Chin. Ann. of Math.
14B: 2(1993), 245-252.

CONSTRUCTION OF PBIB DESIGNS BY USING DESCARTES PRODUCT OF KNOWN PBIB DESIGNS

WEI WANDI* XU JUYONG**

Abstract

The author discuss the Descartes product of a number of association schemes, introduce the Descartes product of $t\,(t \geq 2)$ given PBIB designs and, finally, solve the general problem: How to obtain a new PBIB design from any $t\,(t \geq 2)$ PBIB designs?

§1. Introduction

The concepts and symbols used but not defined in this paper are adopted from [1] or [2].

Let $[a, b]$ be the set of the integers neither smaller than a nor larger than b. Let $D_q(q \in [1, t])$ be a PBIB design with w_q associate classes and with the parameters

$$v_q,\ b_q,\ r_q,\ k_q,\ n_{qi_q},\ \lambda_{qi_q},\ p^{i_q}_{qj_ql_q},\ i_q,\ j_q,\ l_q \in [1, w_q]. \tag{1.1}$$

As the usual convention in the PBIB design theory

$$\left.\begin{aligned} n_{q0} &= 1, \quad \lambda_0 = r, \\ p^{0}_{qj_ql_q} &= n_{qj_q}\delta_{j_ql_q}, \\ p^{i_q}_{q0j_q} &= p^{i_q}_{qj_q0} = \delta_{i_qj_q}, \end{aligned}\right\} \tag{1.2}$$

where

$$\delta_{xy} = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

Denote by S_q the set of treatments of D_q, by

$$R_{qi_q}\ \ (i_q \in [1, w_q])$$

the i_q-th associate class, and by $\mathcal{B}^q = \{B^q_1, B^q_2 \cdots, B^q_{b_q}\}$ the family of blocks of D_q. Let A_q be the incidence matrix of D_q:

$$A_q = (a^q_{i_qj_q}),\ \ i_q \in [1, v_q],\ \ j_q \in [1, b_q],$$

where $a^q_{i_qj_q}$ is the number of times the i-th treatment of D_q occurs in the j_q-th block of D_q.

In order to construct a new PBIB design from two given PBIB designs, Vartak[3] introduced the concept of the Kronecker product of two PBIB designs, say D_1 and D_2. Let

$$A = A_1 \times A_2$$

Manuscript received June 8, 1990.

*Department of Mathematics, Shchuan University, Chengdu 610064, Sichuan, China.
**Department of Basic Science, Wuhan Urban Construction Institute, Wuhan 430074, Hubei, China .

be the Kronecker product of the matrices A_1 and A_2 defined by (1.2). Then Vartak[3] proved that the matrix A can be viewed as the matrix of some PBIB design with $w_1 + w_2 + w_1w_2$ associate classes and with the parameters

$$v = v_1v_2,\ b = b_1b_2,\ r = r_1r_2,\ k = k_1k_2, \tag{1.3}$$

$$n_{i_2} = n_{2i_2},\ n_{w_2+i_1} = n_{1i_1},\ n_{w_2+i_1+i_2w_1} = n_{2i_2}n_{1i_1}, \tag{1.4}$$

$$\lambda_{i_2} = r_1\lambda_{2i_2},\ \lambda_{w_2+i_1} = r_2\lambda_{1i_1},\ \lambda_{w_2+i_1+i_2w_1} = \lambda_{2i_2}\lambda_{1i_1}, \tag{1.5}$$

$$\left(p_{yz}^{i_2}\right) = \begin{pmatrix} \left(p_{2j_2l_2}^{i_2}\right) & O_{w_2\times w_1} & O_{w_2\times w_1w_2} \\ O_{w_1\times w_2} & O_{w_1\times w_1} & (\delta_{i_2j_2})\times(n_{1j_1}\delta_{j_1l_1}) \\ O_{w_1w_2\times w_2} & (\delta_{j_2l_2})\times(n_{1j_1}\delta_{j_1l_1}) & \left(p_{2j_2l_2}^{i_2}\right)\times(n_{1j_1}\delta_{j_1l_1}) \end{pmatrix} \tag{1.6}$$

$$\left(p_{yz}^{w_2+i_1}\right) = \begin{pmatrix} O_{w_2\times w_2} & O_{w_2\times w_1} & (n_{2j_2}\delta_{j_2l_2})\times(\delta_{i_1j_1}) \\ O_{w_1\times w_2} & \left(p_{1j_1l_1}^{i_1}\right) & O_{w_1\times w_1w_2} \\ (n_{2j_2}\delta_{j_2l_2})\times(\delta_{i_1j_1}) & O_{w_1w_2\times w_1} & (n_{2j_2}\delta_{j_2l_2})\times\left(p_{1j_1l_1}^{i_1}\right) \end{pmatrix} \tag{1.7}$$

$$\left(p_{yz}^{w_2+i_1+i_2w_1}\right) = \begin{pmatrix} O_{w_2\times w_2} & (\delta_{i_2j_2})\times(\delta_{i_1j_1}) & \left(p_{2j_2l_2}^{i_2}\right)\times(\delta_{i_1j_1}) \\ (\delta_{i_2j_2})\times(\delta_{i_1j_1}) & O_{w_1\times w_1} & (\delta_{i_2j_2})\times\left(p_{1j_1l_1}^{i_1}\right) \\ \left(p_{2j_2l_2}^{i_2}\right)\times(\delta_{i_1j_1}) & (\delta_{i_2j_2})\times\left(p_{1j_1l_1}^{i_1}\right) & \left(p_{2j_2l_2}^{i_2}\right)\times\left(p_{1j_1l_1}^{i_1}\right) \end{pmatrix} \tag{1.8}$$

where

$$i_1,\ j_1,\ l_1 \in [1, w_1],\ i_2,\ j_2,\ l_2 \in [1, w_2],\ y, z, \in [1, w_1 + w_2 + w_1w_2].$$

The general problem naturally arises: how to obtain a PBIB design from any t ($t \geq 2$) PBIB designs? From the complicated expressions in (1.4)-(1.8) for the parameters of the Kronecker product of two given PBIB designs, it can be seen that if one employed the Vartak's method to the general problem, then the parameters of the Kronecker product of t given PBIB designs would be very difficult both in computing and in expressing. We introduce in the present paper the Descartes product of t given PBIB designs, and then solve the above general problem. We start with the Descartes product of a number of association schemes.

§2. Descartes Product of Association Schemes

Let $M_q (q \in [1, t])$ be an association scheme with w_q associate classes and with the parameters

$$v_q,\ n_{qi_q},\ p_{qj_ql_q}^{i_q},\ i_q,\ j_q,\ l_q \in [1, w_q]. \tag{2.1}$$

Denote by S_q the symbol set of M_q, and by R_{qi_q} $(i_1 \in [1, w_q])$ the associate classes of M_q. Put

$$R_{q0} = \{(s_q, s_q) | s_q \in S_q\}, \tag{2.2}$$

which is called the 0-th associate class. Let

$$S = S_1 \times S_2 \times \cdots \times S_t, \tag{2.3}$$

$$I_0 = [0, w_1] \times [0, w_2] \times \cdots \times [0.w_t], \tag{2.4}$$

$$I = I_0 \backslash \{(0, 0, \cdots, 0)\}, \tag{2.5}$$

$$w = (w_1 + 1)(w_2 + 1) \cdots (w_t + 1). \tag{2.6}$$

Suppose ψ is an arbitrarily given one-one mapping from I_0 to $[0, w-1]$

$$\psi : I_0 \to [0, w-1] \tag{2.7}$$

with the property

$$\psi(0, 0, \cdots, 0) = 0. \tag{2.8}$$

Difinition 2.1. *Let $(i_1, i_2, \cdots, i_t) \in I_0$. Two elements*

$$s^{(1)} = (s_1^{(1)},\ s_2^{(1)}, \cdots, s_t^{(1)}),\ s^{(2)} = (s_1^{(2)},\ s_2^{(2)}, \cdots, s_t^{(2)})$$

are said to be the $\psi(i_1, i_2, \cdots, i_t)$-th associates of each other, if $s_q^{(1)}$ and $s_q^{(2)}$ are the i_q-th associates of each other of M_q for each $i_q \in [0, w_q]$. And the set

$$R_{\psi(i_1, i_2, \cdots, i_t)} = \left\{ (s^{(1)}, s^{(2)}) \,\middle|\, s^{(1)}, s^{(2)} \text{ are the } (i_1, i_2, \cdots, i_t)\text{-th associates} \right\} \tag{2.9}$$

($(i_1, i_2, \cdots, i_t) \in I_0$) is called the $\psi(i_1, i_2, \cdots, i_t)$-th associate class on the set S.

Then we have

Lemma 2.1. (1) *For any $(i_1, i_2, \cdots, i_t) \in I_0$, $R_{\psi(i_1, i_2, \cdots, i_t)}$ is symmetric, i. e.,*

$$(s^{(1)}, s^{(2)}) \in R_{\psi(i_1, i_2, \cdots, i_t)}$$

if and only if $(s^{(2)}, s^{(1)}) \in R_{\psi(i_1, i_2, \cdots, i_t)}$.

(2) *$R_{\psi(i_1, i_2, \cdots, i_t)} \neq \emptyset$, $(i_1, i_2, \cdots, i_t) \in I_0$.*

(3) *Suppose that $(i_1, i_2, \cdots, i_t)$ and $(j_1, j_2, \cdots, j_t)$ are any two distinct elements of I_0. Then*

$$R_{\psi(i_1, i_2, \cdots, i_t)} \cap R_{\psi(j_1, j_2, \cdots, j_t)} = \emptyset.$$

(4) *$S \times S = \cup R_{\psi(i_1, i_2, \cdots, i_t)}$, $(i_1, i_2, \cdots, i_t) \in I_0$.*

Proof. These properties are immediately derived from the correspondent ones for R_{qi_q} ($i_q \in [0, w_q]$, $q \in [1, t]$).

Lemma 2.2. *Let s be a given element of S, and $(i_1, i_2, \cdots, i_t)$ a given element of I. Then the cardinality of the set*

$$C_s(i_1, i_2, \cdots, i_t) = \{s' \in S | (s, s') \in R_{\psi(i_1, i_2, \cdots, i_t)}\} \tag{2.10}$$

is

$$\prod_{q=1}^{t} n_{qi_q},$$

which does not depend on the choice of s.

Proof. Let $s = (s_1, s_2, \cdots, s_t)$. By Definition 2.1,

$$((s_1, s_2, \cdots, s_t),\ (s'_1, s'_2, \cdots, s'_t)) \in R_{\psi(i_1, i_2, \cdots, i_t)} \tag{2.11}$$

if and only if

$$(s_q, s'_q) \in R_{qi_q}, \quad q \in [1, t]. \tag{2.12}$$

Since the number of s'_q satisfying (2.2) is n_{qi_q}, the number of $(s'_1, s'_2, \cdots, s'_t)$ satisfying (2.11) is (2.10). Evidently, the value of (2.10) does not depend on the choice of s. This proves the lemma.

Lemma 2.3. *Suppose that $(i_1, i_2, \cdots, i_t)$, $(j_1, j_2, \cdots, j_t)$ and $(l_1, l_2, \cdots, l_t)$ be any three given elements of I. Let s^1, s^2 be two distinct elements of S such that*

$$(s^1, s^2) \in R_{\psi(i_1, i_2, \cdots, i_t)}. \tag{2.13}$$

Then the cardinality of the set

$$\{s \in S | (s, s^1) \in R_{\psi(j_1, j_2, \cdots, j_t)},\ (s, s^2) \in R_{\psi(l_1, l_2, \cdots, l_t)}\}$$

is

$$\prod_{q=1}^{t} p^{i_q}_{qj_q l_q}, \tag{2.14}$$

which does not depend on the choice of s^1 and s^2 provied (2.13) holds.

Proof. Let

$$s^1 = (s^1_1,\ s^1_2, \cdots, s^1_t),\ s^2 = (s^2_1,\ s^2_2, \cdots, s^2_t).$$

We know that the relations

$$\begin{aligned} &((s_1, s_2, \cdots, s_t),\ (s^1_1,\ s^1_2, \cdots, s^1_t)) \in R_{\psi(j_1, j_2, \cdots, j_t)}, \\ &((s_1, s_2, \cdots, s_t),\ (s^2_1,\ s^2_2, \cdots, s^2_t) \in R_{\psi(l_1, l_2, \cdots, l_t)} \end{aligned} \tag{2.15}$$

hold if and only if

$$(s_q, s^1_q) \in R_{qj_q},\quad (s_q,\ s^2_q) \in R_{ql_q}\quad (q \in [1, t]). \tag{2.16}$$

Noting the convention (1.2), we see that the number of s_q satisfying (2.16) is (2.14). Evidently, the value of (2.14) does not depend on the choices of s^1 and s^2 providd they satisfy (2.15). This proves the lemma.

Combining these lemmas, we have

Theorem 2.1. *The set S with all the associate classes defined in Definition 2.1 is an association scheme with $w - 1$ associate classes and with the parameters:*

$$v = \prod_{q=1}^{t} v_q, \tag{2.17}$$

$$n_{\psi(i_1, i_2, \cdots, i_t)} = \prod_{q=1}^{t} n_{qi_q}, \tag{2.18}$$

$$p^{\psi(i_1, i_2, \cdots, i_t)}_{\psi(j_1, j_2, \cdots, j_t),\ \psi(l_1, l_2, \cdots, l_t)} = \prod_{q=1}^{t} p^{i_q}_{qj_q l_q}. \tag{2.19}$$

$$(i_1, i_2, \cdots, i_t),\ (j_1, j_2, \cdots, j_t),\ (l_1, l_2, \cdots, l_t) \in I.$$

§3. Descartes Product of PBIB Designs

Based on the association scheme of Theorem 2.1, we now turn to consturct a PBIB design. Let D_q $(q = 1, 2, \cdots, t)$ be t PBIB designs given at the beginning of §1. Let

$$S = S_1 \times S_2 \times \cdots \times S_t$$

whose elements are called treatments, and

$$\mathcal{B} = \mathcal{B}^1 \times \mathcal{B}^2 \times \cdots \mathcal{B}^t$$

whose elements are called blocks. A treatment (say $s = (s_1, s_2, \cdots, s_t)$) of S is said to be arranged in a block (say $B = (B_1, B_2, \cdots, B_t)$) of $\mathcal{B}$, if for each $q \in [1,t]$, s_q has been arranged in the block B_q of the design D_q. Then we can prove that this leads to a PBIB design.

It is easy to see that

$$|\mathcal{B}| = \prod_{q=1}^{t} b_q,$$

and that for any block (say $B = (B_1, B_2, \cdots, B_t)$) in $\mathcal{B}$

$$|B| = \prod_{q=1}^{t} |B_q| = \prod_{q=1}^{t} k_q \tag{3.1}$$

which is a constant.

Let $s = (s_1, s_2, \cdots, s_t)$ be an element of S. Since for each $q \in [1,t]$ the element s_q of S_q occurs in exactly r_q blocks in $\mathcal{B}^q$, the element s of S occurs in exactly

$$\prod_{q=1}^{t} r_q \tag{3.2}$$

blocks in $\mathcal{B}$.

Let

$$\{s^1, s^2\} = \{(s_1^1, s_2^1, \cdots, s_t^1),\ (s_1^2, s_2^2, \cdots, s_t^2)\}$$

be a 2-subset of S such that

$$(s^1, s^2) \in R_{\psi(i_1, i_2, \cdots, i_t)}, \tag{3.3}$$

where $(i_1, i_2, \cdots, i_t) \in I$. Let $q \in [1,t]$.

Since $(s_q^1, s_q^2) \in R_{qi_q}$, the 2-subset $\{s_q^1, s_q^2\}$ of S_q occurs in exactly λ_{i_q} (resp. r_q) blocks in $\mathcal{B}^q$ when $i_q \neq 0$ (resp. $i_q = 0$.) Therefore, the 2-subset $\{s^1, s^2\}$ which satisfies (3.3) occurs in exactly

$$\prod_{i_q \neq 0} \lambda_{i_q} \cdot \prod_{i_q = 0} r_q \tag{3.4}$$

blocks in $\mathcal{B}$.

Thus we have proved

Theorem 3.1. *Let D_q ($q \in [1,t]$) be the PBIB design described at the beginning of §1. Taking $S_1 \times S_2 \times \cdots \times S_t$ as the set of treatments, and $\mathcal{B}^1 \times \mathcal{B}^2 \times \cdots \mathcal{B}^t$ as the family of blocks, and defining a treatment to be arranged in a block if the element of $S_1 \times S_2 \times \cdots \times S_t$ which is taken as the treatment is in the subset which is taken as the block, we obtain a PBIB design with $w-1$ associate classes and with the parameters*

$$v = \prod_{q=1}^{t} v_q,\ b = \prod_{q=1}^{t} b_q,\ r = \prod_{q=1}^{t} r_q,\ k = \prod_{q=1}^{t} k_q, \tag{3.5}$$

$$n_i = \prod_{q=1}^{t} n_{qi_q}, \tag{3.6}$$

$$p_{jl}^{i} = \prod_{q=1}^{t} p_{qj_ql_q}^{i_q}, \tag{3.7}$$

$$\lambda_i = \prod_{i_q \neq 0} \lambda_{i_q} \cdot \prod_{i_q=0} r_q, \tag{3.8}$$

where ψ is the mapping described by (2.7) *and* (2.8), *and*

$$i = \psi(i_1, i_2, \cdots, i_t) \neq 0,\ j = \psi(j_1, j_2, \cdots, j_t) \neq 0,\ l = \psi(l_1, l_2, \cdots, l_t) \neq 0.$$

We now turn to some special cases of Theorem 3.1.

For the case $t = 2$, we have

Corollary 3.1. *When $t = 2$, the design D obtained in Theorem 2.1 has $w_1 + w_2 + w_1w_2$ associate classes and the parameters*

$$v = v_1v_2,\ b = b_1b_2,\ k = k_1k_2,\ r = r_1r_2, \tag{3.9}$$

$$n_i = n_{1i_1}n_{2i_2},\ i = \psi(i_1, i_2) \neq 0, \tag{3.10}$$

$$p_{jl}^{i} = p_{1j_1l_1}^{i_1} \cdot p_{2j_2l_2}^{i_2},\quad \begin{aligned} i &= \psi(i_1, i_2) \neq 0, \\ j &= \psi(j_1, j_2) \neq 0, \\ l &= \psi(l_1, l_2) \neq 0, \end{aligned} \tag{3.11}$$

$$\lambda_i = \begin{cases} \lambda_{i_1} r_2, & i = \psi(i_1, 0) \neq 0, \\ \lambda_{i_2} r_1, & i = \psi(0, i_2) \neq 0, \\ \lambda_{i_1}\lambda_{i_2}, & i = \psi(i_1, i_2),\ i_1 \neq 0,\ i_2 \neq 0, \end{cases} \tag{3.12}$$

$$i,\ j,\ l \in [1, w_1 + w_2 + w_1w_2],$$

where ψ is any non-one mapping from $\{(i_1, i_2) | i_1 \in [0, w_1],\ i_2 \in [0, w_2]\}$ to $[0, w_1+w_2+w_1w_2]$ with the property $\psi(0, 0) = 0$.

Put

$$\begin{aligned} \varphi(0, i_2) &= i_2, \quad i_2 \in [1, w_2], \\ \varphi(i_1, 0) &= w_2 + i_1, \quad i_1 \in [1, w_1], \\ \varphi(i_1, i_2) &= w_2 + i_1 + i_2w_1, \quad i_1 \in [1, w_1],\ i_2 \in [1, w_2], \\ \varphi(0, 0) &= 0. \end{aligned}$$

If we take ψ in Corollary 3.1 as φ and note the convention (1.2), then (3.9), (3.10), (3.11) and (3.12) become (1.3), (1.4), (1.6)–(1.8) and (1.5), respectively. So Vartak's result is a special case of our Theorem 3.1. And it is clear that the expressions in (3.9)–(3.11) are much simpler than those in (1.3)–(1.8).

Since a BIB design can be viewed as a specal PBIB design with only one associate class, for the Descartes product of t BIB designs we have

Corollary 3.2 *Let D_q ($q \in [1, t]$) be a BIB design with the parameters*

$$b_q,\ v_q,\ r_q,\ k_q,\ \lambda_q,\ q \in [1, t].$$

Then the Descartes product of D_1, D_2, $D_3, \cdots, D_t$ *is a PBIB design with* $2^t - 1$ *associate classes and with the parameters*

$$b = \prod_{q=1}^{t} b_q, \ v = \prod_{q=1}^{t} v_q, \ r = \prod_{q=1}^{t} r_q, \ k = \prod_{q=1}^{t} k_q,$$

$$\lambda_j = \prod_{j_q=1} \lambda_q \cdot \prod_{j_q=0} r_q, \quad \text{if } j = \psi(j_1, j_2, \cdots, j_t), \ j \in [1, 2^t - 1],$$

$$n_j = \prod_{j_q=1} (v_q - 1), \quad \text{if } j = \psi(j_1, j_2, \cdots, j_t), \ j \in [1, 2^t - 1], \tag{3.13}$$

$$p_{jl}^i = \prod_{\substack{i_q=0 \\ j_q=l_q=1}} (v_q - 1) \cdot \prod_{\substack{i_q=0 \\ j_q \neq l_q}} 0 \cdot \prod_{\substack{i_q=1 \\ j_q=l_q=1}} (v_q - 2) \cdot \prod_{\substack{i_q=1 \\ j_q=l_q=0}} 0 \tag{3.14}$$

if $i = \psi(i_1, i_2, \cdots, i_t)$, $j = \psi(j_1, j_2, \cdots, j_t)$, $l = \psi(l_1, l_2, \cdots, l_t)$, $i, \ j, \ l \in [1, 2^t - 1]$.

The PBIB design which is given in Corollary 3.2 is a $EGD/2^t - 1$-PBIB design as the definition which was given by K.Hinkelmen and O.Kemptharne in [4] and [5]. Obviously, Corollary 3.2 given a general method for constructing $EGD/2^t - 1$-PBIB designs. The parameter p_{jl}^i has clear expression in Corollary 3.2, but K.Hinkelmen and O.Kemptharne was not able to do so in [4] and [5].

Furtherfore, we have

Corollar 3.3. *If the BIB design* D_q $(q = 1, 2, \cdots, t)$ *in Corollary 3.2 are all cyclic, then the family of blocks of the Descartes product of* D_1, $D_2, \cdots$ *and* D_t *is determind "component-wise cyclicly" from any one block (say* $\{(s_1^1, s_2^1, \cdots, s_t^1), (s_1^2, s_2^2, \cdots, s_t^2), \cdots, (s_1^k, s_2^k, \cdots, s_t^k)\}$*) in the following way:*

$$\{((\langle s_1^1 + i_1 \rangle_{v_1}, \langle s_2^1 + i_2 \rangle_{v_2}, \cdots, \langle s_t^1 + i_t \rangle_{v_t}), \ ((\langle s_1^2 + i_1 \rangle_{v_1}, \langle s_2^2 + i_2 \rangle_{v_2}, \cdots, \langle s_t^2 + i_t \rangle_{v_t}),$$
$$\cdots, ((\langle s_1^k + i_1 \rangle_{v_1}, \langle s_2^k + i_2 \rangle_{v_2}, \cdots, \langle s_t^k + i_t \rangle_{v_t})\}, \quad i_q \in [0, v_q - 1], \ q \in [1, t],$$

where the symbol $\langle x \rangle_y$ *represents the smallest non-negative residue of* x *modulo* y.

It is known that for any integer $v > 2$, there exists a $(v, v-1, v-2)$-cyclic difference set. So we have

Corollar 3.4. *Let* $v_q > 2$ $(q \in [1, t])$. *Then there exists a component-wise cyclic PBIB design with the block*

$$\{(s_1, s_2, \cdots, s_t) | 0 \le s_q \le v_q - 2 \ (q \in [1, t])\}$$

and with the parameters n_1, p_{jl}^i *as in* (3.13) *and* (3.14) *and*

$$b = v = \prod_{q=1}^{t} v_q, \ k = r = \prod_{q=1}^{t} k_q = \prod_{q=1}^{t} (v_q - 1),$$

$$\lambda_j = \prod_{j_q=1} (v_q - 2) \cdot \prod_{j_q=0} (v_q - 1), \quad \text{if } j = \psi(j_1, j_2, \cdots, j_t), \ j \in [1, 2^t - 1].$$

REFERENCES

[1] Raghavarao, D., Construction and combinatorial problems in design of experiments, Wcley, New York, 1971.

[2] Wei, W. D., Combinatorial theory (vol. 2): Combinatorial designs, Science Press, Beijing, 1987.

[3] Vartak, M. N., On an apllication of Kronecker product of matrices to statistical designs, *Ann. Math. Statist.*, **26** (1955), 420-438.

[4] Hinkelmen, K. & Kemptharne, O., Two classes of group divisible partial diallel crosses, *Biometrika*, **50** (1963), 281-291.

[5] Hinkelmen, K., Extended group divisible partially balanced incomplete block designs, *Ann. Math. Statist.*, **36** (1964), 681-695.