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LARGE DEVIATION PROPERTY FOR RIEMANNIAN BROWNIAN MOTION ON A COMPLETE MANIFOLD**

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Abstract

Let M be a connected, complete Riemannian manifold with Ricci curvature bounded from below, $p(t,x,y)$ be the transition density function of the Riemannian Brownian motion, and for each $\epsilon > 0$, $(P_{\epsilon,x})$ be the diffusion measure family associated with the transition density function $p(\epsilon t, x, y)$. In this paper, it is shown that $(P_{\epsilon,x})$ has strongly large deviation properties, as $\epsilon \to 0$.

§1. Introduction

Suppose that (M,g) is a d-dimensional, noncompact, connected and complete Riemannian manifold which is stochastically complete, Δ is the Laplace-Betrami operator. The Riemannian Brownian motion (X_t) on M is the manifold-valued diffusion process associated with the infinitesimal generator $\frac{1}{2}\Delta$ (see [5]). That is, let $p(t,x,y)$ denote the minimal fundamental solution of the heat equation $\frac{1}{2}\Delta - \frac{\partial}{\partial t}$, then the Riemannian Brownian motion (X_t) is a continuous Markov process with the state space M and the transition density function $p(t,x,y)$ with respect to the Riemannian volume measure. In particular, (X_t) is conservative under the assumption that M is stochastically complete. Let Ω denote $C([0,1];M)$ with the topology of uniform convergence on the time interval $[0,1]$, $\mathcal{F}$ be its Borel σ-field. For each $\epsilon \in (0,1]$, denote by $(P_{\epsilon,x})_{x\in M}$ the diffusion measure family on $(\Omega,\mathcal{F})$ associated with the process $(X_\epsilon(t))_{0\le t\le 1}$, where $X_\epsilon(t) \equiv X(\epsilon t)$. Then M. I. Freidlin and A. D. Wentzell[3] proved that $(P_{\epsilon,y})$ has strongly large deviation property as $\epsilon \to 0$ and $y \to x$, under the additional condition that M has bounded geometry and Δ is locally uniformly elliptic. More precise, M. I. Freidlin and A. D. Wentzell proved the following (see Theorem 3.2 in [3])

Theorem. *Assume that M has the following "bounded geometry" property: There is a positive number r_0, and for every ball $B(x,r_0) \subset M$, there is a chart $\phi_x: B(x,r_0) \to R^d$ satisfying the uniform quasi-isometry condition*

$$C_0^{-1} d(y,z) \le |\phi_x(y) - \phi_x(z)| \le C_0 d(y,z), \tag{1.1}$$

$\forall x \in M$, $\forall y,z \in B(x,r_0)$, with a constant C_0 independent of x, Moreover, assume Δ is locally uniformly elliptic operator with constant ellipticity $\mu > 0$. Then $(P_{\epsilon,y})$ has strongly large deviation property (see Theorem 3.1).

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By the Rauch comparison theorem[2], and by use of normal coordinates with respect to $x \in M$, the property (1.1) holds if the sectional curvatures of M are bounded from above and from below, and if the injectivity radius at each $x \in M$ is larger then a positive constant.

In this paper, we prove that $(P_{\epsilon,y})$ has strongly large deviation property as $\epsilon \to 0$ and $y \to x$ provided that the Ricci curvature is bounded from below. We mainly use the heat estimates established by Peter Li and S.T.Yau[7] and the comparison theorems for the solutions of stochastic differential equations to prove above result. We note that under the condition (1.1), the volume $\mathrm{Vol}(B_y(1))$ of a unit geodesic ball centered at a point $y \in M$ is larger than a fixed positive constant independent of y. On the other hand, there are complete Riemannian manifolds with Ricci curvatures bounded from below (even non-negative) satisfying $\inf_{y\in M} \mathrm{Vol}(B_y(1)) = 0$.

Remark. It was proved by R.Azencott (see Grandes deviations et applications, in Lecture Notes in Mathematics 774) that $(P_{\epsilon,x})$ has large deviation property as $\epsilon \to 0$.

§2. Several Facts

In this section, we assume that (M,g) is a connected, complete Riemannian manifold (without boundary) with Ricci curvature bounded from below, and assume that $Ricc(M) \geq -(d-1)k^2$, $k \geq 0$ is a constant.

Theorem 2.1 (Comparison theorem for the heat kernel). *Let $\mathcal{M}(d,-k^2)$ denote the d-dimensional space form with constant section curvatures $-k^2$, and p_k be the heat kernel of $\mathcal{M}(d,-k^2)$ which solves the heat equation $\frac{1}{2}\Delta - \frac{\partial}{\partial t}$ on $\mathcal{M}(d,-k^2)$, given by*

$$p_k(t,\overline{x},\overline{y}) = \mathcal{E}_k(t, d_k(\overline{x},\overline{y})) \tag{2.1}$$

for $\overline{x},\overline{y} \in \mathcal{M}(d,-k^2)$, where d_k denotes the distance in $\mathcal{M}(d,-k^2)$ and

$$\mathcal{E}_k\colon\ (0,\infty)\times[0,\infty) \to R.$$

Then

$$p(t,x,y) \geq \mathcal{E}_k(t, d(x,y)) \tag{2.2}$$

for all $(t,x,y) \in (0,\infty)\times M \times M$, where $p(t,x,y)$ denotes the transition density function of the Riemannian Brownian motion on M, and $d(x,y)$ denotes the geodesic distance from x to y, respectively.

For the upper bound of the heat kernel, we have

Theorem 2.2. *For any $\delta \in (0,1)$, there are constants $C(d,\delta)$ and $C(\delta)$ such that*

$$p(t,x,y) \leq \frac{C(d,\delta)}{V_x^{\frac{1}{2}}(\sqrt{t})V_y^{\frac{1}{2}}(\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{2(1+\delta)t} + C(d)\delta k^2 t\right) \tag{2.3}$$

for all $(t,x,y) \in (0,\infty)\times M\times M$. Moreover, $C(d,\delta) \to \infty$ when $\delta \to 0$. Here $V_x(\sqrt{t})$ denotes the volume of the geodesic ball centered at x of radius $\sqrt{t}$.

For the proofs of Theorems 2.1 and 2.2, we refer to I. Chavel[2] and Peter Li and S. T. Yau[7], respectively.

Lemma 2.1[10]. (1) *For a fixed point $x_0 \in M$, let $\rho(x) = d(x,x_0)$. Then there exists a constant C depending only on n, k and $B_{x_0}(1)$, such that*

$$V_x(1) \geq e^{-C\rho(x)}. \tag{2.4}$$

(2) For $0 < r \le R$

$$\frac{V_x(R)}{V_x(r)} \le \frac{\overline{V}(R)}{\overline{V}(r)}, \tag{2.5}$$

where $\overline{V}(R)$ denotes the geodesic ball of radius R in $\mathcal{M}(d, -k^2)$.

Now we can prove the following Theorem 2.3, which is an improvement of the famous S. R. S. Varadhan's theorem.

Theorem 2.3[9].

$$\lim_{t\to 0} 2t \ln p(t, x, y) = -d^2(x, y) \tag{2.6}$$

uniformly on the geodesic ball $B(x_0, r)$ for any $x_0 \in M, r > 0$.

Proof. By Theorem 2.1, we know

$$\liminf_{t\to 0} 2t \ln p(t, x, y) \ge -d^2(x, y)$$

uniformly over $x, y \in M$ such that $d(x, y)$ is bounded. By Theorem 2.2 and Lemma 2.1 we get

$$\begin{aligned} 2t \ln p(t, x, y) \le& 2t \ln C(d, \delta) + 2t \ln \frac{\overline{V}(1)}{\overline{V}(\sqrt{t})} \\ &+ 2tCd(x, x_0) + 2tC(\delta)\delta k^2 - \frac{d^2(x, y)}{(1 + \delta)}. \end{aligned}$$

Hence we have

$$\limsup_{t\to 0} 2t \ln p(t, x, y) \ge -d^2(x, y)$$

uniformly on $B(x_0, r)$, for any $x_0 \in M, r > 0$.

Now we turn our attention to the Riemannian Brownian motion. Let $\Omega = C([0, 1]; M)$ with the topology of uniform convergence on $[0, 1]$, $x(t)$ be the coordinate process and $\mathcal{F}, \mathcal{F}_t$ be the natural σ-field. It is well known that for each $x \in M$, there is a unique probability measure P_x on $(\Omega, \mathcal{F})$, such that $(\Omega, \mathcal{F}, \mathcal{F}_t, x(t), \theta_t, P_x)$ is a strongly Feller continuous, Markov process with the transition density function $p(t, x, y)$. For each $\epsilon > 0$, denote by $P_{\epsilon,x}$ the diffusion measure family on $(\Omega, \mathcal{F})$ associated with the transition density function $p_\epsilon(t, x, y) = p(\epsilon t, x, y)$.

For each $x \in M$, consider the radial process $\rho(x(t)) = d(x(t), x)$. By Kendall [6], there is a standard Brownian motion (with initial value zero) β_t over the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$ and a nondecreasing process L_t with initial value zero which increases only when $x(t)$ belongs to the cut-locus $C(x)$ of x, such that

$$\rho(x(t)) = \beta_t + \frac{1}{2}\int_0^t \Delta\rho(x(s))ds - L_t. \tag{2.7}$$

By the comparison theorem for Laplacian's (see [10]), we have

$$\Delta\rho \le (n-1)k \coth k\rho \tag{2.8}$$

in the weak sense. For any $r > 0$, let γ_t be the process on $[0, 1]$ determined by the stochastic differential equation

$$\gamma_t = \frac{1}{r}\beta_t + \frac{d-1}{2}\int_0^t k \coth k\gamma_s ds. \tag{2.9}$$

By a comparison theorem for solutions of stochastic differential equations (see Ikeda and Watanabe[5]), we get

$$P_x(\sup_{s\in[0,t]} \rho(x(s)) \geq r) \leq P_x(\sup_{s\in[0,t]} \gamma_s \geq 1). \tag{2.10}$$

Theorem 2.4. *Let $\alpha = (2(d-1)k \coth \frac{k}{4})^{-1} \wedge 1$, which is a positive constant depending only on d and k. There exists a constant C depending only on d, such that*

$$P_x(\sup_{s\in[0,t]} d(x(t),x) \geq r) \leq C \exp(-\frac{r^2}{Ct}) \tag{2.11}$$

for all $(t,x) \in (0,\alpha] \times M$.

Proof. Consider the solution γ_t of the stochastic differential equation (2.9). Define a sequence of stopping times

$$\begin{aligned} T_0 &= \inf\{t > 0: \gamma_t > 1/2\}, \\ T_1 &= \inf\{t > 0: \gamma < 1/4\}, \\ T_{2m} &= T_0 \circ \theta_{T_{2m-1}}, \\ T_{2m+1} &= T_1 \circ \theta_{T_{2m}}, \\ T &= \inf\{t \geq 0: \gamma_t = 1\}. \end{aligned}$$

It is easy to see that $\gamma_t \leq 1/2$ on $[T_{2m-1}, T_{2m}]$, hence T is in one of the intervals $[T_{2m}, T_{2m+1}]$, but $\gamma_t \geq 1/4$ on $[T_{2m}, T_{2m+1}]$. Thus we have

$$\gamma_T - \gamma_{T_{2m}} = \frac{1}{r}(\beta_T - \beta_{T_{2m}}) + \frac{d-1}{2}\int_{T_{2m}}^{T} k \coth k\gamma_s ds.$$

Noting that $\gamma_T = 1$, we get

$$1 \leq \frac{1}{2} + \frac{1}{r}(\beta_T - \beta_{T_{2m}}) + \frac{d-1}{2} Tk \coth \frac{k}{4},$$

hence we have

$$\sup_{s\in[0,t]} \frac{2}{r}|\beta_s| \geq \frac{1}{2} - \frac{d-1}{2} tk \coth \frac{k}{4}$$

on $\{T \leq t\}$. Hence for any $t \in (0,\alpha]$, we have

$$\begin{aligned} P_x(\sup_{s\in[0,t]} \gamma_s \geq 1) &= P_x(T \leq t) \\ &\leq P_x(\sup_{s\in[0,t]} |\beta_s| \geq \frac{r}{8}) \\ &\leq C \exp(-\frac{r^2}{Ct}) \end{aligned}$$

for some positive constant. Using 2.10, we get (2.11).

§3. Strongly Large Deviation Property

In this section, we assume that M is a connected, complete Riemannian manifold which is stochastically complete. For any $\omega \in \Omega \equiv C([0,1]; M)$, define

$$I(\omega) = \frac{1}{2}\int_0^1 |\dot{\omega}(t)|^2 dt \leq +\infty, \tag{3.1}$$

where $\dot{\omega}$ denotes the generalized derivatives of ω. Then I is a rate function on Ω, that is, for any $K \geq 0$, $\{\omega: I(\omega) \leq K\}$ is a compact subset of Ω. It is well known that

$$\frac{1}{2}d^2(x,y) = \inf_{\substack{\omega:\omega(0)=x \\ \omega(1)=y}} I(\omega) \tag{3.2}$$

and easy to check that for $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$

$$\inf_{\substack{\omega(t_i)=x_i \\ i=1,\cdots,n}} I(\omega) = \frac{1}{2}\sum_{i=1}^{n-1} \frac{d^2(x_{i+1},x_i)}{t_{i+1}-t_i}. \tag{3.3}$$

Now we can state the strongly large deviation property for the Riemannian Brownian motion in a small time interval.

Theorem 3.1. *Assume that*

(1)$\lim_{t\to 0} 2t \ln p(t,x,y) = -d^2(x,y)$ *uniformly on* $B(x_0,r)$ *for any* $x_0 \in M, r > 0$.

(2) *There are constants* $\alpha \in (0,1], C > 0$ *such that*

$$P_x(\sup_{s\in[0,t]} d(x(t),x) \geq r) \leq C\exp(-\frac{r^2}{Ct})$$

for all $(t,x) \in (0,\alpha] \times M$.

Then $(P_{\epsilon,x})$ *has strongly large deviation property with rate function* I, *that is , for any closed subset* $C \subset \Omega$ *and open subset* $G \subset \Omega$, *we have*

$$\limsup_{\substack{\epsilon\to 0 \\ y\to x}} P_{\epsilon,y}(C) \leq -\inf_{\omega\in C_x} I(\omega), \tag{3.4}$$

$$\liminf_{\substack{\epsilon\to 0 \\ y\to x}} P_{\epsilon,y}(G) \geq -\inf_{\omega\in G_x} I(\omega), \tag{3.5}$$

where $C_x = \{\omega \in C: \omega(0) = x\}$.

We say a complete Riemannian manifold M has strongly large deviation property if (3.4) and (3.5) hold for $(P_{\epsilon,x})$, where $(P_{\epsilon,x})$ denotes the diffusion measure family associated with the transition density function $p(\epsilon t,x,y)$, and $p(t,x,y)$ denotes the heat kernel of the manifold M. Then Theorems 2.3 and 2.4 imply the following

Theorem 3.2. *If* M *is a connected, complete Riemannian manifold with Ricci curvature bounded from below, then* M *has strongly large deviation property.*

We can follow the idea given by S. R. S. Varadhan[9] to prove Theorem 3.1, so we will merely outline the proof, giving details only at those points where new difficulties arise.

Let $\pi: 0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of the time interval $[0,1]$, denote by T_π the map: $\Omega \to M \times \cdots \times M$ defined by

$$T_\pi\omega = \{\omega(t_0),\cdots,\omega(t_n)\}. \tag{3.6}$$

Then for any Borel set $A \subset M \times \cdots \times M$, we have

$$P_{\epsilon,y}(T_\pi^{-1}A) = \int_{A_y} \prod_{j=1}^{n} p_\epsilon(t_j - t_{j-1}, y_{j-1}, y_j)dy_j, \tag{3.7}$$

where $A_y \equiv \{(y_0,\cdots) \in A: y_0 = y\}$.

Lemma 3.1. *Let* $A \subset M \times \cdots \times M$ *be a closed subset and* $C = T_\pi^{-1}A$. *Then*

$$\limsup_{\substack{\epsilon\to 0 \\ y\to x}} \epsilon \ln P_{\epsilon,y}(C) \leq -\inf_{\omega\in C_x} I(\omega). \tag{3.8}$$

Proof. If A is bounded, then using (2.6) and (3.3), one can easily check that (3.8) holds. Since $d(x(t),y) \le d(x(t),x)+d(x,y)$, we have

$$P_{\epsilon,y}(\sup_{s\in[0,1]} d(x(t),x) \ge r) \le P_{\epsilon,y}(\sup_{s\in[0,1]} d(x(t),y) \ge r/2)$$

for any $x,y\in M$ such that $d(x,y)\le r/2$ and $r>0$. Hence for any $\epsilon\in(0,\alpha]$, $r\ge 1$, we have

$$\sup_{d(y,x)\le 1/2} P_{\epsilon,y}(\sup_{s\in[0,1]} d(x(t),x)\ge r) \le C\exp(-\frac{r^2}{C\epsilon}).$$

Choose a constant $r>0$ such that $\frac{r^2}{4C} > \inf_{\omega\in C_x} I(\omega)$. Denote

$$A_1 = A\cap \overline{B(x,r)},\quad A_2 = A\cap \overline{B(x,r)}^c.$$

Then A_1 is a bounded set. Hence we have

$$\limsup_{\substack{\epsilon\to0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(T_\pi^{-1}A_1) \le -\inf_{\omega\in C_x} I(\omega)$$

and

$$\begin{aligned}\limsup_{\substack{\epsilon\to0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(T_\pi^{-1}A_2) &\le \limsup_{\substack{\epsilon\to0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(T_\pi^{-1}B(x,r)^c)\\ &= \limsup_{\substack{\epsilon\to0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(\sup_{s\in[0,1]} d(x(s),x)\ge r)\\ &\le -\frac{r^2}{4C}.\end{aligned}$$

Thus we have proved (3.8) since

$$P_{\epsilon,y}(C) \le P_{\epsilon,y}(T_\pi^{-1}A_1) + P_{\epsilon,y}(T_\pi^{-1}A_2).$$

For any positive intergal number n, denote by π_n the partition: $0=t_0<t_1<\cdots<t_n=1$ with $t_i=\frac{i}{n}$, and by $x_n(t)$ the process such that $x_n(t_i)=x(t_i)$, and join the successive ones by geodesics (which is well defined because M is complete). Using the same method, we can define a $\omega_n\in\Omega$ such that $x_n(t;\omega)=x(t;\omega_n)$.

Lemma 3.2. *For any* $\delta>0$,

$$\limsup_{n\to\infty}\limsup_{\epsilon\to0}\sup_{x}\epsilon\ln P_{\epsilon,x}\{\sup_{t\in[0,1]} d(x(t),x_n(t))\ge\delta\} = -\infty. \tag{3.9}$$

Proof. Using Corollary 2.4, by the same method used in the proof of Lemma 3.2 in [9], we can prove the lemma. In fact, one can easily check

$$\sup_{t\in[0,1]} d(x(t),x_n(t)) \le 2\sup_{0\le j\le n-1}\sup_{t_j\le t\le t_{j+1}} d(x(t),x(t_j)).$$

For any positive intergal number $n > \frac{1}{\alpha}$, using Markov property we have

$$\begin{aligned}
&P_{\epsilon,x}\left(\sup_{t\in[0,1]} d(x(t),x_n(t)) \geq \delta\right) \\
\leq &P_{\epsilon,x}\left[\sup_{0\leq j\leq n-1}\sup_{t_j\leq t\leq t_{j+1}} d(x(t),x(t_j)) \geq \delta/2\right] \\
\leq &\sum_{j=1}^{n-1} P_{\epsilon,x}\left[\sup_{t_j\leq t\leq t_{j+1}} d(x(t),x(t_j)) \geq \delta/2\right] \\
\leq &n\sup_x P_{\epsilon,x}\left(\sup_{t\in[0,\frac{1}{n}]} d(x(t),x(0)) \geq \delta/2\right) \\
\leq &nC\exp\left(-\frac{n\delta^2}{4C\epsilon}\right).
\end{aligned}$$

Now (3.9) follows immediately.

It is a routine matter to prove the upper bound (3.4) by using (3.8) and (3.9) (see [9]). For completion, we outline the proof as follows.

Let $C \subset \Omega$ be a closed set. For any $\delta > 0$, set

$$\begin{aligned}
&C_\delta = \{\omega \in C: d(\omega(0),x) < \delta\}, \\
&C_\delta^\delta = \{\gamma: \sup_{t\in[0,1]} d(\gamma(t),\omega(t)) < \delta \text{ for some } \omega \in C_\delta\}, \\
&I^\delta(\omega) = \inf_{\gamma:\sup_{t\in[0,1]} d(\gamma(t),\omega(t))<\delta} I(\gamma), \\
&\alpha_\delta = \inf_{\omega\in C_\delta^\delta} I(\omega).
\end{aligned}$$

It is easy to see that $\omega \in C_\delta$ implies $I^\delta(\omega) \geq \alpha_\delta$. We have

$$\begin{aligned}
P_{\epsilon,y}(C) &= P_{\epsilon,y}(C_\delta) \leq P_{\epsilon,y}(I^\delta(\omega) \geq \alpha_\delta) \\
&\leq P_{\epsilon,y}\left(\omega: \sup_{t\in[0,1]} d(\omega(t),\omega_n(t)) \geq \delta) + P_{\epsilon,y}(I(\omega_n) \geq \alpha_\delta\right)
\end{aligned}$$

for any $\delta > 0, n$ and y such that $d(y,x) < \delta$. We note that

$$\{I(\omega_n) \geq \alpha_\delta\} = \left\{\omega: \frac{1}{2}\sum_{j=1}^{n-1}\frac{d^2(\omega(t_j),\omega(t_{j+1}))}{t_{j+1}-t_j} \geq \alpha_\delta\right\}.$$

Using (3.8), we get

$$\limsup_{\substack{\epsilon\to 0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(I(\omega_n) \geq \alpha_\delta) \leq -\alpha_\delta$$

for any n. Using (3.9), we have

$$\limsup_{\substack{\epsilon\to 0\\ y\to x}} \epsilon\ln P_{\epsilon,y}(C) \leq -\alpha_\delta$$

for any $\delta > 0$. Letting $\delta \to 0$ in above inequality, we get the upper bound by the fact that $\lim_{\delta\to 0}\alpha_\delta = \inf_{\omega\in C_x} I(\omega)$.

On the other hand, the lower bound is a consequence of Theorem 3.2, (3.9) and the upper bound (3.4) (see the proof of Lemma 3.4 of [9]).

This completes the proof.

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