^p - (|a|^p + |b|^p)\big| \le p2^{p-1}(|a||b|^{p-1} + |b||a|^{p-1}), \tag{2.8}$$

$$|a||b|^{p-1} \le (1/p)^{1/p}(1-1/p)^{1-1/p}(|a|^p + |b|^p). \tag{2.9}$$

Proof. All of the above inequalities except (2.9) can be found in [1] Lemma 8.3. For (2.9), if we let

$$\chi = |a|^p/(|a|^p + |b|^p)$$

then, (2.9) is equivalent to say

$$\chi^{1/p}(1-\chi)^{1-1/p}, \qquad \chi \in [0,1]$$

attains its maximum at point $\chi = 1/p$, which is easy to see.

We next formulate the analogy of Lemma 8.1 of [1]

By Lemma 1.2 of [1], letting $f(s)$ have support on $s > 0$ and $h(t,x)$ be defined on $t > |x|$, $r = |x|$, we have

$$\Box (f(t-r)h(t,x)) = f(t-r)\Box h(t,r) + f'(t-r)[2r^{-1/2}(\partial_t + \partial_r)r^{1/2}h(t,x)]. \tag{2.10}$$

We take h such that

$$\Box h(t,x) = 0 \tag{2.11}$$

and

$$h(t,x) = |x|^{-q}h(t/|x|), \qquad 0 \le |x| < t. \tag{2.12}$$

Then by a direct calculation we see that h satisfies the following o.d.e.

$$(\xi^2 - 1)h''(\xi) + (2q+1)\xi h'(\xi) + q^2 h(\xi) = 0. \tag{2.13}$$

Moreover, by (2.10) we see that

$$\Box (f(t-r)|x|^{-q}h(t/|x|)) = f'(t-r)|x|^{-q-1}G(t/|x|), \tag{2.14}$$

where

$$G(\xi) = 2(1-\xi)h'(\xi) + (1-2q)h(\xi). \tag{2.15}$$

From (2.13) and (2.15) we can easily derive the o.d.e. satisfied by G.

$$(\xi^2 - 1)G''(\xi) + [(2q+1)\xi - 2]G'(\xi) + q^2 G(\xi) = 0. \tag{2.16}$$

We see that both h and G satisfy a Riemann P equation (see [6])

$$\begin{aligned} h &= P\left\{\begin{matrix} 1 & -1 & +\infty & \\ 0 & 0 & q & \xi \\ 1/2-q & 1/2-q & q & \end{matrix}\right\} \\ &= (\xi+1)^{-q} P\left\{\begin{matrix} 0 & 1 & +\infty & \\ 0 & 0 & q & 2/(\xi+1) \\ 0 & 1/2-q & 1/2 & \end{matrix}\right\}. \end{aligned} \tag{2.17}$$

So we may take

$$h(\xi) = (\xi+1)^{-q} F(1/2, q, 1, 2/(\xi+1)), \tag{2.18}$$

where for $0 < q < 1$

$$F(1/2, q, 1, z) = \Gamma(1)[\Gamma(q)\Gamma(1-q)]^{-1} \int_0^1 t^{q-1}(1-t)^{-q}(1-zt)^{-1/2} dt. \tag{2.19}$$

We may also write, for $0 \le q \le 1$,

$$F(1/2, q, 1, z) = \Gamma(1)[\Gamma(1/2)]^{-2} \int_0^1 t^{-1/2}(1-t)^{-1/2}(1-zt)^{-q} dt. \tag{2.20}$$

Similarly

$$\begin{aligned} G &= P\begin{Bmatrix} 1 & -1 & +\infty & \\ 0 & 0 & q & \xi \\ 3/2-q & -1/2-q & q & \end{Bmatrix} \\ &= (\xi+1)^{-q} P\begin{Bmatrix} 0 & 1 & +\infty & \\ 0 & 0 & q & 2/(\xi+1) \\ 0 & 3/2-q & -1/2 & \end{Bmatrix}, \end{aligned} \tag{2.21}$$

so we may assume

$$G(\xi) = (\xi+1)^{-q} G'(2/(\xi+1)). \tag{2.22}$$

From (2.15), (2.18) one easily calculate $G'(0) = 1$. So we get

$$G'(z) = F(-1/2, q, 1, z) = \Gamma(1)[\Gamma(q)\Gamma(1-q)]^{-1} \int_0^1 t^{q-1}(1-t)^{-q}(1-t)^{1/2} dt \tag{2.23}$$

for $0 < q < 1$ and

$$G'(z) = 1 \quad \text{if } q = 0,$$
$$G'(z) = (1-z)^{1/2} \quad \text{if } q = 1.$$

Lemma 2.2. *Let $J(s)$ have support on $s \ge 0$, and suppose $0 < q < 1$. Define*

$$h_q(t, x) = (t+|x|)^{-q} F(1/2, q, 1, 2|x|/(t+|x|)), \tag{2.24}$$
$$G_q(t, x) = (t+|x|)^{-q} F(-1/2, q, 1, 2|x|/(t+|x|)). \tag{2.25}$$

Then one has, for $r = |x|$,

$$E * J(t-r) G_q(t, x)/r = \int_0^{t-r} J(s) ds \; h_q(t, x). \tag{2.26}$$

Moreover, G_q, h_q, satisfy

$$ct^{-q} \le G_q(t, x) \le Ct^{-q}, \tag{2.27}$$

$$ct^{-q} \le h_q(t, x) \le Ct^{-q}, \quad 0 < q < 1/2, \tag{2.28}$$

$$ct^{-1/2} \log[2(t+r)/(t-r)] \le h_{1/2}(t, x) \le Ct^{-1/2} \log[2(t+r)/(t-r)], \tag{2.29}$$

$$ct^{-1/2}(t-r)^{1/2-q} \le h_q(t,x) \le Ct^{-1/2}(t-r)^{1/2-q}, \qquad 1/2 < q < 1, \tag{2.30}$$

and

$$|r^{1/2}(\partial_t + \partial_r)r^{1/2}h_q(t,x)| \le ch_q(t,x), \tag{2.31}$$

$$h_{\alpha+\beta}(t,x) \le h_q(t,x)(t-r)^{-\beta}, \qquad 0 < \beta < 1-q, \tag{2.32}$$

and for $q \ge 1/2$,

$$(t+r)^{-1}G_q(t,x) \le (t-r)^{-q}G_1(t,x). \tag{2.33}$$

Proof. (2.26) follows from (2.14) with $J = f'$.

We have (2.27) because $F(-1/2,q,1,z)$ is positive and $F(-1/2,q,1,1)$ exists, so

$$c \le F(-1/2,q,1,z) \le C. \tag{2.34}$$

We have (2.28) because of the same reason. Similarly, (2.29) and (2.30) follow from

$$c\log[2/(1-z)] \le F(1/2,1/2,1,z) \le C\log[2/(1-z)], \tag{2.35}$$

$$c(1-z)^{1/2-q} \le F(1/2,q,1,z) \le C(1-z)^{1/2-q}, \qquad 1/2 < q < 1. \tag{2.36}$$

(2.31) follows from the identity

$$r^{1/2}(\partial_t + \partial_r)r^{1/2}h_q(t,x) = G_q(t,x).$$

(2.32) follows from (2.20). (2.33) is equivalent to

$$(1-z)^q F(-1/2,q,1,z) \le F(-1/2,1,1,z). \tag{2.37}$$

We have, for $0 \le z \le 1$,

$$(1-z)^q F(-1/2,q,1,z) < (1-z)^q \le (1-z)^{1/2} = F(-1/2,1,1,z). \tag{2.38}$$

Lemma 2.3. *Let $F_0(\rho,\omega)$ be the Feidlander radiation field of u_L, defined by*

$$F_0(\rho,\omega) = \lim_{r\to\infty} r^{-1/2}u_L(r+\rho, r\omega), \tag{2.39}$$

and let

$$G_0(t,x) = F_0(t-r,\omega)/r^{1/2}, \qquad r = |x|, \omega = x/r, \tag{2.40}$$

$$\mu = \int |F_0(\rho,\omega)|^p d\rho\ \delta(t-r)r^{-p/2}, \tag{2.41}$$

$$u^0 = E * \mu. \tag{2.42}$$

Then for any $0 < \alpha < p/2 - 1$ we have

$$|E * |G_0|^p H(2r-t) - u^0| \le c\int \rho^\alpha \sup_\omega |F_0(\rho,\omega)|^p d\rho\ (t-r)^{-\alpha}h_{p/2-1}(t,x), \tag{2.43}$$

where H is the Heaviside function.

Proof. This lemma follows from Lemma 2.2 and Lemma 1.5 of [1] in the same way that Lemma 7.4 of [1] follows from Lemma 1.7 of [1] and Lemma 1.5 of [1].

Lemma 2.4. *Let u^0 be defined by (2.42), T_p be the life span of $J_p(u^0)$. Then*

$$0 < T_p < +\infty, \qquad 2 < p < p_0. \tag{2.44}$$

Proof. The existence part follows from Theorem 8.6 of [1], the blow up part follows from Lemma 8.8 of [1].

Theorem 2.1. *If u_i^0 satisfy $u_i^0 \in L^p_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$, $E * |u_i^0|^p \in L^\infty_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$, $\lambda_i \geq 0$, $i = 1, \cdots, N$, $\sum \lambda_i = 1$ and $\sum \lambda_i u_i^0 \geq 0$, then*

$$J_p(\sum \lambda_i u_i^0) \leq \sum \lambda_i J_p(u_i^0). \tag{2.45}$$

Proof. This theorem follows from Lemma 8.5 of [1] in the same way that Theorem 3.11 of [1] follows from Theorem 3.8 of [1].

Lemma 2.5. *Let α satisfy*

$$0 < \alpha < \min(p/2 - 1, (3-p)/2) \qquad \text{if } 2 < p < 3, \tag{2.46}$$

$$0 < \alpha < 1/2 \qquad \text{if } 3 \leq p \leq p_0. \tag{2.47}$$

Then

$$|E * |u_L|^p - u^0| \leq c_\alpha (t-r)^{-\alpha} h_{p/2-1}(t,x). \tag{2.48}$$

Proof.
$$\begin{aligned} &|E * |u_L|^p - u^0| \\ \leq & |E * |u_L|^p H(t-2r)| + |E * |u_L|^p H(2r-t) - E * |G_0|^p H(2r-t)| \\ &+ |E * |G_0|^p H(2r-t) - u^0| \\ = & \text{I+II+III}. \end{aligned}$$

By Lemma 7.1 of [1]

$$|u_L| \leq C(t-r)^{-1/2} t^{-1/2} H(t-r-M). \tag{2.49}$$

So we get

$$\begin{aligned} \text{I} &\leq cE * (t-r)^{-p/2} t^{-p/2} H(t-2r) H(t-r-M) \\ &\leq 2^\alpha cE * (t-r)^{-p/2-\alpha} t^{-p/2-\alpha-1} r^{-1} H(t-r-M) \\ &\leq c_\alpha h_{p/2+\alpha-1} \\ &\leq c_\alpha (t-r)^{-\alpha} h_{p/2-1}. \end{aligned} \tag{2.50}$$

Again by Lemma 7.1 of [1]

$$|G_0(t,x)| \leq c(t-r)^{-1/2} r^{-1/2} H(t-r-M), \tag{2.51}$$

$$\begin{aligned} |u_L - G_0| &\leq c(t-r)^{-1/2} r^{-1/2} t^{-1} \\ &\leq c_\alpha (t-r)^{-1/2} r^{-1/2} t^{-\alpha}. \end{aligned} \tag{2.52}$$

So by Lemma 2.1

$$\begin{aligned} \text{II} &\leq c_\alpha E * (t-r)^{-p/2} r^{-p/2} t^{-\alpha} H(2r-t) H(t-r-M) \\ &\leq c_\alpha E * (t-r)^{-p/2} t^{-(\alpha+p/2-1)} r^{-1} H(t-r-M) \\ &\leq c_\alpha h_{\alpha+p/2-1}. \end{aligned} \tag{2.53}$$

By Lemma 2.3, this lemma is proved.

Lemma 2.6. *There exists a positive constant c_1 such that*

(I) when $2 < p < 3$ and

$$(t-r)t^{p(3-p)/2}\varepsilon^{p(p-1)} \le c_1(p), \tag{2.54}$$

solutions of (1.1) (2.1) exist and satisfy

$$|u - \varepsilon u_L| \le c\varepsilon^p h_{p/2-1}; \tag{2.55}$$

(II) when $p = 3$ and

$$\varepsilon^{p(p-1)}t^\gamma(t-r)^{1-\gamma} \le c_1(p,\gamma), \qquad \forall 0 < \gamma < 1/3, \tag{2.56}$$

solution of (1.1) (2.1) *exists and satisfies* (2.55) ;

(III) when $3 < p < p_0$ and

$$(t-r)^{1+p(3-p)/2}\varepsilon^{p(p-1)} \le c_1(p), \tag{2.57}$$

solutions of (1.1) (2.1) *exist and satisfy* (2.55);

(IV) when $p = p_0$ and

$$\log(t-r)\varepsilon^{p(p-1)} \le c_1(p), \tag{2.58}$$

solution of (1.1) (2.1) *exists and satisfies* (2.55).

Remark. This lemma immediately gives a lower bound $T_\varepsilon \ge c\varepsilon^{-k(p)}$ when $2 < p < p_0$. $T_\varepsilon \ge \exp\{\kappa_1\varepsilon^{-p_0(p_0-1)}\}$ when $p = p_0$.

Proof. Let

$$w = u - \varepsilon u_L. \tag{2.59}$$

Then (1.1) (2.1) is equivalent to

$$w = E * |\varepsilon u_L + w|^p. \tag{2.60}$$

By Lemma 2.1

$$E * |\varepsilon u_L + w|^p \le 2^{p-1}(\varepsilon^p E * |u_L|^p + E * |w|^p). \tag{2.61}$$

By (2.49)

$$\begin{aligned} E * |u_L|^p &\le cE * (t-r)^{-p/2}t^{-p/2}H(t-r-M) \\ &\le cE * (t-r)^{-p/2}t^{1-p/2}r^{-1}H(t-r-M) \\ &\le ch_{p/2-1}. \end{aligned} \tag{2.62}$$

Therefore, we get

$$0 \le w \le c_0\varepsilon^p h_{p/2-1} + 2^{p-1}E * |w|^p. \tag{2.63}$$

We first prove the lemma for the case $2 < p < 3$. We assume presumably

$$0 \le w \le 4c_0\varepsilon^p h_{p/2-1}. \tag{2.64}$$

Then

$$\begin{aligned}2^{p-1}E*|w|^p &\le c\varepsilon^{p^2}E*|h_{p/2-1}|^p\\ &\le c\varepsilon^{p^2}E*t^{-(p/2-1)p}\\ &\le c\varepsilon^{p^2}E*r^{-1}t^{1-(p/2-1)p}\\ &\le c\varepsilon^{p^2}(t-r)t^{1-(p/2-1)p}\\ &\le c\varepsilon^p h_{p/2-1}\varepsilon^{p(p-1)}(t-r)t^{p(3-p)/2}.\end{aligned} \tag{2.65}$$

So, when (2.54) is satified for sufficiently small $c_1(p)$, we get from (2.65)

$$2^{p-1}E*|w|^p \le c_0\varepsilon^p h_{p/2-1}. \tag{2.66}$$

So we get from (2.63)

$$0\le w\le 2c_0\varepsilon^p h_{p/2-1}. \tag{2.67}$$

which is a sharper estimate than (2.64). So we have proved the lemma for the case $2<p<3$.

We next consider the case $p=3$. As before we assume (2.64), from which we get

$$\begin{aligned}w &\le ct^{-1/2}\log[4t/(t-r)]\\ &\le c_\gamma t^{-1/2}[4t/(t-r)]^{\gamma/3}.\end{aligned} \tag{2.68}$$

Therefore

$$\begin{aligned}2^{p-1}E*|w|^3 &\le c_\gamma\varepsilon^9 E*t^{\gamma-3/2}(t-r)^{-\gamma}\\ &\le c_\gamma\varepsilon^9 E*r^{-1}t^{\gamma-1/2}(t-r)^{-\gamma}\\ &\le c_\gamma\varepsilon^9 t^{\gamma-1/2}(t-r)^{1-\gamma}\\ &\le c_\gamma\varepsilon^3 h_{1/2}\varepsilon^{3(3-1)}t^\gamma(t-r)^{1-\gamma}.\end{aligned} \tag{2.69}$$

So, when (2.56) is satified for sufficiently small $c_1(p,\gamma)$, we get from (2.69)

$$2^{p-1}E*|w|^3 \le c_0\varepsilon^3 h_{1/2}. \tag{2.70}$$

By (2.63), we have

$$0\le w\le 2c_0\varepsilon^3 h_{1/2}, \tag{2.71}$$

which is a sharper estimate than (2.64). So we have proved the lemma for $p=3$.

The case $3<p\le p_0$ can be dealt with similarly.

Lemma 2.7. *Let*

$$T(\varepsilon)=\begin{cases}(T_p+1)\varepsilon^{-k(p)}, & 2<p\le 3,\\ +\infty, & 3<p\le p_0.\end{cases} \tag{2.72}$$

Then there exists $\theta(\varepsilon)$, $\theta(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that for $t\le T(\varepsilon)$.

$$|E*|\varepsilon u_L+w|^p - E*|\varepsilon u_L|^p - E*|w|^p| \le \theta(\varepsilon)(\varepsilon^p u^0 + E*|w|^p). \tag{2.73}$$

Proof. We only prove the theorem for the case $2<p<3$, the case $3\le p\le p_0$ can be treated in a similar way.

Let

$$\rho(\varepsilon) = \kappa\varepsilon^{-k(p)} \tag{2.74}$$

where κ satisfy

$$\kappa(T_p + 1)^{p(3-p)/2} \leq c_1(p). \tag{2.75}$$

Let χ_ρ denote the characteristic function of the set $\{(t,x)|0 \leq t - |x| \leq \rho(\varepsilon), t \leq T(\varepsilon)\}$. Then by Lemma 2.6, we have

$$\begin{aligned}|\chi_\rho w| &\leq c\varepsilon^p \chi_\rho h_{p/2-1} \\ &\leq c\varepsilon^p \chi_\rho t^{1-p/2} \\ &\leq c\varepsilon^p [T(\varepsilon)]^{(3-p)/2} [\rho(\varepsilon)]^\beta t^{-1/2} (t-r)^{-\beta} \\ &\leq c\varepsilon [\rho(\varepsilon)]^{\beta - 1/p} t^{-1/2} (t-r)^{-\beta}.\end{aligned} \tag{2.76}$$

Therefore, by Lemma 2.1

$$\begin{aligned}&|E * |\varepsilon u_L + w|^p - E * |\varepsilon u_L|^p - E * |w|^p| \\ \leq& 2^{p-1} p E * (|\varepsilon u_L||w|^{p-1} + |\varepsilon u_L|^{p-1}|w|) \\ =& cE * (|\varepsilon u_L||\chi_\rho w|^{p-1} + |\varepsilon u_L|^{p-1}|\chi_\rho w|) \\ &+ cE * (|(1-\chi_\rho)\varepsilon u_L||w|^{p-1} + |(1-\chi_\rho)\varepsilon u_L|^{p-1}|w|) \\ =& \text{I+II}.\end{aligned} \tag{2.77}$$

By (2.76) and (2.49), we get

$$\begin{aligned}\text{I} \leq& c\varepsilon^p [\rho(\varepsilon)]^{(\beta - 1/\rho)(p-1)} E * t^{-p/2} (t-r)^{1/2 + (p-1)\beta} \\ &+ c\varepsilon^p [\rho(\varepsilon)]^{(\beta - 1/p)} E * t^{-p/2} (t-r)^{(p-1)/2 + \beta}.\end{aligned} \tag{2.78}$$

We choose β such that

$$\beta - 1/p < 0, \qquad 1/2 + (p-1)\beta > 1, \qquad (p-1)/2 + \beta > 1. \tag{2.79}$$

Then by Lemma 2.2

$$\text{I} \leq \theta_1(\varepsilon)\varepsilon^p u^0, \tag{2.80}$$

where $\theta_1(\varepsilon) = c[\rho(\varepsilon)]^{\beta - 1/p}$.

Let

$$\theta_2(\varepsilon) = [\rho(\varepsilon)]^{-\alpha/p}, \tag{2.81}$$

where $0 < \alpha < p/2 - 1$. Then by Lemma 2.1

$$\begin{aligned}\text{II} &\leq c\{\theta_2^{1-p} E * |\varepsilon(1-\chi_\rho)u_L|^p + \theta_2 E * |w|^p + \theta_2^{-1/(p-1)} E * |\varepsilon(1-\chi_p)u_L|^p + \theta_2 E * |w|^p\} \\ &\leq c\{\theta_2^{1-p} E * |\varepsilon(1-\chi_\rho)u_L|^p + \theta_2 E * |w|^p\} \\ &\leq c\{\varepsilon^p \theta_2^{1-p} [\rho(\varepsilon)]^{-\alpha} E * t^{-p/2} (t-r)^{-p/2+\alpha} H(t-r-M) + \theta_2 E * |w|^p\} \\ &\leq c\theta_2(\varepsilon^p u^0 + E * |w|^p).\end{aligned} \tag{2.82}$$

Combining (2.80) and (2.82), we see that the lemma is true.

The next lemma concerns with the lower bound estimates of the life span of $J_p(\varepsilon^\sigma(t-r)^{-\alpha}h_{p/2-1})$, we want to show that it is longer than that of $J_p(\varepsilon^p h_{p/2-1})$ when $\sigma < p$ but sufficiently close to p. Here we take α to be small

$$0<\alpha<1/p, \qquad 2<p\le 3 \quad \text{or} \quad p=p_0, \tag{2.83}$$

$$\alpha p+p(p-3)/2<1, \qquad 3<p<p_0. \tag{2.84}$$

Lemma 2.8. *Let α satisfy (2.83) (2.84). Then*

(I) when $2+3p-p^2-2\alpha(p-1)<0$ and ε is small enough, $J_p(\varepsilon^\sigma(t-r)^{-\alpha}h_{p/2-1})$ exist globally and satisfy

$$J_p(\varepsilon^\sigma(t-r)^{-\alpha}h_{p/2-1})\le 2\varepsilon^\sigma(t-r)^{-\alpha}h_{p/2-1}; \tag{2.85}$$

(II) when $2+3p-p^2-2\alpha(p-1)>0$ and ε is small enough, $J_p(\varepsilon^\sigma(t-r)^{-\alpha}h_{p/2-1})$ exist for $t\le c_2\varepsilon^{-l(p,\alpha,\sigma)}$ and satisfy (2.85), where

$$l(p,\alpha,\sigma)=2\sigma(p-1)/(2+3p-p^2-2\alpha(p-1)). \tag{2.86}$$

Proof. This can be proved similarly as Lemma 2.6.

§3. Main Result

As in [1] Theorem 1.2 follows easily from Lemma 2.5, Lemma 2.7, Lemma 2.8 and Theorem 2.1.

The existence part of Theorem 1.3 follows from Lemma 2.6. We now prove the blow up part.

By Lemma 2.5 and Lemma 2.7, we get for $w=u-\varepsilon u_L$

$$w\ge \varepsilon^p(1-\theta(\varepsilon))u^0+(1-\theta(\varepsilon))E*|w|^p-c\varepsilon^p(t-r)^{-\alpha}u^0. \tag{3.1}$$

So there exists t_0 such that when $t-r\ge t_0$ and ε is small enough

$$w>c\{\varepsilon^{p_0}(t+r)^{-1/2}(t-r)^{(3-p_0)/2}+E*|w|^{p_0}\}. \tag{3.2}$$

Let Q be defined on $t-r\ge t_0$ satisfying

$$Q\le c\{\varepsilon^{p_0}(t+r)^{-1/2}(t-r)^{(3-p_0)/2}+E*|Q|^{p_0}H(t-r-t_0)\}. \tag{3.3}$$

Then we have

$$w(t,x)>Q(t,x), \qquad t-r\ge t_0. \tag{3.4}$$

As in [3], we take Q to be of the form of

$$Q=l(t-r-t_0)(t+r)^{-1/2}(t-r)^{(3-p_0)/2}, \tag{3.5}$$

where $l(s)=0$ when $s<0$. Then

$$\begin{aligned}
&2E*|Q|^{p_0}H(t-r-t_0)\\
=&2E*|l(t-r-t_0)|^{p_0}(t+r)^{-p_0/2}(t-r)^{-1}\\
=&E*|l(t-r-t_0)|^{p_0}r^{-1}(t+r)^{1-p_0/2}[(t-r)^{-1}-(t+r)^{-1}]\\
\ge& cE*|l(t-r-t_0)|^{p_0}r^{-1}G_{p_0/2-1}[(t-r)^{-1}-(t+r)^{-1}].
\end{aligned} \tag{3.6}$$

By Lemma 2.2,

$$G_{p_0/2-1}(t,x)(t+r)^{-1} \le G_1(t,x)(t-r)^{1-p_0/2}. \tag{3.7}$$

Therefore, we get from (3.6)

$$\begin{aligned}
&E * |Q|^{p_0} H(t-r-t_0)\\
\ge & cE * |l(t-r-t_0)|^{p_0} r^{-1}[G_{p_0/2-1}(t-r)^{-1} - G_1(t-r)^{\beta-1}]\\
= & c\{h_{p_0/2-1}\int_0^{t-r-t_0}(s+t_0)^{-1}|l(s)|^{p_0}ds\\
& - h_1 \int_0^{t-r-t_0}(s+t_0)^{\beta-1}|l(s)|^{p_0}ds\}\\
\ge & ch_{p_0/2-1}\{\int_0^{t-r-t_0}(s+t_0)^{-1}|l(s)|^{p_0}ds\\
& - (t-r)^{-\beta}\int_0^{t-r-t_0}(s+t_0)^{\beta-1}|l(s)|^{p_0}ds\},
\end{aligned} \tag{3.8}$$

where $\beta = 2 - p_0/2$. Therefore, if l satisfies

$$\begin{aligned}
l(\tau) = & c\varepsilon^{p_0} + c'\{\int_0^{\tau}(s+t_0)^{-1}|l(s)|^{p_0}ds\\
& - (\tau+t_0)^{-\beta}\int_0^{\tau}(s+t_0)^{\beta-1}|l(s)|^{p_0}ds\}
\end{aligned} \tag{3.9}$$

for sufficiently small c', then we have (3.3).

Let

$$l(\tau) = v(\ln[(\tau+t_0)/t_0]\beta^{-1}). \tag{3.10}$$

Then it is easy to see that v satisfies an o.d.e. of the form

$$\begin{cases} v''(z) + v'(z) = c''|v(z)|^{p_0},\\ v(0) = c\varepsilon^{p_0}, v'(0) = 0. \end{cases} \tag{3.11}$$

By Lemma 2.6 of [3], we know that v blows up before a time $\kappa_3\varepsilon^{-p_0(p_0-1)}$. Therefore, by (3.10) and (3.4) we conclude that

$$T_\varepsilon \le \exp\{\kappa_2\varepsilon^{p_0(p_0-1)}\}. \tag{3.12}$$

Thus we have proved Theorem 1.3.

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