Chin. Ann. of Math.
14B: 3(1993),253-278.

STATIONARY SOLUTIONS OF THE RELATIVISTIC VLASOV-MAXWELL SYSTEM OF PLASMA PHYSICS

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Abstract

The authors consider the stationary relativistic coupled system consisting of Vlasov's equation for the distribution function of charged particles and Maxwell's equations for the electric and magnetic fields of a plasma. With different tools of nonlinear functional analysis the existence of solutions is proved, in which, according to different geometries and symmetries, the distribution function depends on one, two or three independent integrals of the motion.

Keywords Relativistic Vlasov-Maxwell system, Stationary solutions, Systems of semilinear elliptic equations

1991 MR Subject Classification 82A45, 35J65

§1. Introduction

The present paper is part of a mathematical description of a collisionless plasma considered as a collection of many fast moving charged particles whose collisions are neglected and which interact only by their charges. The basic equations for the time development of the species of electrons consist of the following system of partial differential equations which is now called the Relativistic Vlasov-Maxwell System (RVMS)

$$\partial_t f + \hat{v}\,\partial_x f - q\,(E + \frac{1}{c}\cdot \hat{v}\times B) = 0, \tag{1.1}$$

$$\frac{1}{c}\partial_t E - \operatorname{curl} B = -\frac{4\pi}{c} j, \tag{1.2}$$

$$\frac{1}{c}\partial_t B + \operatorname{curl} E = 0, \tag{1.3}$$

$$\operatorname{div} E = 4\pi\rho, \tag{1.4}$$

$$\operatorname{div} B = 0. \tag{1.5}$$

Here $f = f(t,x,v) \geq 0$ denotes the distribution function of the electrons depending upon the time $t \geq 0$, the space coordinate $x = (x_1,x_2,x_3) \in I\!R^3$ and the momentum $v = (v_1,v_2,v_3) \in I\!R^3$, $E = E(t,x)$ and $B = B(t,x)$ are the electric and magnetic fields and $\hat{v} := \frac{v}{\sqrt{m^2+v^2/c^2}}$ is the relativistic speed of a particle, q and m denote the charge and mass of a particle, c is the speed of light and $\rho = \rho(t,x)$ and $j = j(t,x)$ are the local charge and current densities.

Manuscript received March 4, 1993.

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These two quantities are related with f through

$$\rho(x) := q \int_{\mathbb{R}^3} f(x,v)\,dv,$$

$$j(x) := q \int_{\mathbb{R}^3} \hat{v}\, f(x,v)\,dv.$$

The associated initial value problem with a prescribed distribution f_0

$$f(0,x,v) = f_0(x,v)$$

has received much attention in recent years. In the classical setting, R.T. Glassey and W.A. Strauss have shown that a local solution is actually global if there exists an a priori bound for the support of $f(t,x,v)$ in the momentum variable on the interval of its existence[8], or more generally, if the kinetic energy density

$$\int_{\mathbb{R}^3} (m^2+v^2)^{1/2} f(t,x,v)\,dv$$

stays bounded[10]. These results were used by R.T. Glassey, J. Schaeffer and W.A. Strauss to obtain the existence of global classical solutions for small initial data[9] or nearly neutral data[11]. G. Rein considered a class of global solutions with a certain asymptotic behavior of the resulting solutions[17]. Global existence of classical solutions for general initial data is still an open problem. Global existence of distributional solutions were proven by P.L. Lions and R. Di Perna[12]. In contrast to the classical situation, there is no uniqueness result for these weak solutions.

The present paper is concerned with the existence of stationary solutions of the RVMS in several geometric configurations arising from known solutions of Vlasov's equation (1.1), that is, from three known integrals of the associated system of ordinary differential equations

$$\dot{x} = \hat{v},$$

$$\dot{v} = q(E + \frac{1}{c} \cdot \hat{v} \times B).$$

Let us denote by $\Phi = \Phi(x)$ and $A = A(x)$ the scalar and vector potential associated with Maxwell's equations (1.2)-(1.5). A first integral is the energy density

$$\mathcal{E}(x,v) := \Phi(x) + \sqrt{1+v^2}.$$

We get a second integral if we assume that Φ and A are cylindrically symmetric, that is, they only depend upon $r(x) := \sqrt{x_1^2+x_2^2}$ and $z := x_3$ (but not upon ϑ in cylindrical coordinates r, ϑ, z), namely

$$F(r,z,v) := r(v_\vartheta + A_\vartheta(r,z)).$$

(Subscripts r, ϑ, z denote the components of a vector in $\mathbb{R}^3$ in the local coordinate system $(e_r(x), e_\vartheta(x), e_z(x))$.) If we assume that Φ and A are translational invariant, such that they do not depend upon z, then the quantity

$$P(r,\vartheta,v) := v_z + A_z(r,\vartheta)$$

is a third integral. These integrals are well known in plasma physics. In a recent note P. Degond has set up the form of systems of equations whose solution might lead to the construction of stationary solutions of the RVMS[5]. In fact, the determination of a distribution f as a solution of Vlasov's equation in one of the following forms

$$\text{Case 1) } f = \varphi(\mathcal{E}), \qquad \text{Case 2) } f = \varphi(\mathcal{E}, F), \qquad \text{Case 3) } f = \varphi(\mathcal{E}, F, P)$$

requires the solution of one semilinear elliptic equation in Case 1), of a system of two semilinear elliptic equations (one containing a singular term for $r = 0$) in Case 2) and a system of three ordinary differential equations of second order (one again containing a singular term) in Case 3), subject to suitable boundary conditions.

It is the purpose of the present paper to prove the existence of solutions for the resulting equations with the boundary conditions of a perfect conductor and thus to get three essentially different families of the stationary RVMS.

The existing literature is not yet very rich in the topic addressed here. However, if we formally let $B = 0$ or let $c \to \infty$[19] then we obtain the well known (relativistic) Vlasov-Poisson system of equations denoted by (R)VPS. Stationary solutions of the RVPS have been constructed in [4] and for the classical VPS in [2] and [3]. These articles have influenced the present investigation. For two species of particles and for given distribution functions depending only upon the energy, G. Rein has recently proven the existence and uniqueness of stationary solutions of the RVMS by variational methods[18]. The articles of J. Dolbeault[6], of F. Poupaud[16] and of the Russian School at Irkutsk[13,14] contribute further to an expanding theory.

§2. Formulation of the Problem

Let $\Omega \subset I\!R^3$ be a domain with boundary $\partial\Omega \in C^1$. For the sake of simplicity we let $q = m = c = 1$ and consider the following system of equations:

$$\hat{v}\partial_x f - (E(x) + \hat{v} \times B(x))\partial_v f = 0, \tag{2.1}$$

$$\operatorname{curl} B(x) = 4\pi j(x), \tag{2.2}$$

$$\operatorname{curl} E(x) = 0, \tag{2.3}$$

$$\operatorname{div} E(x) = 4\pi\rho(x), \tag{2.4}$$

$$\operatorname{div} B(x) = 0, \quad x \in \Omega, v \in I\!R^3 \tag{2.5}$$

together with

$$\rho(x) := \int_{I\!R^3} f(x,v)\,dv, \tag{2.6}$$

$$j(x) := \int_{I\!R^3} \hat{v}\, f(x,v)\,dv, \quad x \in \Omega. \tag{2.7}$$

We shall impose the boundary conditions of an ideal conductor, that is,

$$E(x) \times \nu(x) = 0, \tag{2.8}$$

$$\langle B(x), \nu(x)\rangle = 0, \quad x \in \partial\Omega, \tag{2.9}$$

where $\nu(x)$ is the outer normal in $x \in \partial\Omega$. A triple of functions (f, E, B) with $f \in C(\overline{\Omega} \times I\!R^3) \cap C^1(\Omega \times I\!R^3)$, $f \geq 0$, $f(x,\cdot) \in L^1(I\!R^3)$ for $x \in \Omega$ and $E, B \in C^1(\overline{\Omega})^3$ satisfying the equations (2.1)-(2.9) will be called a stationary solution of the RVMS on Ω. We shall have to relax these conditions in certain situations. We introduce the scalar potential Φ and the vector potential A by

$$E(x) = -\partial_x \Phi(x), \tag{2.10}$$

$$B(x) = \operatorname{curl} A(x) \tag{2.11}$$

with the Lorentz gauge

$$\operatorname{div} A(x) = 0. \tag{2.12}$$

Then (2.10) implies (2.3), (2.11) implies (2.5), (2.4) is equivalent to

$$-\Delta\Phi(x) = 4\pi\rho(x) \tag{2.13}$$

in view of (2.12) and the well known relation $\operatorname{curl}\operatorname{curl} A = \partial_x \operatorname{div} A - \Delta A$, (2.2) is equivalent to

$$-\Delta A(x) = 4\pi j(x). \tag{2.14}$$

The boundary conditions (2.8) and (2.9) are satisfied if Φ and A satisfy

$$\Phi(x) = \alpha, \quad x \in \partial\Omega, \tag{2.15}$$

$$\langle \operatorname{curl} A(x), \nu(x)\rangle = 0, \quad x \in \partial\Omega \tag{2.16}$$

for a constant $\alpha \in \mathbb{R}^3$. We observe that if for given ρ and j the potentials $\Phi \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and $A \in C^1(\overline{\Omega})^3 \cap C^2(\Omega)^3$ satisfy (2.13)-(2.16) then the fields $E, B \in C(\overline{\Omega})^3 \cap C^1(\Omega)^3$ given by (2.10)-(2.11) satisfy (2.2)-(2.9). We find solutions f of (2.1) in three different situations.

Case 1. Let $A = 0$. The energy density

$$\mathcal{E}(x,v) := \sqrt{1+v^2} + \Phi(x)$$

obviously satisfies (2.1). Hence for any $\varphi \in C^1(\mathbb{R})$, $f(x,v) := \varphi(\mathcal{E}(x,v))$ is a solution of (2.1). If $f(x,\cdot) \in L^1(\mathbb{R}^3)$ for all $x \in \Omega$, then

$$\begin{aligned} 4\pi\rho(x) &= 4\pi \int_{\mathbb{R}^3} f(x,v)\,dv = 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1+v^2} + \Phi(x))\,dv \\ &= h_\varphi(\Phi(x)) \end{aligned}$$

with

$$h_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi(t+\xi)\, t\sqrt{t^2-1}\,dt. \tag{2.17}$$

Furthermore,

$$j(x) = \int_{\mathbb{R}^3} \hat{v}\, f(x,v)\,dv = \int_{\mathbb{R}^3} \hat{v}\,\varphi(\sqrt{1+v^2} + \Phi(x))\,dv = 0,$$

because the integrand is odd in v. Hence our choice $A = 0$ is compatible with (2.14) and (2.16). We deduce: To get a stationary solution of the RVMS in Case 1, it is sufficient to solve the problem

$$\begin{aligned} -\Delta\Phi &= h_\varphi(\Phi) \quad \text{in } \Omega, \\ \Phi &= \alpha \quad \text{on } \partial\Omega \end{aligned} \tag{2.18}$$

for $\alpha \geq 0$, with h_φ defined by (2.17). A solution will be given by Theorem 3.3 in Section 3.

Case 2. Here we assume Ω to be cylindrically symmetric, that is, invariant with respect to all rotations about the x_3-axis Z. We use cylindrical coordinates (r, ϑ, z), $r(x) := \sqrt{x_1^2 + x_2^2}$, $z := x_3$ for $x \in \overline{\Omega}$. For $x \in \overline{\Omega} \backslash Z$ define the local vector basis

$$e_r(x) := \frac{1}{r(x)} \cdot (x_1, x_2, 0), \quad e_\vartheta(x) := \frac{1}{r(x)} \cdot (-x_2, x_1, 0), \quad e_z(x) := (0, 0, 1).$$

Any vector function $K : \overline{\Omega} \backslash Z \to \mathbb{R}^3$ has a decomposition $K(x) = K_r(x) e_r(x) + K_\vartheta(x) e_\vartheta(x) + K_z(x) e_z(x)$ with

$$\begin{aligned} K_r(x) &:= \langle K(x), e_r(x) \rangle, \\ K_\vartheta(x) &:= \langle K(x), e_\vartheta(x) \rangle, \\ K_z(x) &:= \langle K(x), e_z(x) \rangle. \end{aligned}$$

We define K to be cylindrically symmetric, if K_r, K_ϑ, K_z are invariant with respect to all rotations about Z, that is

$$K_r = K_r(r, z), \quad K_\vartheta = K_\vartheta(r, z), \quad K_z = K_z(r, z)$$

do not depend upon ϑ.

Lemma 2.1. *If Φ and A are cylindrically symmetric, then*

$$F(x, v) := r(x)(v_\vartheta(x) + A_\vartheta(r, z))$$

is a solution of (2.1) and for $v \in \mathbb{R}^3$, $F(\cdot, v)$ is cylindrically symmetric.

It follows from Lemma 2.1, that for any $\varphi \in C^1(\mathbb{R}^2), \varphi \geq 0$, the function $f := \varphi(\mathcal{E}, F)$ is a solution of (2.1). If $f(x, \cdot) \in L^1(\mathbb{R}^3)$ for all $x \in \overline{\Omega}$, then

$$\begin{aligned} 4\pi\rho(x) &= 4\pi \int_{\mathbb{R}^3} f(x, v)\, dv \\ &= 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1 + v^2} + \Phi(x), r(x)(v_\vartheta + A_\vartheta(x)))\, dv \\ &= h_\varphi(r(x), \Phi(x), A_\vartheta(x)), \\ 4\pi j_\vartheta(x) &= 4\pi \int_{\mathbb{R}^3} \hat{v}\, f(x, v)\, dv \\ &= 4\pi \int_{\mathbb{R}^3} \hat{v}_\vartheta\, \varphi(\sqrt{1 + v^2} + \Phi(x), r(x)(v_\vartheta + A_\vartheta(x)))\, dv \\ &= 8\pi^2 \int_{\mathbb{R}} \int_0^\infty \frac{v_\vartheta}{\sqrt{1 + v_\vartheta^2 + q^2}} \varphi(\sqrt{1 + v_\vartheta^2 + q^2} + \Phi(x), r(x)(v_\vartheta + A_\vartheta(x)))\, q\, dq dv_\vartheta \\ &= g_\varphi(r(x), \Phi(x), A_\vartheta(x)) \end{aligned}$$

with

$$h_\varphi(r,\xi,\eta) := 8\pi^2 \int_{\mathbb{R}} \int_0^\infty \varphi(\sqrt{1+v_\vartheta^2+q^2}+\xi, r(v_\vartheta+\eta))\, q\, dq\, dv_\vartheta,$$

$$g_\varphi(r,\xi,\eta) := 8\pi^2 \int_{\mathbb{R}} \int_0^\infty \frac{v_\vartheta}{\sqrt{1+v_\vartheta^2+q^2}} \varphi(\sqrt{1+v_\vartheta^2+q^2}+\xi, r(v_\vartheta+\eta))\, q\, dq\, dv_\vartheta$$

after introducing cylindrical coordinates ($q = \sqrt{v_r^2+v_z^2}, \gamma, v_\vartheta$) with the axis given by e_ϑ. The substitution $t := \pm\sqrt{1+v_\vartheta^2+q^2}$ yields the unified representation

$$\begin{pmatrix} h_\varphi \\ g_\varphi \end{pmatrix}(r,\xi,\eta) = 8\pi^2 \int_{\mathbb{R}} \int_{-\sqrt{t^2-1}}^{+\sqrt{t^2-1}} \begin{pmatrix} t \\ s \end{pmatrix} \varphi(t+\xi, r(s+\eta))\, ds\, dt. \tag{2.19}$$

Furthermore

$$\begin{aligned} j_r(x) &= \int_{\mathbb{R}^3} \hat{v}_r\, f(x,v)\, dv \\ &= \int_{\mathbb{R}^3} \hat{v}_r\, \varphi(\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}+\Phi(x), r(x)(v_\vartheta+A_\vartheta(x)))\, dv_r dv_\vartheta dv_z \\ &= 0 \end{aligned} \tag{2.20}$$

because the integrand is odd in v_r, and for a similar reason,

$$j_z(x) = 0. \tag{2.21}$$

We now express $-\Delta A(x) = \operatorname{curl}\operatorname{curl} A(x)$ in the system $(e_r(x), e_\vartheta(x), e_z(x))$. It is well known that

$$\left.\begin{aligned} (\operatorname{curl} A)_r &= \frac{1}{r}\partial_\vartheta A_z - \partial_z A_\vartheta, \\ (\operatorname{curl} A)_\vartheta &= \partial_z A_r - \partial_r A_z, \\ (\operatorname{curl} A)_z &= \frac{1}{r}(\partial_r(rA_\vartheta)) - \frac{1}{r}\partial_\vartheta A_r. \end{aligned}\right\} \tag{2.22}$$

If A is also cylindrically symmetric, then $\partial_\vartheta A_r = \partial_\vartheta A_\vartheta = \partial_\vartheta A_z = 0$. Hence

$$\left.\begin{aligned} -(\Delta A)_r &= -\partial_z(\partial_z A_r - \partial_r A_z), \\ -(\Delta A)_\vartheta &= \partial_z(-\partial_z A_\vartheta) - \partial_r(\frac{1}{r}\partial_r(rA_\vartheta)) = -\Delta A_\vartheta + \frac{A_\vartheta}{r^2}, \\ -(\Delta A)_z &= \frac{1}{r}(\partial_r(r(\partial_z A_r - \partial_r A_z))). \end{aligned}\right\} \tag{2.23}$$

Let us now choose $A_r = A_z = 0$ and A_ϑ to be cylindrically symmetric. This is compatible with (2.12) because

$$\operatorname{div} A(x) = \frac{1}{r}\partial_r(rA_r(x)) + \frac{1}{r}\partial_\vartheta A_\vartheta(x) + \partial_z A_z(x) = 0.$$

As for (2.11) our choice implies

$$-(\Delta A)_r = 0 = j_r, \quad -(\Delta A)_z = 0 = j_z$$

with (2.23), (2.20), (2.21). Equation (2.13) and the remaining part of (2.14) now read

$$\begin{aligned} -\Delta\Phi &= h_\varphi(r,\Phi,A_\vartheta), \\ -\Delta A_\vartheta + \frac{A_\vartheta}{r^2} &= g_\varphi(r,\Phi,A_\vartheta) \quad \text{in } \Omega. \end{aligned} \tag{2.24}$$

We impose the boundary conditions

$$\Phi = 0, \quad A_\vartheta = 0 \quad \text{on } \partial\Omega. \tag{2.25}$$

The fields E and B are then rediscovered from Φ and A by the general formula

$$(\operatorname{grad}\Phi)_r = \partial_r\Phi, \quad (\operatorname{grad}\Phi)_\vartheta = \frac{1}{r}\partial_\vartheta\Phi, \quad (\operatorname{grad}\Phi)_z = \partial_z\Phi \tag{2.26}$$

and (2.22). If cylindrically symmetric solutions are continuously differentiable near $x \in \partial\Omega\backslash Z$, they satisfy the boundary conditions (2.15), (2.16). In fact, as for (2.16), we note that $\operatorname{grad} A_\vartheta(x) = \partial_r A_\vartheta(x)\cdot e_r(x) + \partial_z A_\vartheta(x)\cdot e_z(x)$ is a scalar multiple of $\nu(x)$ if $\operatorname{grad} A_\vartheta(x) \neq 0$, and we get with (2.22) and (2.25) for some $c \in I\!R$

$$\langle \operatorname{curl} A(x), \nu(x)\rangle = \frac{1}{cr}A_\vartheta(x)\partial_z A_\vartheta(x) = 0.$$

If $\operatorname{grad} A_\vartheta(x) = 0$ then $\operatorname{curl} A(x) = 0$ with (2.22) and (2.25).

Note that (2.24) contains a singularity in the term $\frac{A_\vartheta}{r^2}$ if $\overline{\Omega}\cap Z \neq \emptyset$. In Section 4 we shall first investigate the regular case $\overline{\Omega}\cap Z = \emptyset$ (Theorem 4.2 gives the existence result). The singular case $\overline{\Omega}\cap Z \neq \emptyset$ requires further preparations and will be treated in Section 7 for the case that Ω is a ball about the origin (see Theorem 7.1).

Case 3. Now let Ω be translation invariant with respect to z. Then we have

Lemma 2.2. *If Φ and A do not depend upon z, then*

$$P(x,v) := v_z(x) + A_z(x_1,x_2)$$

is a solution of (2.1),*and for all $v \in I\!R^3$, P does not depend upon z.*

An interesting case arises if Ω is both cylindrically symmetric and translation invariant, and Φ and A are cylindrically symmetric and independent of z. Then r is the only remaining variable. If $\varphi \in C^1(I\!R^3), \varphi \geq 0$, is a given function, then $f := \varphi(\mathcal{E},F,P)$ is a solution of

(2.1). If $f(x,\cdot)\in L^1(\mathbb{R}^3)$ for all $x\in\Omega$, then

$$
\begin{aligned}
4\pi\rho(x) &= 4\pi\int\limits_{\mathbb{R}^3} f(x,v)\,dv\\
&= 4\pi\int\limits_{\mathbb{R}^3}\varphi(\sqrt{1+v^2}+\Phi(x),r(x)(v_\vartheta+A_\vartheta(x)),v_z+A_z(x))\,dv\\
&= h_\varphi(r(x),\Phi(x),A_\vartheta(x),A_z(x)),\\
4\pi j_\vartheta(x) &= 4\pi\int\limits_{\mathbb{R}^3}\hat v_\vartheta\, f(x,v)\,dv\\
&= 4\pi\int\limits_{\mathbb{R}^3}\hat v_\vartheta\,\varphi(\sqrt{1+v^2}+\Phi(x),r(x)(v_\vartheta+A_\vartheta(x)),v_z+A_z(x))\,dv\\
&= g_\varphi(r(x),\Phi(x),A_\vartheta(x),A_z(x)),\\
4\pi j_z(x) &= 4\pi\int\limits_{\mathbb{R}^3}\hat v_z\, f(x,v)\,dv\\
&= 4\pi\int\limits_{\mathbb{R}^3}\hat v_z\,\varphi(\sqrt{1+v^2}+\Phi(x),r(x)(v_\vartheta+A_\vartheta(x)),v_z+A_z(x))\,dv\\
&= k_\varphi(r(x),\Phi(x),A_\vartheta(x),A_z(x)),
\end{aligned}
$$

where

$$
\begin{aligned}
h_\varphi(r,\xi,\eta,\zeta) &:= 4\pi\int\limits_{\mathbb{R}^3}\varphi(\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}+\xi,r(v_\vartheta+\eta),v_z+\zeta)\,dv_r\,dv_\vartheta\,dv_z,\\
g_\varphi(r,\xi,\eta,\zeta) &:= 4\pi\int\limits_{\mathbb{R}^3}v_\vartheta\,\frac{\varphi(\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}+\xi,r(v_\vartheta+\eta),v_z+\zeta)}{\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}}dv_r\,dv_\vartheta\,dv_z,\\
k_\varphi(r,\xi,\eta,\zeta) &:= 4\pi\int\limits_{\mathbb{R}^3}v_z\,\frac{\varphi(\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}+\xi,r(v_\vartheta+\eta),v_z+\zeta)}{\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}}dv_r\,dv_\vartheta\,dv_z,
\end{aligned}
$$

and with the substitution $t:=\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}$ one gets the unified representation

$$
\begin{pmatrix} h_\varphi\\ g_\varphi\\ k_\varphi\end{pmatrix}(r,\xi,\eta,\zeta)=8\pi\int\limits_{\mathbb{R}^2}\int\limits_{\sqrt{1+v_\vartheta^2+v_z^2}}^{\infty}\begin{pmatrix} t\\ v_\vartheta\\ v_z\end{pmatrix}\frac{\varphi(t+\xi,r(v_\vartheta+\eta),v_z+\zeta)}{\sqrt{t^2-(1+v_\vartheta^2+v_z^2)}}\,dt\,dv_\vartheta\,dv_z.
$$

Furthermore, similarly as above

$$
\begin{aligned}
j_r(x) &= \int\limits_{\mathbb{R}^3}\hat v_r\, f(x,v)\,dv\\
&= \int\limits_{\mathbb{R}^3}\hat v_r\,\varphi(\sqrt{1+v_r^2+v_\vartheta^2+v_z^2}+\Phi(x),r(x)(v_\vartheta+A_\vartheta(x)),v_z+A_z(x))\,dv_r dv_\vartheta dv_z\\
&= 0. \qquad (2.27)
\end{aligned}
$$

If we choose $A_r=0$ and A_ϑ and A_z to depend on r only, this choice is again compatible with (2.12) and it follows from (2.27) and (2.23) that

$$(-\Delta A)_r = 0 = j_r.$$

Now (2.13) and the two equations remaining in (2.14) read

$$\left.\begin{aligned} &-\Phi'' - \frac{\Phi'}{r} = h_\varphi(r, \Phi, A_\vartheta, A_z),\\ &-A_\vartheta'' - \frac{A_\vartheta'}{r} + \frac{A_\vartheta}{r^2} = g_\varphi(r, \Phi, A_\vartheta, A_z),\\ &-A_z'' - \frac{A_z'}{r} = k_\varphi(r, \Phi, A_\vartheta, A_z), \end{aligned}\right\} \tag{2.28}$$

where $(\cdot)'$ denotes differentiation with respect to r. We shall prove the existence of solutions $\Phi, A_\vartheta, A_z \in C^2[0, R]$ in Section 6 under the boundary conditions $\Phi'(0) = A_\vartheta(0) = A_z'(0) = 0$ in connection with

$$\text{a) } \Phi(0) = A_\vartheta'(0) = A_z(0) = 0 \quad \text{or} \quad \text{b) } \Phi(R) = A_\vartheta(R) = A_z(R) = 0$$

(see Theorem 6.1). Of course, a vanishing derivative at $r = 0$ guarantees the C^2-extendability to functions of x in Ω; Φ, A_ϑ and A_z are constant on $\partial\Omega$ and hence (2.15) and (2.16) are satisfied (see the above argument).

With the methods presented here, further possibilities could be investigated, e.g. the case $f = \varphi(\mathcal{E}, P)$. Let us note that our distribution functions have the following interesting property.

Corollary 2.1. *Any distribution function $f = f(x, v)$ which only depends upon one or more of the integrals $\mathcal{E}, F, P$, satisfies the boundary condition of specular reflection at each $x \in \partial\Omega$, that is,*

$$f(x, \tilde{v}(x)) = f(x, v) \text{ for } \tilde{v}(x) := v - 2\langle v, \nu(x)\rangle\nu(x).$$

In fact, we have $\tilde{v}(x)^2 = v^2, \tilde{v}_\vartheta(x) = v_\vartheta(x)$ (because $\langle \nu(x), e_\vartheta(x)\rangle = 0$ in the case of cylindrical symmetry) and $\tilde{v}_z(x) = v_z(x)$ (in view of $\langle \nu(x), e_z(x)\rangle = 0$ in the case of translation invariance).

Throughout the paper, universal constants (elements of $\mathbb{R}$) will be denoted by C, constants which depend on φ or R or φ and R, $\cdots$ will be denoted by C_φ, C_R, $C_{\varphi,R}$, $\cdots$, and they may vary from line to line.

§3. Distribution Functions Depending Upon $\mathcal{E}$

Our solution of problem (2.18) will be based on the following lemma, which is a slightly specialized version of Theorem 9.6 in [1] (p.649). For notations and the assumptions, see also [1] (p.633-634, p.646-647).

Lemma 3.1 (H. Amann). *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega \in C^{2+\mu}$ for some $\mu \in (0,1)$. Let $g \in C^{2+\mu}(\partial\Omega), g \geq 0$. Let $h \in C^\mu(\overline{\Omega} \times [0, \infty)), h(\cdot, 0) \geq 0$, be such that there is a $\gamma \geq 0$ with*

$$h(x, \xi) - h(x, \xi') > -\gamma(\xi - \xi')$$

for all $x \in \Omega$ and all ξ, ξ' with $\xi > \xi' \geq 0$. Assume there is a function $H \in C(\overline{\Omega})$ and a constant $\lambda_1 > 0$ such that

$$h(x, \xi) \leq H(x) + \lambda_1\xi, \quad x \in \overline{\Omega},\ \xi \geq 0.$$

Then the boundary value problem

$$-\Delta\Phi = h(x,\Phi) \quad in\ \Omega,$$
$$\Phi = g \quad on\ \partial\Omega$$

has a minimal nonnegative solution $\Phi \in C^2(\overline{\Omega})$ *provided* $\lambda_1 < \lambda_0$, *where* λ_0 *denotes the smallest (positive) eigenvalue of the linear eigenvalue problem*

$$-\Delta u = \lambda u \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega.$$

We are going to define a class of functions φ in Case 1 so that the associated function h_φ defined in (2.17) satisfies the assumptions of Lemma 3.1.

Lemma 3.2. *Let* $\varphi \in C^1[1,\infty)$ *be nonnegative and satisfy the following two conditions:*

i) $\forall\xi \geq 0 : (t \mapsto \varphi(t+\xi)t\sqrt{t^2-1} \in L^1(1,\infty)$,

ii) $\exists m \in L^1(1,\infty)\ \forall\xi \geq 0\ \forall t \geq 1 : |\varphi'(t+\xi)t\sqrt{t^2-1}| \leq m(t)$.

Then h_φ *given by*

$$h_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi(t+\xi)t\sqrt{t^2-1}\,dt, \quad \xi \geq 0$$

is nonnegative, monotonically decreasing to zero for $\xi \to \infty$ *and continuously differentiable on* $[0,\infty)$ *with bounded derivative*

$$h'_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi'(t+\xi)t\sqrt{t^2-1}\,dt, \quad \xi \geq 0.$$

The proof of Lemma 3.2 is straightforward. Lemmas 3.1 and 3.2 yield

Theorem 3.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with boundary* $\partial\Omega \in C^{2+\mu}$ *for some* $\mu \in (0,1)$. *Let* $\varphi \in C^1[1,\infty)$ *satisfy the assumptions of Lemma 3.2 and let* $\alpha \geq 0$. *Then the problem*

$$-\Delta\Phi = h_\varphi(\Phi) \quad in\ \Omega,$$
$$\Phi = \alpha \qquad on\ \partial\Omega$$

has a nonnegative solution $\Phi \in C^2(\overline{\Omega})$. *Consequently, every such* φ *induces a stationary solution* (f, E, B) *of the RVMS on* Ω *such that* $f = \varphi(\mathcal{E}), E = -\partial_x\Phi$ *and* $B = 0$.

Proof. We may apply Lemma 3.2 to see that h_φ satisfies the conditions of Lemma 3.1. In fact, we may define $\gamma := \sup_{\xi\geq 0} |h'_\varphi(\xi)|$, $H(x) := h_\varphi(0)$ and $\lambda_1 = 0$, and the assertion follows from Lemma 3.1.

§4. Distribution Functions Depending Upon $\mathcal{E}$ and F (Regular Case)

In this section we are going to solve the system (2.24) with the boundary conditions (2.25). By applying methods and theorems of nonlinear functional analysis in ordered Banach spaces it is possible to generalize Lemma 3.1 to an existence theorem for semilinear elliptic systems (see [1, p.654]). However, the main assumption is that the right hand side of the system has to be increasing in the "off-diagonal" variables, and there do not seem to exist examples φ for which the resulting right hand side (h_φ, g_φ) would satisfy this condition. The following theorem essentially goes back to P.J. McKenna and W. Walter[15,p.209]. The version given here asserts a slightly stronger regularity of the solution which we need later. Because not all details of the proof can be found in [15] and because it will be necessary to conclude the existence of cylindrically symmetric solutions in Theorem 4.2 we are going to provide the main arguments here. In the following, inequalities between vectors in $\mathbb{R}^n$ are to be understood componentwise.

Theorem 4.1 (P.J. McKenna - W. Walter). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain ($n \in \mathbb{N}$) with boundary $\partial\Omega \in C^{2+\mu}$ for some $\mu \in (0,1)$. Let $F : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following condition: For all $\eta > 0$ there exists $C_\eta > 0$ such that for all $x, x_i \in \overline{\Omega}$, $y, y_i \in \mathbb{R}^n$ with $|y|, |y_i| \le \eta, i = 1, 2$ one has*

$$|F(x_1, y) - F(x_2, y)| \le C_\eta |x_1 - x_2|^\mu,$$
$$|F(x, y_1) - F(x, y_2)| \le C_\eta |y_1 - y_2|.$$

Assume further that there exists a pair of vector functions $v, w \in C^1(\overline{\Omega})^n \cap C^2(\Omega)^n$ with $v \le w$ in $\overline{\Omega}$ and $v \le 0 \le w$ on $\partial\Omega$, such that for $i = 1, \cdots, n$:

$$\forall x \in \Omega, \forall z \in \mathbb{R}^n, v(x) \le z \le w(x), z_i = v_i(x) : -\Delta v_i(x) \le F_i(x, z),$$
$$\forall x \in \Omega, \forall z \in \mathbb{R}^n, v(x) \le z \le w(x), z_i = w_i(x) : -\Delta w_i(x) \ge F_i(x, z).$$

Then there exists a solution $u \in C^{2+\mu}(\overline{\Omega})^n$ of the problem

$$-\Delta u = F(x, u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

and $v \le u \le w$ pointwise on $\overline{\Omega}$.

Proof. 1. We introduce a cut-off $P : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_i(x, z) \mapsto \begin{cases} w_i(x) & \text{if } z_i \ge w_i(x), \\ v_i(x) & \text{if } z_i \le v_i(x), \\ z_i & \text{else} \end{cases}$$

and let

$$G(x, z) := F(x, P(x, z)) + \arctan(P(x, z) - z)$$

(we define $\arctan z := (\arctan z_i)_{i=1..n}$ for $z \in \mathbb{R}^n$). Then G is bounded, has the same regularity properties as F, and $G(x, \cdot) = F(x, \cdot)$ on $[w(x), v(x)]$.

2. We prove the existence of a solution $u \in C^{2+\mu}(\overline{\Omega})^n$ of $-\Delta u = G(x, u)$ in Ω, $u = 0$ on $\partial\Omega$. Define the Nemytskii-Operator

$$\tilde{G} : C(\overline{\Omega})^n \to C(\overline{\Omega})^n \text{ by } \psi \mapsto G(\cdot, \psi(\cdot)). \tag{$\star$}$$

In particular,

$$\tilde{G} : C^1(\overline{\Omega})^n \to C^\mu(\overline{\Omega})^n \tag{$\star\star$}$$

and $\tilde{G}$ maps bounded sets into bounded sets[1,p.647]. The inverse $(-\Delta)^{-1} : C^\mu(\overline{\Omega})^n \to C^{2+\mu}(\overline{\Omega})^n$ is defined with respect to zero boundary values[1,p.635] and has a unique extension to a compact operator $C(\overline{\Omega})^n \to C^\sigma(\overline{\Omega})^n$ for all $\sigma \in [0, 2)$, again denoted by $(-\Delta)^{-1}$ (see [1, p.635]) ($\star\star\star$).

Hence $K := (-\Delta)^{-1} \circ \tilde{G} : C(\overline{\Omega})^n \to C(\overline{\Omega})^n$ is compact. $\tilde{G}$ maps $C(\overline{\Omega})^n$ into some open ball B about 0. The mapping $\lambda \mapsto \mathrm{id} - \lambda K$ on $[0, 1]$ is a homotopy on $[0, 1]$, and

$0 \notin (\mathrm{id} - \lambda K)(\partial B)$. Hence the Leray-Schauder degree $D(\mathrm{id} - \lambda K, B, 0) = 1$. There exists $u \in C(\overline{\Omega})^n$ with $u = Ku$. By $(\star)$, $\tilde{G}u \in C(\overline{\Omega})^n$. By $(\star\star\star)$, $u = Ku \in C^1(\overline{\Omega})^n$. By $(\star\star)$, $\tilde{G}u \in C^\mu(\overline{\Omega})^n$, so that $u = Ku \in C^{2+\mu}(\overline{\Omega})^n$.

3.We show $v \le u \le w$. In fact, assume there exists $i \in \{1, \cdots, n\}$ (we let $i = 1$) such that $\min\limits_{\overline{\Omega}}(w_1 - u_1) < 0$. There exists $x_0 \in \Omega$ such that the minimum is given by $w_1(x_0) - u_1(x_0)$ (for $x_0 \in \partial\Omega$ implies $w_1(x_0) \ge 0$ and $u_1(x_0) = 0$). Then, for $\hat{z} := (P_2(x_0, u(x_0)), \cdots, P_n(x_0, u(x_0)))$ we have

$$\begin{aligned} 0 &\le \Delta(w_1 - u_1)(x_0) = \Delta w_1(x_0) + G_1(x_0, u(x_0)) \\ &\le G_1(x_0, u(x_0)) - F_1(x_0, (w_1(x_0), \hat{z})) \\ &= (F_1(x_0, P(x_0, u(x_0))) - F_1(x_0, (w_1(x_0), \hat{z}))) + \arctan(P_1(x_0, u(x_0)) - u_1(x_0)) \\ &< 0, \end{aligned}$$

which is a contradiction. The inequality $v \le u$ is proven similary. It follows that u is the desired solution.

We define $C_{cyl}(\overline{\Omega}) := \{f \in C(\overline{\Omega}) : f \circ R = f$ for all rotations R about $Z\}$.

Corollary 4.1. *In addition to the hypotheses of Theorem 4.1 we assume for $n = 3$: Ω is cylindrically symmetric and $F(\cdot, y) \in C_{cyl}(\overline{\Omega})$ for all $y \in I\!R^3$ and $v, w \in C_{cyl}(\overline{\Omega})$. Then there exists a solution $u \in C_{cyl}(\overline{\Omega})^3 \cap C^{2+\mu}(\Omega)^3$ such that $v \le u \le w$.*

Proof. We observe that the class $C_{cyl}(\overline{\Omega})$ is a closed subspace of $C(\overline{\Omega})$ and that the arguments of the proof of Theorem 4.1 can be carried through in $C_{cyl}(\overline{\Omega})$.

We now make our choice of suitable functions φ which allow an application of the foregoing results.

Lemma 4.1. *Let $\varphi \in C^1([1, \infty) \times I\!R)$ be nonnegative and satisfy the following condition: $\exists m \in L^1(1, \infty) \forall \mathcal{E} \ge 1 \; \forall F \in I\!R$:*

$$\begin{aligned} \varphi(\mathcal{E}, F)\mathcal{E}\sqrt{\mathcal{E}^2 - 1} &\le m(\mathcal{E}), \\ |\partial_{\mathcal{E}}\varphi(\mathcal{E}, F)|\mathcal{E}\sqrt{\mathcal{E}^2 - 1} &\le m(\mathcal{E}), \\ |\partial_F\varphi(\mathcal{E}, F)|\mathcal{E}(\mathcal{E}^2 - 1) &\le m(\mathcal{E}). \end{aligned}$$

Then the functions h_φ, g_φ given by

$$\begin{pmatrix} h_\varphi \\ g_\varphi \end{pmatrix}(x, \xi, \eta) = 8\pi^2 \int_1^\infty \int_{-\sqrt{t^2-1}}^{\sqrt{t^2-1}} \begin{pmatrix} t \\ s \end{pmatrix} \varphi(t + \xi, r(x)(s + \eta)) \, ds dt, \quad x \in I\!R^3, \xi \ge 0, \eta \in I\!R$$

are continuous on $I\!R^3 \times [0, \infty) \times I\!R$ together with their derivatives with respect to ξ and η, $\partial_x h_\varphi$ and $\partial_x g_\varphi$ exist as continuous functions on $I\!R^3 \backslash Z \times [0, \infty) \times I\!R$. We have the following estimates:

$$\begin{aligned} & 0 \le \frac{1}{2}h_\varphi, |g_\varphi|, \frac{1}{2}|\partial_\xi h_\varphi|, |\partial_\xi g_\varphi| \le 8\pi^2 \parallel m \parallel_1, \\ & \frac{1}{2}|\partial_\eta h_\varphi(x, \xi, \eta)|, |\partial_\eta g_\varphi(x, \xi, \eta)| \le 8\pi^2 \parallel m \parallel_1 \cdot r(x), \\ \textit{and for } r(x) > 0: \quad & \frac{1}{2}|\partial_x h_\varphi(x, \xi, \eta)|, |\partial_x g_\varphi(x, \xi, \eta)| \le 8\pi^2 \parallel m \parallel_1 \cdot (1 + |\eta|). \end{aligned}$$

Furthermore, $g_\varphi(0, \xi, \eta) = 0$.

Proof. We note

$$
\begin{aligned}
h_\varphi(x,\xi,\eta) &\le 8\pi^2 \int_1^\infty \int_{-\sqrt{t^2-1}}^{\sqrt{t^2-1}} t\,\frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}}\,dsdt \\
&= 8\pi^2 \int_1^\infty 2t\sqrt{t^2-1}\,\frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}}\,dt \\
&\le 16\pi^2 \int_1^\infty m(t+\xi)\,dt \le 16\pi^2 \parallel m \parallel_1, \\
|\partial_x h_\varphi(x,\xi,\eta)| &= 8\pi^2 \left| \int_1^\infty \int_{-\sqrt{t^2-1}}^{\sqrt{t^2-1}} t(s+\eta)\partial_F\varphi(t+\xi, r(x)(s+\eta))\,dsdt \cdot e_r(x) \right| \\
&\le 8\pi^2|\eta| \parallel m \parallel_1 + 8\pi^2 \int_1^\infty t(t^2-1)\,\frac{m(t+\xi)}{(t+\xi)((t+\xi)^2-1)}\,dsdt \\
&\le 8\pi^2(|\eta|+1) \parallel m \parallel_1 \ \text{ for } r(x) > 0.
\end{aligned}
$$

The other estimates are similar.

We can now treat the regular case of the system (2.24),(2.25), in which $\overline{\Omega}$ does not contain points of the z-axis Z.

Theorem 4.2. *Let $\Omega \subset I\!R^3$ be a cylindrically symmetric bounded domain with $\partial\Omega \in C^{2+\mu}$ for some $\mu \in (0,1)$ and assume $\overline{\Omega} \cap Z = \emptyset$. Let $\varphi \in C^1([1,\infty) \times I\!R)$ satisfy the assumptions of Lemma* 4.1. *Then the problem*

$$
\begin{aligned}
-\Delta\Phi &= h_\varphi(r,\Phi,A_\vartheta), \\
-\Delta A_\vartheta &= g_\varphi(r,\Phi,A_\vartheta) - \frac{A_\vartheta}{r^2} \quad \text{in } \Omega, \\
\Phi = 0, &\quad A_\vartheta = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{4.1}
$$

has a cylindrically symmetric solution $\Phi, A_\vartheta \in C^{2+\mu}(\overline{\Omega})$. Consequently, every such φ induces a stationary solution (f,E,B) of the RVMS on Ω such that $f = \varphi(\mathcal{E},F), E = -\partial_x\Phi$ and $B = \operatorname{curl} A$ with $A_r = A_z = 0$.

Proof. It follows from Lemma 4.1 that the right hand side of the system (4.1) satisfies the regularity assumptions of Theorem 4.1. Our next concern is the construction of the sub- and supersolution v and w assumed in Theorem 4.1.

Let $R := \max_{x\in\overline{\Omega}} r(x)$. We know from Lemma 5.2

$$0 \le h_\varphi \text{ and } \frac{1}{2} \parallel h_\varphi \parallel, \parallel g_\varphi \parallel \le 8\pi^2 \parallel m \parallel_1 < \infty.$$

We solve the boundary value problems

$$-\frac{1}{r}(rv_1')' = 0, \quad v_1'(0) = v_1(R) = 0,$$

$$-\frac{1}{r}(rw_1')' = \parallel h_\varphi \parallel, \quad w_1'(0) = w_1(R) = 0,$$

$$-\frac{1}{r}(rv_2')' + \frac{v_2}{r^2} = -\parallel g_\varphi \parallel, \quad v_2(0) = v_2(R) = 0,$$

$$-\frac{1}{r}(rw_2')' + \frac{w_2}{r^2} = \parallel g_\varphi \parallel, \quad w_2(0) = w_2(R) = 0$$

and get

$$v_1 = 0, \quad w_1(r) = \frac{1}{4} \parallel h_\varphi \parallel (R^2 - r^2), \quad w_2(r) = \frac{1}{3} \parallel h_\varphi \parallel r(R-r), \quad v_2 = -w_2$$

(see Lemmas 5.1 and 5.2). We may define $v := (v_1, v_2), w := (w_1, w_2)$. In fact, we have $v \le 0 \le w$ and for all $z \in I\!R^2$ with $z_1 \ge 0$

$$-\Delta v_1(x) = 0 \le h_\varphi(x,z),$$

$$-\Delta v_2(x) = -\parallel g_\varphi \parallel - \frac{v_2(x)}{r^2} \le g_\varphi(x,z) - \frac{z_2}{r^2}, \quad \text{if } z_2 = v_2(x)$$

$$-\Delta w_1(x) = \parallel h_\varphi \parallel \ge h_\varphi(x,z),$$

$$-\Delta w_2(x) = \parallel g_\varphi \parallel - \frac{w_2(x)}{r^2} \ge g_\varphi(x,z) - \frac{z_2}{r^2}, \quad \text{if } z_2 = w_2(x).$$

The existence of a solution in $C_{cyl}(\overline{\Omega})^2 \cap C^{2+\mu}(\Omega)^2$ now follows from Corollary 4.1.

§5. Explicit Solutions of Particular Singular Second Order Equations

In this section we collect some results on certain ordinary differential equations of second order with singular coefficients at $r = 0$. The statements made will be needed in the following two sections. Our general assumption is $f \in C[0,R]$ for some $R > 0$. For $a, b \in I\!R$, we let $[a \le s \le b]$ be the characteristic function (in s) of the interval $[a,b]$.

Lemma 5.1. *Consider the equation*

$$u'' + \frac{u'}{r} = f(r), \quad 0 < r \le R.$$

a) *The solution $u_0 \in C^2[0,R]$ with $u_0(0) = u_0'(0) = 0$ is given by*

$$u_0(r) = \int_0^r \frac{1}{s} \int_0^s \sigma f(\sigma)\, d\sigma ds = \int_0^R [s \le r] s \ln \frac{r}{s} f(s)\, ds.$$

We have $u_0''(0) = \frac{1}{2} f(0)$.

b) *The general solution $u \in C^2(0,R]$ is*

$$u(r) = a_1 + a_2 \log r + u_0(r), \quad a_1, a_2 \in I\!R.$$

The following conditions are equivalent:

i) *u or u' is bounded at $r = 0$ or has a finite limit for $r \to 0$,*

ii) $\lim\limits_{r \to 0} u'(r) = 0$,

iii) $\lim\limits_{r \to 0} r \cdot u'(r) = 0$,

iv) $a_2 = 0$.

In this case $u \in C^2[0,R]$, $u''(0) = \frac{1}{2} f(0)$; and $u(R) = 0$ iff u is

$$u_R(r) = -u_0(R) + u_0(r)$$

$$= \int_0^R \left\{ [s \le r] s \log \frac{r}{R} + [s \ge r] s \log \frac{s}{R} \right\} s f(s)\, ds.$$

(For $f(r) = \lambda$ one has $u_R(r) = -\frac{\lambda}{4}(R^2 - r^2)$.)

Lemma 5.2. *Consider the equation*

$$v'' + \frac{v'}{r} - \frac{v}{r^2} = f(r), \quad 0 < r \le R. \tag{5.1}$$

a) *The solution $v_0 \in C^2[0,R]$ with $v_0(0) = v_0'(0) = 0$ is given by*

$$v_0(r) = \frac{1}{r}\int_0^r s \int_0^s f(\sigma)\, d\sigma ds = \frac{1}{2}\int_0^R [s \le r](1 - \frac{s^2}{r^2}) r f(s)\, ds.$$

We have $v_0''(0) = \frac{2}{3} f(0)$.

b) *The general solution $v \in C^2(0,R]$ is*

$$v(r) = b_1 r + b_2 \frac{1}{r} + v_0(r), \quad b_1, b_2 \in \mathbb{R}.$$

The following conditions are equivalent:

i) *v or v' is bounded at $r = 0$ or has a finite limit for $r \to 0$,*

ii) $\lim\limits_{r\to 0} v(r) = 0$,

iii) $\lim\limits_{r\to 0} r \cdot v(r) = 0$,

iv) $b_2 = 0$.

In this case $v \in C^2[0,R]$, $v''(0) = -\frac{1}{3} f(0)$; and $v(R) = 0$ iff v is

$$\begin{aligned} v_R(r) &= -v_0(R)\frac{r}{R} + v_0(r) \\ &= -\frac{1}{2}\int_0^R \left\{[s \le r](\frac{1}{r} - \frac{r}{R^2})s^2 + [s \ge r](1 - \frac{s^2}{R^2})r\right\} f(s)\, ds. \end{aligned}$$

(For $f(r) = \lambda$ one has $v_R(r) = -\frac{\lambda}{3} r(R - r)$.)

Lemma 5.3. *For $0 < \delta \le R$ consider the equation*

$$w'' + \frac{w'}{r} - \frac{w}{\delta^2} = f(r), \quad 0 < r \le R. \tag{5.2}$$

a) *The solution $w_0 \in C^2[0,R]$ with $w_0(0) = w_0'(0) = 0$ is given by*

$$w_0(r) = z_\delta(r)\int_0^r (I_\delta(r) - I_\delta(s))\, z_\delta(s)\, s\, f(s) ds,$$

where

$$z_\delta(r) := \sum_{k=0}^{\infty} \frac{(r/\delta)^{2k}}{[(2k)!!]^2}, \quad I_\delta(r) := \int_\delta^r \frac{ds}{s z_\delta^2(s)}.$$

Here $z_\delta \in C^2[0,R]$ is a solution of the homogeneous equation with

$$z_\delta(0) = 1, \quad z_\delta'(0) = 0, \quad z_\delta''(0) = \frac{1}{2\delta^2},$$

$$z_\delta(\delta) = S, \quad z_\delta'(\delta) = \frac{S_1}{\delta},$$

with $S := \sum\limits_{k=0}^{\infty} [(2k)!!]^{-2}$, $S_1 := \sum\limits_{k=1}^{\infty} 2k[(2k)!!]^{-2}$.

We have $w_0''(0) = \frac{2}{3} f(0)$.

b) *The general solution* $w \in C^2(0,R]$ *is*

$$w(r) = c_1 z_\delta(r) + c_2 I_\delta(r) z_\delta(r) + w_0(r), \quad c_1, c_2 \in \mathbb{R}.$$

The following conditions are equivalent:

i) w *or* w' *is bounded at* $r = 0$ *or has a finite limit for* $r \to 0$,

ii) $\lim_{r\to 0} w'(r) = 0$,

iii) $c_2 = 0$.

In this case $w \in C^2[0,R]$, $w''(0) = \frac{c_1}{2\delta^2} + f(0)$.

Now we define

$$r_\delta(r) := \begin{cases} \delta & \text{for } 0 \le r \le \delta \\ r & \text{for } r \ge \delta. \end{cases} \tag{5.3}$$

Lemma 5.4. *For* $0 < \delta \le R$ *the solution* $v_\delta \in C^2[0,R]$ *of*

$$v'' + \frac{v'}{r} - \frac{v}{r_\delta^2} = f(r), \quad 0 < r \le R \tag{5.4}$$

with $v_\delta(R) = 0$ *is given by*

$$\begin{aligned} v_\delta(r) &= c_\delta z_\delta(r) + w_0(r) \\ &= z_\delta(r)\left(c_\delta + \int_0^r (I_\delta(r) - I_\delta(s)) z_\delta(s) s f(s)\, ds\right), \quad 0 \le r \le \delta, \\ v_\delta(r) &= b_\delta\left(\frac{1}{r} - \frac{r}{R^2}\right) + v_R(r), \quad \delta \le r \le R, \end{aligned}$$

where w_0 *and* v_R *are defined in Lemma* 5.3 *and* 5.2 *respectively, and*

$$\begin{aligned} c_\delta := &-\frac{1}{N}\int_0^R \left\{[s \le \delta]\frac{R^2 - \delta^2}{S} z_\delta(s) s + [s \ge \delta](R^2 - s^2)\delta\right\} f(s)\, ds \\ &- \int_0^\delta (I_\delta(\delta) - I_\delta(s)) z_\delta(s) s f(s)\, ds, \end{aligned} \tag{5.5}$$

$$\begin{aligned} b_\delta := &-\frac{1}{2}\int_0^R \Big\{[s \le \delta](\frac{2R^2}{N}\delta z_\delta(s) - s)\, s \\ &+ [s \ge \delta]\frac{S - S_1}{N}(R^2 - s^2)\delta^2\Big\} f(s)\, ds, \end{aligned} \tag{5.6}$$

with $N := (S + S_1)R^2 + \delta^2(S - S_1)$.

Proof. Lemma 5.3 implies that $cz_\delta + w_0$, $c \in \mathbb{R}$, is the general bounded solution of (5.2), and Lemma 5.2 says that $b(\frac{1}{r} - \frac{r}{R^2}) + v_R$, $b \in \mathbb{R}$, is the general solution of (5.1) which vanishes at $r = R$. We can determine the constants in such a way that the solutions and their first derivatives have the same value at $r = \delta$ and thus obtain the formulas (5.5), (5.6). Because r_δ is continuous this implies the continuity of the second derivative at $r = \delta$ and $v_\delta \in C^2[0,R]$ follows.

Corollary 5.1. *There exists a constant* $C_R^* > 0$ *(only depending upon* R*) such that for* *all* $f \in C[0,R]$ *with* $f \leq 0$ *one has*

$$0 \leq v_R(r), v_\delta(r) \leq C_R^* \parallel f \parallel \delta, \quad 0 \leq r \leq \delta,$$

$$0 \leq v_R(r) \leq v_\delta(r) \leq v_R(r) + C_R^* \parallel f \parallel \delta, \quad \delta \leq r \leq R,$$

for small $\delta > 0$.

Proof. The representation of v_R in Lemma 5.2 b) implies $v_R \geq 0$ on $[0,R]$ and $v_R \leq C_R^* \parallel f \parallel \delta$ on $[0,\delta]$. The relations

$$\int_0^r (I_\delta(r) - I_\delta(s))z_\delta(s)s\,ds = O(\delta^2)\ (\delta \to 0) \text{ uniformly for } r \leq \delta,$$

$$c_\delta = O(\delta \cdot \parallel f \parallel), \quad b_\delta = O(\delta^2 \cdot \parallel f \parallel)\ (\delta \to 0)$$

are obvious. The integral kernel which represents $-b_\delta$ is nonnegative because for $s \leq \delta$

$$2R^2\delta z_\delta(s) - Ns \geq R^2\delta\left(2 - \left[S + S_1 + \frac{S - S_1}{R^2}\delta^2\right]\right) > 0$$

for all $\delta > 0$ such that $S + S_1 + \frac{S-S_1}{R^2}\delta^2 < 2$ (note that $S + S_1 < 2$ and $S - S_1 > 0$). Hence $b_\delta \geq 0$ and because $c_\delta \geq 0$ the representation of v_δ in Lemma 5.4 gives the result.

§6. Distribution Functions Depending Upon $\mathcal{E}$, F and P

This section is devoted to the study of the system (2.28). The following lemma gives sufficient conditions on φ such that existence can be proven later by Schauder's fixed point theorem.

Lemma 6.1. *Let* $\varphi \in C^1([1,\infty) \times I\!R \times [0,\infty))$ *be nonnegative and satisfy the following condition:* $\exists m \in L^1(1,\infty)\ \forall \mathcal{E} \geq 1\ \forall F \in I\!R\ \ \forall P \geq 0$:

$$\varphi(\mathcal{E},F,P)\,\mathcal{E}\sqrt{\mathcal{E}^2-1} \leq m(\mathcal{E}),$$
$$|\partial_\mathcal{E}\varphi(\mathcal{E},F,P)|\,\mathcal{E}\sqrt{\mathcal{E}^2-1}, |\partial_P\varphi(\mathcal{E},F,P)|\,\mathcal{E}\sqrt{\mathcal{E}^2-1} \leq m(\mathcal{E}),$$
$$|\partial_F\varphi(\mathcal{E},F,P)|\,\mathcal{E}\,(\mathcal{E}^2-1) \leq m(\mathcal{E}).$$

Then the functions $h_\varphi, g_\varphi, k_\varphi$ *given by*

$$\begin{pmatrix} h_\varphi \\ g_\varphi \\ k_\varphi \end{pmatrix}(x,\xi,\eta,\zeta) := 8\pi \int_{I\!R^2} \int_{\sqrt{1+v_\vartheta^2+v_z^2}}^{\infty} \begin{pmatrix} t \\ v_\vartheta \\ v_z \end{pmatrix} \frac{\varphi(t+\xi, r(v_\vartheta+\eta), v_z+\zeta)}{\sqrt{t^2-(1+v_\vartheta^2+v_z^2)}}\,dt dv_\vartheta dv_z$$

are continuous on $I\!R^3 \times [0,\infty) \times I\!R^2$ *together with their derivatives with respect to* ξ, η *and* ζ; $\partial_x h_\varphi, \partial_x g_\varphi$ *and* $\partial_x k_\varphi$ *exist as continuous functions on* $I\!R^3\backslash Z \times [0,\infty) \times I\!R^2$. *We have the following estimates:*

$$0 \leq \frac{1}{2}h_\varphi,\ |g_\varphi|,\ |k_\varphi|,\ \frac{1}{2}|\partial_\xi h_\varphi|,\ |\partial_\xi g_\varphi|,\ |\partial_\xi k_\varphi|,$$
$$\frac{1}{2}|\partial_\zeta h_\varphi|,\ |\partial_\zeta g_\varphi|,\ |\partial_\zeta k_\varphi| \leq 8\pi^2 \parallel m \parallel_1,$$
$$\frac{1}{2}|\partial_\eta h_\varphi|,\ |\partial_\eta g_\varphi|,\ |\partial_\eta k_\varphi| \leq 8\pi^2 \parallel m \parallel_1 r(x),$$

and for $r(x) > 0$:

$$\frac{1}{2}|\partial_x h_\varphi(x,\xi,\eta,\zeta)|,\ |\partial_x g_\varphi(x,\xi,\eta,\zeta)|,\ |\partial_x k_\varphi(x,\xi,\eta,\zeta)| \le 8\pi^2 \parallel m \parallel_1 (1+|\eta|).$$

Furthermore, $g_\varphi(0,\xi,\eta,\zeta) = 0$.

Proof. For $a > 0$ and $k = 0,1,2,3$ we have

$$I_k(a) := \int_0^a \frac{s^k}{\sqrt{a^2-s^2}}\,ds = a^k I_k(1),$$

where $I_0(1) = \frac{\pi}{2}$, $I_1(1) = 1$, $I_2(1) = \frac{\pi}{4}$, $I_3(1) = \frac{2}{3}$. Hence

$$\begin{aligned}
0 \le h_\varphi(x,\xi,\eta,\zeta) &\le 8\pi \int_{\mathbb{R}^2} \int_{\sqrt{1+v_\vartheta^2+v_z^2}}^{\infty} \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \frac{t}{\sqrt{t^2-(1+v_\vartheta^2+v_z^2)}}\,dt dv_\vartheta dv_z \\
&= 16\pi^2 \int_0^\infty \int_{\sqrt{1+\rho^2}}^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \frac{t}{\sqrt{t^2-(1+\rho^2)}}\,dt\,\rho\,d\rho \\
&= 16\pi^2 \int_1^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \int_0^{\sqrt{t^2-1}} \frac{\rho}{\sqrt{t^2-1-\rho^2}}\,d\rho\,t\,dt \\
&= 16\pi^2 \int_1^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}}\,t\sqrt{t^2-1}\,dt \\
&\le 16\pi^2 \parallel m \parallel_1 .
\end{aligned}$$

Similarly, with the substitution $\sigma := v_\vartheta$, $\rho := \sqrt{v_\vartheta^2+v_z^2}$,

$$\begin{aligned}
|g_\varphi(x,\xi,\eta,\zeta)| &\le 32\pi \int_0^\infty \int_0^\infty \int_{\sqrt{1+v_\vartheta^2+v_z^2}}^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \frac{v_\vartheta}{\sqrt{t^2-(1+v_\vartheta^2+v_z^2)}}\,dt dv_\vartheta dv_z \\
&= 32\pi \int_0^\infty \int_0^\rho \int_{\sqrt{1+\rho^2}}^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \frac{\sigma}{\sqrt{t^2-(1+\rho^2)}} \frac{\rho}{\sqrt{\rho^2-\sigma^2}}\,dt\,d\sigma\,d\rho \\
&= 32\pi \int_0^\infty \int_{\sqrt{1+\rho^2}}^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \frac{\rho}{\sqrt{t^2-(1+\rho^2)}}\,\rho\,dt\,d\rho \\
&= 32\pi \int_1^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}} \int_0^{\sqrt{t^2-1}} \frac{\rho^2}{\sqrt{t^2-(1+\rho^2)}}\,d\rho\,dt \\
&= 8\pi^2 \int_1^\infty \frac{m(t+\xi)}{(t+\xi)\sqrt{(t+\xi)^2-1}}(t^2-1)\,dt \\
&\le 8\pi^2 \parallel m \parallel_1 .
\end{aligned}$$

The remaining estimates follow in a similar way.

We remark that a radial function defined on an interval $[0,a]$ is C^2 on a neighborhood of

0 in $I\!R^n$ if it belongs to $C^2[0,a]$ and its radial derivative vanishes at $r=0$.

Theorem 6.1. *Let $\Omega \subset I\!R^3$ be a cylindrical domain of the form $\Omega = \{x \in I\!R^3 : r(x) < R\}$ for some $R > 0$. Let $\varphi \in C^1([1,\infty) \times I\!R \times [0,\infty))$ satisfy the assumptions of Lemma 6.1. Then the system*

$$\left.\begin{aligned} -\Phi'' - \frac{\Phi'}{r} &= h_\varphi(r, \Phi, A_\vartheta, A_z) \\ -A_\vartheta'' - \frac{A_\vartheta'}{r} + \frac{A_\vartheta}{r^2} &= g_\varphi(r, \Phi, A_\vartheta, A_z) \\ -A_z'' - \frac{A_z'}{r} &= k_\varphi(r, \Phi, A_\vartheta, A_z), \quad 0 < r \le R \end{aligned}\right\} \tag{6.1}$$

in connection with $\Phi'(0) = A_\vartheta(0) = A_z'(0) = 0$ and

$$\text{a) } \Phi(0) = A_\vartheta'(0) = A_z(0) = 0 \quad \textit{or} \quad \text{b) } \Phi(R) = A_\vartheta(R) = A_z(R) = 0$$

has a solution $(\Phi, A_\vartheta, A_z) \in C^2[0,R]^3$. Consequently, every such φ induces a stationary solution (f, E, B) such that $f = \varphi(\mathcal{E}, F, P)$; E, B only depend upon r, and $f \in C(\overline{\Omega} \times I\!R^3) \cap C^1((\overline{\Omega}\backslash Z) \times I\!R^3)$, $E, B \in C^1(\overline{\Omega})$.

Proof. Let K_1 and K_2 be the kernels in the integral representation of u and v in any case a) or b) according to Lemmas 5.1 and 5.2 respectively. Then the system (6.1) has a solution (Φ, A_ϑ, A_z) in $C^2[0,R]^3$ if and only if $(\Phi, A_\vartheta, A_z) \in C[0,R]^3$ and for $0 \le r \le R$

$$\Phi(r) = -\int_0^R K_1(r,s) h_\varphi(s, \Phi(s), A_\vartheta(s), A_z(s))\, ds,$$

$$A_\vartheta(r) = -\int_0^R K_2(r,s) g_\varphi(s, \Phi(s), A_\vartheta(s), A_z(s))\, ds,$$

$$A_z(r) = -\int_0^R K_1(r,s) k_\varphi(s, \Phi(s), A_\vartheta(s), A_z(s))\, ds.$$

Lemma 6.1 yields the a priori estimates

$$|\Phi(r)| \le 16\pi^2 \parallel m \parallel_1 \int_0^R K_1(r,s)\, ds,$$

$$|A_\vartheta(r)| \le 8\pi^2 \parallel m \parallel_1 \int_0^R K_2(r,s)\, ds,$$

$$|A_z(r)| \le 8\pi^2 \parallel m \parallel_1 \int_0^R K_1(r,s)\, ds.$$

Because of the continuity of the kernels K_1, K_2 on $[0,R]^2$ and the Lipschitz-continuity of $h_\varphi, g_\varphi, k_\varphi$, we may apply Schauder's fixed point theorem and get a solution of (6.1). The regularity of E, B follows from (2.22),(2.26),

$$\begin{aligned} B_z'(r) &= (A_\vartheta'(r) + \frac{A_\vartheta(r)}{r})' = A_\vartheta''(r) + \frac{A_\vartheta'(r)}{r} - \frac{A_\vartheta(r)}{r^2} \\ &= g_\varphi(r, \Phi(r), A_\vartheta(r), A_z(r)), \end{aligned}$$

and $g_\varphi(0,\xi,\eta,\zeta)=0$.

§7. Distribution Functions Depending Upon $\mathcal{E}$ and F (Singular Case)

We shall now investigate the system (2.24) in the case that the z-axis Z intersects $\overline{\Omega}$. For the sake of simplicity we assume that Ω is the ball $\mathbb{B}=\{x\in\mathbb{R}^3:|x|<R\}$. We shall first consider regularized problems by replacing the singular term $\frac{1}{r^2}$ by $\frac{1}{r_\delta^2}$, where $r_\delta(x):=r_\delta(\sqrt{x_1^2+x_2^2})$, $x\in\mathbb{R}^3$ (see (5.3)), and then we let $\delta\to 0$. For the regularization, the restriction to $\Omega=\mathbb{B}$ is not necessary.

Lemma 7.1. *Let $\Omega\subset\mathbb{R}^3$ be a cylindrically symmetric bounded domain with $\partial\Omega\in C^{2+\mu}$ for some $\mu\in(0,1)$. Let $\varphi\in C^1([1,\infty)\times\mathbb{R})$ satisfy the assumptions of Lemma* 4.1. *Then for each sufficiently small $\delta>0$ the problem*

$$-\Delta\Phi=h_\varphi(r,\Phi,A_\vartheta),$$
$$-\Delta A_\vartheta=g_\varphi(r,\Phi,A_\vartheta)-\frac{A_\vartheta}{r_\delta^2}\quad in\ \Omega,$$
$$\Phi=0,\quad A_\vartheta=0\quad on\ \partial\Omega$$

has a cylindrical symmetric solution $\Phi_\delta,A_{\vartheta,\delta}\in C^{2+\mu}(\overline{\Omega})$ (with a similar statement for $(f,E,\mathbb{B})$ as in Theorem 4.2). *We have the uniform estimates*

$$0\le\Phi_\delta(x)\le w_1(r(x)),$$

$$-W_\delta(r(x))\le A_{\vartheta,\delta}(x)\le +W_\delta(r(x)),\quad x\in\overline{\Omega}\backslash Z,\tag{7.1}$$

where $w_1(r):=\frac{1}{4}\parallel h_\varphi\parallel(R^2-r^2)$, $W_\delta(r):=v_\delta(r)+C_R^*\parallel g_\varphi\parallel\delta$, and v_δ is given by Lemma 5.4 *for $f(r):=-\parallel g_\varphi\parallel$ (C_R^* is the constant of Corollary* 5.1).

Proof. We want to apply Theorem 4.1 and Corollary 4.1 and we need to construct sub- and supersolutions in $C^1(\overline{\Omega})^2\cap C^2(\Omega)^2$. We can use $v_1=0$ and w_1 as in the proof of Theorem 4.2 because $w_1'(0)=0$ (see the remark preceeding Theorem 6.1). From Corollary 5.1 we have $W_\delta\ge 0$ and $W_\delta'(0)=v_\delta'(0)=c_\delta z_\delta'(0)+w_0'(0)=0$ by Lemma 5.3. For $z_1\ge 0$ and $z_2:=W_\delta(x)$,

$$\begin{aligned}-\Delta W_\delta(x)=-\Delta v_\delta(x)&=\parallel g_\varphi\parallel-\frac{v_\delta(x)}{r_\delta^2}\\&\ge g_\varphi(x,z)-\frac{W_\delta(x)-C_R^*\parallel g_\varphi\parallel\delta}{r_\delta^2}\\&\ge g_\varphi(x,z)-\frac{z_2}{r_\delta^2},\quad x\in\Omega.\end{aligned}$$

Hence $(0,-W_\delta)$ and (w_1,W_δ) are sub- and supersolutions.

By the uniqueness of the solution for Poisson's equation with right hand sides

$$h_{\varphi,\delta}(x):=h_\varphi(r,\Phi_\delta(x),A_{\vartheta,\delta}(x))$$
$$g_{\varphi,\delta}(x):=g_\varphi(r,\Phi_\delta(x),A_{\vartheta,\delta}(x))$$

and with homogeneous boundary conditions we have for the ball $I\!B$

$$\Phi_\delta(x) = \int\limits_{I\!B} G(x,y)\, h_{\varphi,\delta}(y)\, dy, \tag{7.2}$$

$$A_{\vartheta,\delta}(x) = \int\limits_{I\!B} G(x,y) \left(g_{\varphi,\delta}(y) - \frac{A_{\vartheta,\delta}(y)}{r_\delta^2(y)} \right) dy, \quad x \in \overline{I\!B}, \tag{7.3}$$

where G is Green's function

$$G(x,y) = \frac{1}{4\pi} \left(\frac{1}{|x-y|} - Q(x,y) \right)$$

with

$$Q(x,y) := \begin{cases} 1/R & \text{for } x = 0, y \in \overline{I\!B}, \\ \dfrac{R}{|x|} \dfrac{1}{|x^* - y|} & \text{for } x \neq 0, y \in I\!B \cup (\partial I\!B \setminus \{x\}), \end{cases}$$

where $x^* := \frac{R^2}{x^2} x$ for $x \neq 0$. The following result is classical. If $f \in C(\overline{I\!B})$ then

$$U(x) := \int\limits_{I\!B} G(x,y)\, f(y)\, dy, \quad x \in \overline{I\!B}$$

is an element of $C^1(\overline{I\!B})$ and

$$|U(x)|, |DU(x)| \le C_R \parallel f \parallel, \quad x \in \overline{I\!B}; \tag{7.4}$$

by $D, D^2, \cdots$ we denote partial derivatives of the respective order. If $f \in C^\alpha(\overline{I\!B})$ for some $0 < \alpha \le 1$, then $U \in C^{2+\alpha}(\overline{I\!B})$, and

$$\partial_{x_i}\partial_{x_j} U(x) = \int\limits_{I\!B} (f(y) - f(x)) \partial_{x_i}\partial_{x_j} G(x,y)\, dy - \frac{1}{3}\delta_{ij} f(x) \tag{7.5}$$

$$|\partial_{x_i}\partial_{x_j} U(x)| \le C_R(\parallel f \parallel + H_\alpha(f)), \quad x \in \overline{I\!B}$$

$$H_\alpha(\partial_{x_i}\partial_{x_j} U) \le C_\alpha H_\alpha(f),$$

where $H_\alpha(f)$ is the Hölder constant of f. This is the content of Müntz' Theorem, a direct proof of which has been given by S. Simoda [20]. In the present situation we can control the Hölder continuity of the derivatives of f and U only away from Z, and we shall have to refine Simoda's arguments. For $x, y \in I\!B \setminus \{0\}$ and for $x, y \in \partial\Omega$ with $x \neq y$ we have

$$\frac{|x|}{R}|x^* - y| = \frac{|y|}{R}|x - y^*|.$$

Hence for $x \in I\!B$

$$Q(x,y) = \frac{R}{|y|} \frac{1}{|x - y^*|}, \quad y \in \overline{I\!B} \setminus \{0\}.$$

Because $|x - y^*| \ge |x - y| \frac{R}{|y|}$, we have for $\lambda \ge 0$

$$\frac{R}{|y|} \frac{1}{|x - y^*|^{1+\lambda}} \ge \frac{1}{|x - y|^{1+\lambda}}, \quad y \in \overline{I\!B}, 0 \neq y \neq x,$$

and this implies

$$|D_x^k G(x,y)| \le \frac{C_k}{|x-y|^{1+k}}, \quad k = 0, 1, 2, 3. \tag{7.6}$$

In the following we let $Z_\eta := \{x \in \overline{B} : r(x) < \eta\}$, and for $f \in C^\alpha(\overline{B}\backslash Z_\eta)$,

$$H_{\alpha,\eta}(f) := \sup\{|f(y) - f(y')| \cdot |y - y'|^{-\alpha} : y, y' \in \overline{B}\backslash Z_\eta\},$$
$$H_{\alpha,\beta,\eta}(f) := \sup\{|f(y) - f(y')| \cdot |y - y'|^{-\alpha} : y, y' \in \overline{B}\backslash Z_\eta,\ |y - y'| \le \beta\eta\},$$

$0 < \eta < R$, $0 < \alpha, \beta \le 1$. For $x \in \mathbb{R}^3$, let $K_a(x) := \{y \in \mathbb{R}^3 \colon |y - x| < a\}$.

Lemma 7.2. *If $f \in C(\overline{B}) \cap C^\alpha(\overline{B}\backslash Z_{\eta/5})$, then $U \in C^{2+\alpha}(\overline{B}\backslash Z_\eta)$, and*

$$|D^2U(x)| \le C_{\alpha,R}(H_{\alpha,\eta/2}(f) + \| f \| (|\log \eta| + 1)), \quad x \in \overline{B}\backslash Z_\eta, \tag{7.7}$$
$$H_{\alpha,1/5,\eta}(D^2U) \le C_{\alpha,R}(H_{\alpha,\eta/5}(f) + \| f \| \eta^{-1-\alpha}). \tag{7.8}$$

Proof. We still have the formula (7.5) for $x \in \overline{B}\backslash Z_\eta$. We estimate over $B \cap K_{\eta/2}(x)$ and $B\backslash K_{\eta/2}(x)$ separatly ($y \in K_{\eta/2}(x)$ implies $r(y) \ge \eta/2$) and we get (7.7). For $p, q \in \overline{B}\backslash Z_\eta$ such that $0 < |p - q| \le \eta/4$ and $K' := K_{|p-q|}(\frac{p+q}{2}) \subset B$ (first case) one estimates

$$\left.\left.\int_B (f(y) - f(x))D^2G(x, y)\, dy\right|\right|_{x=q}^{x=p} = \left.\int_{B\cap K'} (f(y) - f(x))D^2G(x, y)\, dy\right|_{x=q}^{x=p}$$
$$+ \left.\int_{B\backslash K'} (f(y) - f(p))D^2G(x, y)\, dy\right|_{x=q}^{x=p} + (f(p) - f(q)) \int_{B\backslash K'} D^2G(q, y)\, dy$$

to establish the inequality

$$H_{\alpha,1/4,\eta}(D^2U) \le C_{\alpha,R}(H_{\alpha,\eta/4}(f) + \| f \| \eta^{-1-\alpha}). \tag{7.9}$$

This is done similarly as in [20]: one uses $H_{\alpha,5\eta/8}(f)$ in the first term, and with (7.6) one sees that the second term is bounded by

$$C_3 \int_0^{|p-q|} \int_{\substack{y\in B \\ |y-(p+sE)| \ge \frac{|p-q|}{2}}} \frac{|f(y) + f(p + sE)|}{|p + sE - y|^4}\, dyds$$
$$+C_3 \int_0^{|p-q|} \int_{\substack{y\in B \\ |y-(p+sE)| \ge \frac{|p-q|}{2}}} \frac{|f(p + sE) - f(p)|}{|p + sE - y|^4}\, dyds,$$

where $E := \frac{p-q}{|p-q|}$. The inner integral of the first of these two terms is estimated over the domains $\{y \in B : |y - (p + sE)| \ge \frac{|p-q|}{2},\ r(y) \le \eta/4\}$ (where $|y - (p + sE)| \ge \eta/2$) and its complement, bounding the dominator by $2 \| f \|$ or by $H_{\alpha,\eta/4}(f)|y - (p + sE)|^\alpha$ respectively. In the second term $H_{\alpha,3\eta/4}(f)$ is used. In the general case $p, q \in \overline{B}\backslash Z_\eta$ such that $0 < |p - q| \le \eta/5$ one defines

$$p^{(0)} := \lambda p, \quad q^{(0)} := \lambda q \quad \text{with } \lambda := \frac{|p| - |p - q|}{|p|}$$

and one sees that $p^{(0)}, q^{(0)}$ belongs to the first case with η replaced by $4\eta/5$, and one can apply the argument with the chain of balls to prove the full assertion as in [20].

We still need a further result.

Lemma 7.3. a) *If $f \in C(\overline{\mathbb{B}})$, then*

$$V(x) := \int_{\mathbb{B}} G(x,y)\frac{f(y)}{r(y)}\,dy, \quad x \in \overline{\mathbb{B}}$$

is an element of $C(\overline{\mathbb{B}}) \cap C^1(\overline{\mathbb{B}}\backslash Z)$ and

$$|V(x)| \leq C_R \parallel f \parallel, \; x \in \overline{\mathbb{B}}, \quad |DV(x)| \leq C_R \parallel f \parallel \cdot |\log \eta|, \; x \in \overline{\mathbb{B}}\backslash Z. \tag{7.10}$$

b) *If $f \in C(\overline{\mathbb{B}}) \cap C^\alpha(\overline{\mathbb{B}}\backslash Z_{\eta/5})$, then $V \in C^{2+\alpha}(\overline{\mathbb{B}}\backslash Z_\eta)$, and*

$$|D^2V(x)| \leq C_{R,\alpha}(H_{\alpha,\eta/2}(f)\eta^{-1} + \parallel f \parallel \eta^{-1-\alpha}),$$

$$H_{\alpha,1/5,\eta}(D^2V) \leq C_{R,\alpha}(H_{\alpha,\eta/5}(f)\eta^{-1} + \parallel f \parallel \eta^{-2-\alpha}). \tag{7.11}$$

Proof. For $\delta > 0$ we define

$$V_\delta(x) := \int_{\mathbb{B}} G(x,y)\frac{f(y)}{r_\delta(y)}\,dy, \quad x \in \overline{\mathbb{B}}.$$

By the remark preceding Lemma 7.2 we have $V_\delta \in C^{2+\alpha}(\overline{\mathbb{B}})$,

$$D\,V_\delta(x) = \int_{\mathbb{B}} D\,G(x,y)\frac{f(y)}{r_\delta(y)}\,dy,$$

$$\partial_{x_i}\partial_{x_j} V_\delta(x) = \int_{\mathbb{B}} \left(\frac{f(y)}{r_\delta(y)} - \frac{f(x)}{r_\delta(x)}\right) \partial_{x_i}\partial_{x_j} G(x,y)\,dy$$
$$- \frac{1}{3}\delta_{ij}\frac{f(x)}{r_\delta(x)}, \quad x \in \overline{\mathbb{B}}. \tag{7.12}$$

It is easy to see that

$$\int_{\mathbb{B}} \frac{1}{|x-y|}\frac{1}{r(y)}\,dy \leq \int_{\mathbb{B}} \frac{1}{|y|}\frac{1}{r(y)}\,dy < \infty, \quad x \in \overline{\mathbb{B}},$$

$$\int_{\mathbb{B}} \frac{1}{|x-y|^2}\frac{1}{r(y)}\,dy \leq \int_{\mathbb{B}} \frac{1}{|(\eta,0,0)-y|^2}\frac{1}{r(y)}\,dy \leq C_R|\log \eta|, \quad x \in \overline{\mathbb{B}}\backslash Z_\eta,$$

$$\int_{\mathbb{B}} \frac{1}{|x-y|^{3-\alpha}}\frac{1}{r(y)}\,dy \leq \int_{\mathbb{B}} \frac{1}{|(\eta,0,0)-y|^{3-\alpha}}\frac{1}{r(y)}\,dy \leq C_{R,\alpha}\eta^{\alpha-2}.$$

Using (7.6), we get $V_\delta \to V$ in $C(\overline{\mathbb{B}})$ and in $C^2(\overline{\mathbb{B}}\backslash Z_\eta)$, and (7.12) is true on $\overline{\mathbb{B}}\backslash Z$ if we omit the index δ. For (7.11), we proceed as in Lemma 7.2, replacing f by f/r. Because

$$\frac{f(y)}{r(y)} - \frac{f(x)}{r(x)} = \frac{f(y)-f(x)}{r(y)} + f(x)\frac{r(x)-r(y)}{r(x)r(y)},$$

we have

$$H_{\alpha,\eta}(\frac{f}{r}) \leq H_{\alpha,\eta}(f)\eta^{-1} + C_R \parallel f \parallel \eta^{-1-\alpha}.$$

Similarly as in (7.9) for $p, q \in \overline{\mathbb{B}}\backslash Z_\eta$ such that $0 < |p-q| \leq \eta/4$

$$C_3 \int_0^{|p-q|} \int_{\substack{|y-(p+sE)|\geq \eta/2 \\ r(y)\leq \eta/4 \\ y \in \mathbb{B}}} \left(\frac{\parallel f \parallel}{r(y)} + \frac{\parallel f \parallel}{r(p+sE)}\right) \frac{dy}{|p+sE-y|^4}\,ds$$

$$\leq C_R |p-q|^\alpha \parallel f \parallel \eta^{-2-\alpha} + \frac{C_3}{\eta} \int\limits_0^{|p-q|} \int\limits_{\substack{|y-(p+sE)|\geq\eta/2 \\ r(y)\leq\eta/4 \\ y\in B}} \parallel f \parallel \frac{dy}{|p+sE-y|^4} ds,$$

and the second term is of the order of the first term.

In the sequel we let $C^{2+1^-}_{loc}(\overline{\mathbb{B}}\backslash Z) := \bigcap\limits_{\eta>0} C^{2+\alpha}(\overline{\mathbb{B}}\backslash Z_\eta)$ for $\alpha = 1$.

Theorem 7.1. *Let $\mathbb{B} := \{x \in \mathbb{R}^3 : |x| < R\}$, $Z := \{(0,0,x_3) : x_3 \in \mathbb{R}\}$. Let $\varphi \in C^1([1,\infty)\times\mathbb{R})$ satisfy the assumptions of Theorem 4.2. Then the problem*

$$\begin{aligned} &-\Delta\Phi = h_\varphi(r,\Phi,A_\vartheta), \\ &-\Delta A_\vartheta = g_\varphi(r,\Phi,A_\vartheta) - \frac{A_\vartheta}{r^2} \quad \text{in } \mathbb{B}\backslash Z, \\ &\Phi = 0, \quad A_\vartheta = 0 \quad \text{on } \partial\mathbb{B} \end{aligned}$$

has a cylindrically symmetric solution

$$\Phi \in C^1(\overline{\mathbb{B}}) \cap C^{2+1^-}_{loc}(\overline{\mathbb{B}}\backslash Z), \; A_\vartheta \in C(\overline{\mathbb{B}}) \cap C^{2+1^-}_{loc}(\overline{\mathbb{B}}\backslash Z).$$

We have

$$\begin{aligned} 0 \leq \Phi(r,z) &\leq \frac{1}{4} \parallel h_\varphi \parallel (R^2-r^2), \\ |A_\vartheta(r,z)| &\leq \frac{1}{3} \parallel g_\varphi \parallel r(R-r). \end{aligned} \tag{7.13}$$

For the corresponding stationary solution such that $f = \varphi(\mathcal{E},F)$ we have $f \in C(\overline{\mathbb{B}}) \cap C^1(\mathbb{B}\backslash Z)$, $E \in C(\overline{\mathbb{B}})^3 \cap C^{1+1^-}_{loc}(\overline{\mathbb{B}}\backslash Z)^3$, $B \in C^{1+1^-}_{loc}(\overline{\mathbb{B}}\backslash Z)^3$ and $|B(r,z)| \leq C_{\varphi,R}|\log r|$.

Proof. It follows from (7.2) and (7.4) that

$$|D\Phi_\delta(x)| \leq C_R \parallel h_{\varphi,\delta} \parallel \leq C_R \parallel h_\varphi \parallel . \tag{7.14}$$

We write (7.3) as

$$A_{\vartheta,\delta}(x) = \int\limits_B G(x,y)\left(g_{\varphi,\delta}(y) - \frac{1}{r(y)}\left(r(y)\frac{A_{\vartheta,\delta}(y)}{r_\delta^2(y)}\right)\right) dy$$

and note that $r(y)\frac{A_{\vartheta,\delta}(y)}{r_\delta^2(y)}$ is a continuous function on $\overline{\mathbb{B}}$ which can be estimated by Lemma 7.1 and Corollary 5.1:

$$\begin{aligned} |A_{\vartheta,\delta}(x)| &\leq W_\delta(r) = v_\delta(r) + C_R^* \parallel g_\varphi \parallel \delta \\ &\leq v_R(r) + 2C_R^* \parallel g_\varphi \parallel \delta \\ &= \parallel g_\varphi \parallel (\frac{1}{3}r(R-r) + 2C_R^*\delta), \end{aligned} \tag{7.15}$$

such that

$$\left| r\frac{A_{\vartheta,\delta}(y)}{r_\delta^2(y)} \right| \leq \parallel g_\varphi \parallel \left(\frac{1}{3}\frac{r^2}{r_\delta^2}(R-r) + 2C_R^*\frac{r\delta}{r_\delta^2}\right) \leq C_R \parallel g_\varphi \parallel .$$

Hence we obtain from Lemma 7.3

$$|DA_{\vartheta,\delta}(x)| \leq C_R \parallel g_\varphi \parallel |\log\eta|, \quad x \in \overline{\mathbb{B}}\backslash Z_\eta. \tag{7.16}$$

Theorem 4.2, the estimates (7.14), (7.15), (7.16) imply with the chainrule

$$|Dh_{\varphi,\delta}(x)|, |Dg_{\varphi,\delta}(x)| \leq C_{\varphi,R,\eta}, \quad x \in \overline{\mathbb{B}}\backslash Z_\eta.$$

Lemmas 7.2 and 7.3 then imply that $\{\Phi_\delta\}$ is bounded in $C^1(\overline{I\!B})$ and in $C^{2+1^-}(\overline{I\!B}\backslash Z_\eta)$ and that $\{A_{\vartheta,\delta}\}$ is bounded in $C^{2+1^-}(\overline{I\!B}\backslash Z_\eta)$ for each $\eta > 0$. By compactness, there exists a sequence $\delta_n \downarrow 0$ and functions $\Phi \in C(\overline{I\!B}) \cap C^{2+1^-}_{loc}(I\!B\backslash Z)$, $A_\vartheta \in C^{2+1^-}_{loc}(I\!B\backslash Z)$ such that $\Phi_{\delta_n} \to \Phi$ in $C(\overline{I\!B})$ and $C^{2+1^-}(\overline{I\!B}\backslash Z_\eta)$ and $A_{\vartheta,\delta_n} \to A_\vartheta$ in $C^{2+1^-}(\overline{I\!B}\backslash Z_\eta)$ for all $\eta > 0$. If we define $A_\vartheta(x) = 0$ for $x \in \overline{I\!B} \cap Z$, then also $A_{\vartheta,\delta_n} \to A_\vartheta$ in $C(\overline{I\!B})$ and (7.13) is valid. This follows from (7.15). The limits satisfy

$$\Phi(x) = \int_{I\!B} G(x,y)\, h_\varphi(y,\Phi(y),A_\vartheta(y))\, dy,$$

$$A_\vartheta(x) = \int_{I\!B} G(x,y)\left(g_\varphi(y,\Phi(y),A_\vartheta(y)) - \frac{A_\vartheta(y)}{r^2(y)}\right) dy, \quad x \in \overline{I\!B} \tag{7.17}$$

and hence Φ, A_ϑ are solutions of the problem. The asserted regularity of f and E is obvious. Using (2.22), we get in $\overline{I\!B}\backslash Z$

$$B_r = -\partial_z A_\vartheta, \quad B_\vartheta = 0, \quad B_z = \frac{1}{r}A_\vartheta + \partial_r A_\vartheta.$$

The logarithmic estimate of B follows from (7.13) and (7.16).

In Theorem 7.1 it might not be excluded that B exists as a continuous function up to the axis Z. To prove the existence of a Green's function for the operator $-\Delta + \frac{1}{r^2}$ and the generalization to arbitrary cylindrically symmetric domains Ω seems to be a topic of further research.

Acknowledgement. The first author wants to thank Professor Li Ta-tsien, Fudan University, for the hospitality during his visit in September 1992.

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