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HOMOMORPHISMS BETWEEN SYMPLECTIC GROUPS**

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Abstract

Let F and F_1 be fields with $|F| > 5$, and n, n_1 integers satisfying $n \geq n_1$. The present paper determines the forms of homomorphisms from $SP_{2n}(F)$ to $SP_{2n}(F_1)$ and that from $PSP_{2n}(F)$ to $PSP_{2n}(F_1)$.

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In 1928, O. Schreier and B. L. van der Waerden[3] started the study of homomorphisms between classical groups. Since then, a lot of results have been obtained on the subject. In [4] , L.N. Vaserstein stated that the problem of homomorphisms between classical groups is one of the three main ones. A. Borel and J. Tits[5], B. Weisfeiler[6] clarified abstract homomorphisms between subgroups of algebraic groups on the condition that the images of the homomorphisms are Zariski dense subsets and the fields over which algebraic groups are defined are infinite. Recently, Yu Chen[2] determined the homomorphisms between two dimensional linear groups over fields without the assumption of Zariski density and of infinitude of fields. Therefore, we have the forms of homomorphisms between two dimensional symplectic groups over fields. The purpose of this paper is to determine the homomorphisms between symplectic groups over fields. Namely, we prove the following

Main Theorem. *Let F and F_1 be fields and n a positive integer, α: $SP_{2n}(F) \to SP_{2n}(F_1)$ a non-trivial homomorphism. Then, α is of one of the following forms* (I) *and* (II).

(I) α: $X \mapsto PX^\tau P^{-1}$, $\forall\, X \in SP_{2n}(F)$,

where $P \in GSP_{2n}(F_1)$ and τ is a homomorphism from F to F_1.

(II) α^*: $L_i(f) \mapsto P\widetilde{L}_i(f^\tau)P^{-1}$, $\quad L_i'(f) \mapsto P\widetilde{L}_i'(f^\tau)P^{-1}$,

$Q_{ij}(f) \mapsto P\widetilde{Q}_{ij}(f^\tau)P^{-1}$, $\quad i \neq j,\ i,j = 1,2.$

$N_{12}(f) \mapsto P\widetilde{N}_{12}(f^\tau)P^{-1}$, $\quad S_{12}(f) \mapsto P\widetilde{S}_{12}(f^\tau)P^{-1}$,

where $n = 2$, $\mathrm{ch}\, F = \mathrm{ch}\, F_1 = 2$, $P \in GSP_4(F_1)$, τ is a homomorphism from F to F_1 satisfying that F_1 is a splitting field for all polynomials $x^2 + a^\tau$, $\forall\, a \in F$, and

$$L_i(f) = I + fE_{2+i,i}, \qquad L_i'(f) = I + fE_{i,i+2},$$

$$N_{12}(f) = I + f(E_{4,1} + E_{3,2}), \qquad S_{12}(f) = I + f(E_{1,4} + E_{2,3})$$

$$Q_{ij}(f) = I + f(E_{j,i} + E_{2+i,2+j}), \quad i \neq j,\ i,j = 1,2,$$

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and

$$\widetilde{L}_1(y)=N_{12}(y^{\frac{1}{2}}),\ \ \widetilde{L}_1'(y)=S_{12}(y^{\frac{1}{2}}),\ \ \widetilde{L}_2(y)=Q_{12}(y^{\frac{1}{2}}),\ \ \widetilde{L}_2'(y)=Q_{21}(y^{\frac{1}{2}});$$
$$\widetilde{Q}_{12}(y)=L_2(y),\ \ \widetilde{Q}_{21}(y)=L_2'(y),\ \ \widetilde{N}_{12}(y)=L_1(y),\ \ \widetilde{S}_{12}(y)=L_1'(y).$$

Proposition 1. *Suppose F and F_1 are fields and n is a positive integer. Then there exists a homomorphism from $SP_{2n}(F)$ to $SP_{2n}(F_1)$ with the form of* (I) *(or* (II) *when* ch $F=$ ch $F_1=2$ *and $n=2$) if and only if there exists a homomorphism from F to F_1.*

Proof. The necessity is clear. We now prove the sufficiency. When ch F=ch F_1=2, $\tau_0: a\mapsto a^2$, $\forall a\in F$, is an endomorphism of F. Then $\tau\tau_0$, for any τ a homomorphism from F to F_1, is a homomorphism from F to F_1 satisfying the condition of (II). Therefore, we have homomorphisms from $SP_4(F)$ to $SP_4(F_1)$ with the form (II). The existence of form (I) is obvious.

Proposition 2. *Let F and F_1 be fields and n, n_1 positive integers. Then there does not exist any non-trivial homomorphism from $SP_{2n_1+2}(F)$ to $SP_{2m_1}(F_1)$ for any m_1 $(1\le m_1\le n_1+1)$, if there is not any non-trivial homomorphism from $SP_{2n}(F)$ to $SP_{2m}(F_1)$ for any m $(1\le m\le n_1)$.*

Proof. Since $SP_{2m_1}(F_1)$ is a subgroup of $SP_{2n_1+2}(F_1)$ up to isomorphism, we need only to prove that any homomorphism from $SP_{2n+2}(F)$ to $SP_{2n_1+2}(F_1)$ is trivial.

Let

$$T_0=\begin{cases}\text{diag }(1,-I^{(n-1)},1,-I^{(n-1)}), & \text{when ch } F\neq 2;\\ I+E_{1,n+1}, & \text{when ch } F=2.\end{cases}$$

$$\Pi=\left\{\left.\begin{pmatrix}1 & & 0 & \\ & A & & B\\ 0 & & 1 & \\ & C & & D\end{pmatrix}\right|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in SP_{2n}(F)\right\}.$$

Clearly, $T_0^2=I$, $T_0\in C(\Pi)$ and $\Pi\simeq SP_{2n}(F)$. If α is non-trivial,

i) when ch F_1=2, by [1, p.474, p.481], we can assume

$$\alpha T_0=\left(\begin{array}{c:cc} I & S & 0\\ & 0 & 0\\ \hdashline 0 & \multicolumn{2}{c}{I}\end{array}\right),\tag{1}$$

where

$$S=\begin{pmatrix}0 & I^{(p)} & & & \\ I^{(p)} & 0 & & & \\ & & 0 & I^{(q)} & \\ & & I^{(q)} & C & \\ & & & & D\end{pmatrix},\qquad 1\le \text{rank } S=r\le n_1+1,$$

and $\begin{pmatrix}C & 0\\ 0 & D\end{pmatrix}$ is definite sign (see [1, p.481]).

Since $\alpha\Pi\subset C_{\mathrm{im}\ \alpha}(\alpha T_0)$, by [1,p.489] elements in $C_{\mathrm{im}\ \alpha}(\alpha T_0)$ are of the following form when $r < n_1 + 1$:

$$\begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ 0 & A_4 & B_3 & B_4 \\ 0 & 0 & S^{-1}A_1S & 0 \\ 0 & C_4 & D_3 & D_4 \end{pmatrix}, \tag{2}$$

where A_1 satisfies $A_1SA_1^t = S$, and $\begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix} \in SP_{2(n_1+1-r)}(F_1)$. When $r = n_1 + 1$, (2) transforms into

$$\begin{pmatrix} A_1 & B_1 \\ 0 & A_1^{t\,-1} \end{pmatrix}. \tag{3}$$

When $pq \neq 0$ and D exists, by [1, p.486] we can assume that A_1 is of the following form after writing the elements in the same blocks as S.

$$A_1 = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ A_{41} & A_{42} & A_{43} & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}, \tag{4}$$

where $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in SP_{2p}(F_1)$. When $pq = 0$, or D does not exist, the elements corresponding to D in (4) disappear. Hence, the following map

$$\beta_1 : \begin{pmatrix} 1 & 0 & & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ 0 & A_4 & B_3 & B_4 \\ 0 & 0 & S^{-1}A_1S & 0 \\ 0 & C_4 & D_3 & D_4 \end{pmatrix} \mapsto \begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix} \tag{5}$$

is a homomorphism from $SP_{2n}(F)$ to $SP_{2(n_1+1-r)}(F_1)$. It is trivial by the hypothesis. Hence

$$\alpha \begin{pmatrix} 1 & 0 & & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ 0 & I & B_3 & 0 \\ 0 & 0 & S^{-1}A_1S & 0 \\ 0 & 0 & D_3 & I \end{pmatrix}. \tag{6}$$

If $p \geq 1$, the following map provided by (4)

$$\beta_2 : \begin{pmatrix} 1 & 0 & & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \mapsto \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ 0 & I & B_3 & 0 \\ 0 & 0 & S^{-1}A_1S & 0 \\ 0 & 0 & D_3 & I \end{pmatrix} \mapsto \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{7}$$

is a homomorphism from $SP_{2n}(F)$ to $SP_{2p}(F_1)$, and it is trivial for $p \leq n_1$. Therefore, the A_1 in (6) is of the following form

$$A_1 = \begin{pmatrix} I & 0 & A_{13} & 0 & 0 \\ 0 & I & A_{23} & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ A_{41} & A_{42} & A_{43} & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Clearly, $A_1^4 = I$, we have

$$\alpha(Y^{16}) = I, \qquad \forall\, Y \in \Pi. \tag{8}$$

But there exist elements whose orders are 3 in $SP_{2n}(F)$, α is trivial by (8). When $p = 0$, it is easy to point out that (8) remains true. Therefore, α is trivial.

ii) When ch $F_1 \neq 2$, by [1, p.454, Theorem 2] we can assume

$$\alpha T_0 = \text{diag}\,(I^{(p)}, -I^{(q)}, I^{(p)}, -I^{(q)}), \quad p+q = n_1+1. \tag{9}$$

Clearly, $\alpha T_0 \neq \pm I$. Hence, $p \geq 1$, $q \geq 1$, and the elements in $C_{\text{im}\,\alpha}(\alpha T_0)$ are of the following form

$$\begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix},$$

where $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in SP_{2(n_1+1-q)}(F_1)$, $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in SP_{2(n_1+1-p)}(F_1)$. Similarly to i), the following maps

$$\begin{aligned} \beta_1' : \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \\ \beta_2' : \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix} \mapsto \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \end{aligned} \tag{10}$$

can be respectively regarded as homomorphisms from $SP_{2n}(F)$ to $SP_{2p}(F_1)$ and from $SP_{2n}(F)$ to $SP_{2q}(F_1)$. Since $p \leq n_1$, $q \leq n_1$, β_1' and β_2' are trivial by the assumption. Consequently, α is trivial.

Corollary 1. *Let F and F_1 be fields with $|F| > 5$ and $n > n_1$. Then no non-trivial homomorphism from $SP_{2n}(F)$ to $SP_{2n_1}(F_1)$ exists.*

Proof. Clearly, $DC(\Pi) \simeq SL_2(F) \simeq \Pi$ when $n=2$ and $n_1 = 1$. Therefore, $\alpha DC(\Pi) \subset C_{\text{im}\,\alpha}(\alpha\Pi)$ for any $\alpha : SP_4(F) \to SP_2(F_1)$. By [2], it is not difficult to prove that α which satisfies the above relations is trivial. When $n \geq 2$, one can easily complete the proof by induction on n.

Corollary 2. *Let F and F_1 be fields with $|F| > 5$ and n positive integer. Then* ch $F =$ ch F_1, *if there is a non-trivial homomorphism from $SP_{2n}(F)$ to $SP_{2n}(F_1)$.*

Proof. By Proposition 2 and Corollary 1, there exists a non-trivial homomorphism from $SP_{2n-2}(F)$ to $SP_{2n-2}(F_1)$. Repeating the manner, we have a non-trivial homomorphism from $SP_2(F)$ to $SP_2(F_1)$. By [2], ch F=ch F_1.

Definition. *Let $T \in SP_{2n}(F)$. T is called* 1*-involution* (1*-inv for short*) *if T satisfies the conditions $T \neq I$, $T^2 = I$ and*

$$\text{rank}\,(T+I) = \begin{cases} 1, & \text{ch } F = 2; \\ 2, & \text{ch } F \neq 2. \end{cases}$$

Set

$$\begin{aligned} &T_1 = \text{diag}\,(1, -I, 1, -I), && T_2 = -T_1 = \text{diag}\,(-1, I, -1, I) \\ &J_1 = I + E_{1,n+1}, && J_2 = I + E_{n+1,1}, \\ &K_1 = \begin{pmatrix} I & H \\ 0 & I \end{pmatrix}, && K_2 = \begin{pmatrix} I & 0 \\ H & I \end{pmatrix}, \end{aligned}$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Clearly, J_i, K_i $(i=1,2)$ are involutions when ch $F = 2$, and so are T_i $(i=1,2)$ when ch $F \neq 2$.

Lemma 1. *Let* $\mathrm{ch}\, F=2$, $K \in SP_4(F)$. *Then there is a* P *in* $C(K_1)$ *satisfying* $PKP^{-1} = K_2$ *if* K *satisfies* $K^2 = I$ *and* $(KK_1)^3 = I$.

Proof. Suppose $K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Taking $X = (A+I)H$, from the conditions one can prove that $P = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ is what we need.

Lemma 2. *Let* F *and* F_1 *be fields with* $|F| > 5$, $\alpha : SP_{2(n+1)}(F) \to SP_{2(n+1)}(F_1)$ *a non-trivial homomorphism. Then* α *maps 1-inv to 1-inv when* $n \geq 2$ *or* $n = 1$ *and* $\mathrm{ch}\, F \neq 2$.

Proof. By Corollary 2, $\mathrm{ch}\, F = \mathrm{ch}\, F_1$. i) When $\mathrm{ch}\, F \neq 2$, one can assume T_1 is of the form of (9). From (9), $\beta_2'(-I^{(2n)}) = -I^{(2q)}$. Hence β_2' is a non-trivial homomorphism from $SP_{2n}(F)$ to $SP_{2q}(F_1)$. By Corollary 1, we have $n = q$, αT_1 is 1-inv. ii) When $\mathrm{ch}\, F = 2$, one can assume αJ_1 is of the form of (1) ($n = n_1$). If $r > 1$, β_1 in (5) is trivial, and (6) or (3) is true. Since $2p \leq n+1 < 2n$, β_2 in (7) is also trivial. That is contrary to (8). Hence, $r = 1$, αJ_1 is 1-inv, and we complete the proof.

Theorem 1. *Let* F *and* F_1 *be fields with* $|F| > 5$ *and* $n \geq 2$, $\alpha : SP_{2n}(F) \to SP_{2n}(F_1)$ *a non-trivial homomorphism. Then* i) α *is of the form of* (I) *if and only if* α *maps* 1-*inv to* 1-*inv;* ii) α *is of the form of* (II) *if and only if* $n = 2$, $\mathrm{ch}\, F = 2$ *and* α *maps 1-inv to non-metabelian 2-inv*[1,p.494].

Proof. i) The necessary is clear, and we need only to prove the sufficiency. The proof is by induction on n.

When $\mathrm{ch}\, F=2$, by Corollary 2 and [1, p.479, Theorem 4], one can assume $\alpha J_i = J_i$. Clearly, $\Pi = C(J_1, J_2)$. α induces a non-trivial homomorphism from $SP_{2n}(F)$ to $SP_{2n}(F_1)$

$$\beta : \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} 1 & & 0 & \\ & A_1 & & B_1 \\ 0 & & 1 & \\ & C_1 & & D_1 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in SP_{2n}(F_1) \tag{11}$$

and β maps 1-inv to 1-inv also. When $\mathrm{ch}\, F \neq 2$, assuming $\alpha T_1 = T_1$ we have $\alpha C(T_1) = C_{\mathrm{im}\,\alpha}(\alpha F_1)$; therefore

$$\beta' : \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} \to \begin{pmatrix} a_1 & & b_1 & \\ & A_1 & & B_1 \\ c_1 & & d_1 & \\ & C_1 & & D_1 \end{pmatrix} \to \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

is a homomorphism from $SP_{2n}(F)$ to $SP_2(F_1)$. By Corollary 1, β' is trivial. Therefore, α satisfies (11). By the induction assumption, there exists τ, a homomorphism from F to F_1, and

$$P_1 = \begin{pmatrix} 1 & & 0 & \\ & P_{11} & & P_{12} \\ 0 & & 1 & \\ & P_{21} & & P_{22} \end{pmatrix} \cdot \mathrm{diag}\,(1, I, b_0, b_0 I)$$

(where $\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in SP_{2n}(F_1)$, $b_0 \in F_1^*$) satisfying

$$\alpha \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix} = P_1 \begin{pmatrix} 1 & & 0 & \\ & A & & B \\ 0 & & 1 & \\ & C & & D \end{pmatrix}^{\tau} P_1^{-1}. \tag{12}$$

Since $\alpha DC(\mathrm{II}) = DC_{\mathrm{im}\ \alpha}(\alpha\Pi)$, by (12) we have

$$\alpha\begin{pmatrix} a & & b & \\ & I & & 0 \\ c & & d & \\ & 0 & & I \end{pmatrix} = \begin{pmatrix} a_1 & & b_1 & \\ & I & & 0 \\ c_1 & & d_1 & \\ & 0 & & I \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SP_2(F). \tag{13}$$

Therefore, α induces a non-trivial homomorphism from $SP_2(F)$ to $SP_2(F_1)$. By [2], there exists τ_1, a homomorphism from F to F_1, and

$$P_2 = \begin{pmatrix} p_{11} & & p_{12} & \\ & I & & 0 \\ p_{21} & & p_{22} & \\ & 0 & & I \end{pmatrix} \mathrm{diag}\,(1, I, a_0, a_0 I)$$

(where $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in SP_2(F_1)$, $a_0 \in F_1^*$) satisfying

$$\alpha\begin{pmatrix} a & & b & \\ & I & & 0 \\ c & & d & \\ & 0 & & I \end{pmatrix} = P_2 \begin{pmatrix} a & & b & \\ & I & & 0 \\ c & & d & \\ & 0 & & I \end{pmatrix}^{\tau_1} P_2^{-1}.$$

Put

$$P^* = \begin{pmatrix} p_{11} & & p_{12} & \\ & p_{11} & & p_{12} \\ p_{21} & & p_{22} & \\ & p_{21} & & p_{22} \end{pmatrix} \cdot \mathrm{diag}\,(I, b_0 I)$$

and $i_{P^*}\alpha = \alpha$ where $i_{P^*} : x \mapsto P^{*-1}XP^*$, $\forall X \in SP_{2n+2}(F_1)$ is an automorphism of $SP_{2n+2}(F_1)$. Then, one can assume

$$\alpha\begin{pmatrix} a & & b & \\ & A & & B \\ c & & d & \\ & C & & D \end{pmatrix} = \begin{pmatrix} a^{\tau_1} & & \rho b^{\tau_1} & \\ & A^{\tau} & & B^{\tau} \\ \rho^{-1}c^{\tau_1} & & d^{\tau_1} & \\ & C^{\tau} & & D^{\tau} \end{pmatrix}, \tag{14}$$

where $\rho = a_0^{-1}b_0$.

Set

$$M_1 = \left\{ \begin{pmatrix} I^{(2)} & & 0 & \\ & I & & B \\ & & I^{(2)} & \\ & & & I \end{pmatrix} \middle|\, B \text{ is } (n-1)\times(n-1) \text{ symmetric matrix composed of } 0, \pm 1 \right\}.$$

$$M_2 = \left\{ \begin{pmatrix} 1 & & & \\ & I & & \\ 0 & & 1 & \\ & B & & I \end{pmatrix} \middle|\, B \text{ is } n\times n \text{ symmetric matrix composed of } 0, \pm 1 \right\}.$$

From (14), α acts identically on M_1 and M_2.

Let

$$T_{12}(f) = \begin{pmatrix} 1 & f & & & & \\ & 1 & & & & \\ & & I & & & \\ & & & 1 & & \\ & & & -f & 1 & \\ & & & & & I \end{pmatrix}.$$

Since $T_{12} \in C(M_1 \cup M_2 \cup \{T_1\})$, we have

$$\alpha T_{12}(f) = \begin{pmatrix} a_1 & a_2 & & b & 0 & \\ 0 & a & & 0 & 0 & \\ & & aI & & & \\ 0 & 0 & & a_1 & 0 & \\ 0 & c & & d_3 & a & \\ & & & & & aI \end{pmatrix},$$

where a_1, a_2, a, b, c, $d_3 \in F_1$. Take $E = \text{diag}\,(\delta I^{(2)}, I, \delta^{-1} I^{(2)}, I)$, where $\delta^2 \neq 0, 1$. Since E commutes with $T_{12}(f)$, by straightforward calculation, we have $b = c = 0$ and $a_1^2 = a^2 = 1$. When ch$F = 2$, we have $a_1 = a = 1$. Therefore, one can assume

$$\alpha T_{12}(1) = T_{12}(t). \tag{15}$$

When ch $F \neq 2$, by the identity $T_{12}(1) = [T_{12}(\frac{1}{2})]^2$, we also have (15). In the similar manner, one can assume $\alpha T_{21}(1) = T_{21}(s)$, where $T_{21}(1) = T_{12}^t(1)$. By the identity $[T_{12}(1)T_{21}(1)]^3 = I$, we have $st = -1$. Put $i_W\alpha = \alpha$, where $W = \text{diag}\,(t, I, t^{-1}, I)$; we have

$$\alpha T_{12}(1) = T_{12}(1), \qquad \alpha T_{21}(1) = T_{21}(1), \qquad \alpha X = X^\tau, \qquad \forall X \in \Pi.$$

Since $SP_{2n+2}(F)$ is generated by $T_{12}(1)$, $T_{21}(1)$ and Π, α is of the form of (I).

Now we consider the situation $n = 2$.

When ch $F = 2$, by the similar discussion to above, we also have (11). When ch $F \neq 2$, β' is a homomorphism from $SP_2(F)$ to $SP_2(F_1)$. Since $\alpha T_1 = T_1$, we have $\beta'(-I) = I$. By [2], it is trivial and we have (11). Hence β is a non-trivial homomorphism from $SP_2(F)$ to $SP_2(F_1)$. By [2], we have (12), (13) and (14). By the same discussion as in the situation $n+1$, α is of the form of (I). By induction, we complete the proof of i).

ii) From i) and Lemma 3, the necessity is clear. Next we prove the sufficiency. Since ch $F = 2$, taking 1-inv $T_0 = J_1$, one can assume αT_0 is of the form in (1) ($n = n_1 = 1$). Since $r = \text{rank}\ S = 2$, the elements in $C_{\text{im}\ \alpha}(\alpha T_0)$ are of the form in (3). If $p = 0$, similar to Lemma 1, we have (8). That is contrary. Hence, $p = 1$, $\alpha J_1 = K_1$. By Lemma 2, one can assume

$$\alpha J_1 = K_1, \qquad \alpha J_2 = K_2. \tag{16}$$

From $\alpha\Pi = \alpha C(J_1, J_2) = C_{\text{im}\ \alpha}(K_1, K_2)$, it is not difficult to point out that the elements in $C_{\text{im}\ \alpha}(K_1, K_2)$ are of the following form

$$\begin{pmatrix} A & 0 \\ 0 & A^{t^{-1}} \end{pmatrix}, \qquad A \in SP_2(F_1).$$

Hence, the following map

$$\begin{pmatrix} 1 & & 0 & \\ & a & & b \\ 0 & & 1 & \\ & c & & d \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} & \\ & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}^{t^{-1}} \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

is a non-trivial homomorphism from $SP_2(F)$ to $SP_2(F_1)$. By [2], there exists τ_1, a homomorphism from F to F_1, and $P_1 \in GL_2(F_1)$ satisfying

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = P_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\tau_1} P_1^{-1}.$$

Take $P_1^* = \mathrm{diag}\,(P_1, \det P_1 \cdot P_1^{-1}) \in GSP_4(F_1)$ and put $i_{P_1^*}\alpha = \alpha$. We have (16) and

$$\alpha\begin{pmatrix}1 & 0 & & \\ & a & & b\\ 0 & & 1 & \\ & c & & d\end{pmatrix} = \begin{pmatrix}\begin{matrix}a & b\\ c & d\end{matrix} & \\ & \begin{pmatrix}a & b\\ c & d\end{pmatrix}^{t^{-1}}\end{pmatrix}^{\tau_1}. \tag{17}$$

From (17), the elements in $C_{\mathrm{im}\,\alpha}(C_{\mathrm{im}\,\alpha}(K_1, K_2))$ are of the form of $\begin{pmatrix}a_1I & b_1H\\ c_1H & d_1I\end{pmatrix}$, where $\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} \in SP_2(F_1)$. Since the following map

$$\begin{pmatrix}a & & b & \\ & 1 & & 0\\ c & & d & \\ & 0 & & 1\end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix}a_1I & b_1H\\ c_1H & d_1I\end{pmatrix} \mapsto \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$$

is another non-trivial homomorphism from $SP_2(F)$ to $SP_2(F_1)$, from [2] we see that there exists τ_2, a homomorphism from F to F_1, and $P_2 = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} \in GSP_2(F_1)$ satisfying $\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} = P_2\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{\tau_2} P_2^{-1}$. Take $P_2^* = \begin{pmatrix}x_1I & x_2I\\ x_3I & x_4I\end{pmatrix} \in GSP_4(F_1)$, We have

$$\alpha\begin{pmatrix}a & & b & \\ & 1 & & 0\\ c & & d & \\ & 0 & & 1\end{pmatrix} = P_2^*\begin{pmatrix}aI & bH\\ cH & dI\end{pmatrix}^{\tau_2} P_2^{*-1}.$$

From (16), we have $x_2 = x_3 = 0$, $x_1 = x_4$. Therefore

$$\alpha\begin{pmatrix}a & & b & \\ & 1 & & 0\\ c & & d & \\ & 0 & & 1\end{pmatrix} = \begin{pmatrix}a^{\tau_2}I & b^{\tau_2}H\\ c^{\tau_2}H & d^{\tau_2}I\end{pmatrix}. \tag{18}$$

Take $f \in F^*$. Since $T_{12}(f) \in C(J_1, I + E_{4,2})$, from (16) and (17) we have

$$\alpha T_{12}(f) = \left(\begin{array}{cc:cc}1 & 0 & 0 & b\\ a_1 & 1 & b & b_1\\ \hdashline & & 1 & a_1\\ & & 0 & 1\end{array}\right).$$

Since

$$\begin{pmatrix}I & fH\\ 0 & I\end{pmatrix} = \begin{pmatrix}1 & 0 & & \\ & 0 & & 1\\ 0 & & 1 & \\ & 1 & & 0\end{pmatrix} T_{12}(f) \begin{pmatrix}1 & 0 & & \\ & 0 & & 1\\ 0 & & 1 & \\ & 1 & & 0\end{pmatrix},$$

by (17) we have

$$\alpha\begin{pmatrix}I & fH\\ 0 & I\end{pmatrix} = \left(\begin{array}{cc:cc}1 & a_1 & b_1 & b\\ 0 & 1 & b & 0\\ \hdashline & & 1 & 0\\ & & a_1 & 1\end{array}\right).$$

Since $T_{12}(f)$ commutes with $\begin{pmatrix}I & fH\\ 0 & I\end{pmatrix}$, we have $a_1 = 0$. By $\alpha\begin{pmatrix}I & fH\\ 0 & I\end{pmatrix} \neq \alpha T_{12}(f)$, we have $b_1 \neq 0$. If $b = 0$, we have $X^4 = I$ for any $X \in C_{\mathrm{im}\,\alpha}\left(\alpha\begin{pmatrix}I & fH\\ 0 & I\end{pmatrix}\right)$. That is contrary.

Hence $b = 0$ and

$$\begin{aligned} &T_{12}(f) = I + b_1 E_{2,4}, \\ &\alpha\begin{pmatrix} I & fH \\ 0 & I \end{pmatrix} = I + b_1 E_{1,3}. \end{aligned} \tag{19}$$

Since

$$T_{21}(f) = \begin{pmatrix} 0 & & 1 & \\ & 1 & & 0 \\ 1 & & 0 & \\ & 0 & & 1 \end{pmatrix} \begin{pmatrix} I & fH \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & & 1 & \\ & 1 & & 0 \\ 1 & & 0 & \\ & 0 & & 1 \end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ fH & I \end{pmatrix} = \begin{pmatrix} 1 & & 0 & \\ & 0 & & 1 \\ 0 & & 1 & \\ & 1 & & 0 \end{pmatrix} \cdot T_{21}(f) \cdot \begin{pmatrix} 1 & & 0 & \\ & 0 & & 1 \\ 0 & & 1 & \\ & 1 & & 0 \end{pmatrix},$$

by (17), (18) and (19) we have

$$\begin{aligned} &\alpha T_{21}(f) = I + b_1 E_{4,2}, \\ &\alpha\begin{pmatrix} I & 0 \\ fH & I \end{pmatrix} = I + b_1 E_{3,1}. \end{aligned} \tag{20}$$

Let α acts on the identity

$$\left(\begin{array}{c:c} I & 0 \\ \hdashline \begin{smallmatrix} 0 & \\ & c \end{smallmatrix} & I \end{array}\right) \begin{pmatrix} I & fH \\ 0 & I \end{pmatrix} \left(\begin{array}{c:c} I & 0 \\ \hdashline \begin{smallmatrix} 0 & \\ & c \end{smallmatrix} & I \end{array}\right) = \left(\begin{array}{c:c} I & \begin{smallmatrix} cf^2 & \\ & 0 \end{smallmatrix} \\ \hdashline 0 & I \end{array}\right) T_{12}(cf) \begin{pmatrix} I & fH \\ 0 & I \end{pmatrix}.$$

From (17), (18) and (19), we have $b_1 = (f^{\tau_2})^2$ and $\tau_1 = \tau_2$. Take $\tau = \tau_1\tau_0$. From (17), (18) and (19), α is of the form of (II).

Remark. The fact that the map of the form (II) is a homomorphism can be proved from the definition relations of $SP_4(F)$.

Theorem 2. *Let F and F_1 be fields with $|F| > 5$ and n a positive integer, $\alpha : SP_{2n}(F) \to SP_{2n}(F_1)$ a non-trivial homomorphism. Then α is of the form of* (I) *or* (II).

Proof. When $n = 1$, the result can be got from [2]. When $n \geq 3$ or $n = 2$ and $\mathrm{ch}F \neq 2$, from Lemma 2, α is of the form of (I). When $n = 2$ and $\mathrm{ch}F = 2$, there exists only 1-inv and 2-inv in $SP_4(F_1)$. By Theorem 1, α is the of form (I) or (II).

Theorem 3. *Let F and F_1 be fields with $|F| > 5$ and $n \geq n_1$ positive integers. Then there exist non-trivial homomorphisms from $SP_{2n}(F)$ to $SP_{2n_1}(F_1)$ if and only if $n = n_1$ and there exist non-trivial homomorphisms from F to F_1.*

Theorem 4. *Let F and F_1 be fields with $|F| > 5$ and n a positive integer, $\alpha : SP_{2n}(F) \to SP_{2n}(F_1)$ a non-trivial homomorphism. Then α is an isomorphism if and only if α is an epimorphism.*

Theorem 5. *Let F and F_1 be fields with $|F| > 5$, $\overline{\alpha} : PSP_{2n}(F) \to PSP_{2n}(F_1)$ a non-trivial homomorphism. Then $\overline{\alpha}$ is induced by a homomorphism from $SP_{2n}(F)$ to $SP_{2n}(F_1)$.*

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