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TOTALLY REAL MINIMAL SUBMANIFOLDS IN A QUATERNION PROJECTIVE SPACE**

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Abstract

Some curvature pinching theorems for compact totally real minimal submanifolds in a quaternion projective space are given, so that the corresponding results due to B. Y. Chen and C. S. Houh in [1] are improved.

Keywords Projective space, Totally real minimal submanifolds, Curvature pinching
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§0. Introduction

A quaternion Kaehler manifold is defined as a $4m$-dimensional Riemannian manifold whose holonomy group is contained in $Sp(m)\cdot Sp(1)$ with the additional condition for $m=1$ that it is a self-dual Einstein space. A quaternion projective space $QP^m(c)$ is a quaternion Kaehler manifold with constant quaternion sectional curvature $c>0$. A complex projective space $CP^m(c)$ with constant holomorphic sectional curvature c can be isometrically imbedded in $QP^m(c)$ as a totally geodesic submanifold.

Let M be an n-dimensional Riemannian manifold and $\mathfrak{J}: M \to QP^m(c)$ an isometric immersion of M into $QP^m(c)$. If each tangent 2-subspace of M is mapped by $\mathfrak{J}$ into a totally real plane of $QP^m(c)$, then M is called a totally real submanifold of $QP^m(c)$. In [1], some fundamental properties of totally real submanifolds in $QP^m(c)$ were studied and the following theorems were shown.

Theorem A([1, Theorem 4]). *Let M be an n-dimensional compact totally real minimal submanifold in $QP^n(c)$. If*

$$\rho \geq \frac{n(3n^2-5n-1)c}{2(6n-1)} \quad \text{or equivalently} \quad \|\sigma\|^2 \leq \frac{3n(n+1)c}{4(6n-1)},$$

then M is totally geodesic, where ρ and $\|\sigma\|^2$ denote the scalar curvature and the length square of the second fundamental form of M, respectively.

Theorem B([1,Theorem 6]). *Let M be an n-dimensional compact totally real minimal submanifold in $QP^n(c)$. If the sectional curvature K_M of M satisfies*

$$K_M \geq \frac{(n-2)c}{4(2n-1)},$$

then either (i) *M is totally geodesic or* (ii) *$n=2$ and M is a flat surface.*

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Unfortunately, there is an error in the proof of Theorems 5 and 6 in [1]. The pinching constant in Theorem B may be adopted only for n-dimensional totally real minimal submanifolds of $CP^n(c)$ (cf. [1], Theorem 8).

In this paper, we shall further study the intrinsic rigidity of compact totally real minimal submanifolds in $QP^n(c)$. In §1 some necessary preliminaries for this article will be given. In §2 an improvement of Theorem A above will be shown (Theorem 2.1), which can be regarded as a generalization of the pinching theorem for the scalar curvature of totally real minimal submanifolds in $CP^n(c)$ given in [6]. In §3 some pinching theorems for the sectional curvature and the Ricci curvature of totally real minimal submanifolds in $QP^n(c)$ will be established (Theorems 3.1 and 3.2), which correct and improve Theorem B above. In the last section, we shall consider 3-dimensional totally real minimal submanifolds in $QP^3(c)$ and obtain some more advantageous pinching constants for the Ricci curvature and the scalar curvature.

§1. Preliminaries

We give here a quick review of basic formulas about totally real submanifolds in a quaternion Kaehler manifold, for details see [1].

Let $(\overline{M},\overline{g})$ be a $4m$-dimensional quaternion Kaehler manifold with almost quaternion structures I, J and K satisfying

$$IJ=K,\quad JK=I,\quad KI=J,\qquad I^2=J^2=K^2=-1.$$

For a unit vector X on $\overline{M}$, let $Q(X)$ denote the 4-plane spanned by X, IX, JX and KX, which is called the quaternion-section determined by X. Any 2-plane in a quaternion-section is called a quaternion-plane, whose sectional curvature is called the quaternion sectional curvature. For any two vectors X and Y on $\overline{M}$, if $Q(X)$ and $Q(Y)$ are mutually orthogonal, the 2-plane spanned by X and Y is called a totally real plane of $\overline{M}$. It is well known that $(\overline{M},\overline{g})$ has constant quaternion sectional curvature c if and only if the curvature tensor $\overline{R}$ of $\overline{M}$ is of the following form:

$$\begin{aligned}\overline{R}(X,Y)Z=&\frac{c}{4}\{\overline{g}(Y,Z)X-\overline{g}(X,Z)Y+\overline{g}(IY,Z)IX-\overline{g}(IX,Z)IY+2\overline{g}(X,IY)IZ\\&+\overline{g}(JY,Z)JX-\overline{g}(JX,Z)JY+2\overline{g}(X,JY)JZ+\overline{g}(KY,Z)KX-\overline{g}(KX,Z)KY\\&+2\overline{g}(X,KY)KZ\}.\end{aligned}$$

Let M be an n-dimensional Riemannian manifold and $\mathfrak{I}:M\to\overline{M}$ an isometric immersion. If each tangent 2-plane of M is mapped by $\mathfrak{I}$ into a totally real plane in $\overline{M}$, then M is called a totally real submanifold of $\overline{M}$.

In the following, let $QP^n(c)$ denote a $4n$-dimensional quaternion projective space with constant quaternion sectional curvature $c>0$. Let M be an n-dimensional totally real manifold in $QP^n(c)$ with $n\geq 2$. We choose a local field of orthonormal frames in $QP^n(c)$:

$$\begin{aligned}&e_1,\cdots,e_n;e_{I(1)}=Ie_1,\cdots,e_{I(n)}=Ie_n;\\&e_{J(1)}=Je_1,\cdots,e_{J(n)}=Je_n;e_{K(1)}=Ke_1,\cdots,e_{K(n)}=Ke_n,\end{aligned}$$

in such a way that, restricted to M, vectors $e_1,\cdots,e_n$ are tangent to M. With respect to this frame field, I, J, K have the following forms:

$$I=\begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{bmatrix}, \quad J=\begin{bmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{bmatrix}, \quad K=\begin{bmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{bmatrix}, \tag{1.1}$$

where E stands for the indentity $(n \times n)$-matrix.

We shall use the following convention on the range of indices unless otherwise stated:

$$A, B, C, \cdots = 1, \cdots, n, I(1), \cdots, I(n), J(1), \cdots, J(n), K(1), \cdots, K(n);$$
$$i, j, k = 1, \cdots, n;$$
$$\alpha, \beta, \gamma, \cdots, = I(1), \cdots, I(n), J(1), \cdots, J(n), K(1), \cdots, K(n);$$
$$\varphi = I, J \text{ or } K.$$

Let ω^A and ω^A_B be the dual frame field and the connection forms with respect to the frame field chosen above. Then, the structure equations of $QP^n(c)$ are

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \qquad \omega^A_B + w^B_A = 0,$$
$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \frac{1}{2}\sum \overline{R}_{ABCD}\omega^C \wedge \omega^D,$$

where

$$\overline{R}_{ABCD} = \frac{c}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}). \tag{1.2}$$

Restricting these forms to M, we have

$$\omega^\alpha = 0, \qquad \omega^\alpha_i = \sum h^\alpha_{ij}\omega^j, \qquad h^\alpha_{ij} = h^\alpha_{ji},$$
$$h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ki} = h^{\varphi(k)}_{ij}. \tag{1.3}$$

The second fundamental form σ of M in $QP^n(c)$ is defined as

$$\sigma = \sum h^\alpha_{ij}\omega^i \otimes \omega^j \otimes e_\alpha, \tag{1.4}$$

whose length square is

$$\|\sigma\|^2 = \sum_{\alpha,i,j}(h^\alpha_{ij})^2.$$

By (1.1) and (1.2), the Gauss-Codazzi-Ricci equations of M in $QP^n(c)$ are

$$R_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{1.5}$$

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{1.6}$$

$$R_{\alpha\beta kl} = \frac{c}{4}(I_{\alpha k}I_{\beta l} - I_{\alpha l}I_{\beta k} + J_{\alpha k}J_{\beta l} - J_{\alpha l}J_{\beta k} + K_{\alpha k}K_{\beta l} - K_{\alpha l}K_{\beta k}) + \sum_i(h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{1.7}$$

If M is minimal in $QP^n(c)$, i.e., trace $\sigma = 0$, then from (1.5) we have

$$R_{ij} = \frac{c}{4}(n-1)\delta_{ij} - \sum_{\alpha,k} h^\alpha_{ik}h^\alpha_{kj}, \tag{1.8}$$

$$\rho = \frac{c}{4}n(n-1) - \|\sigma\|^2, \tag{1.9}$$

where R_{ij} and ρ are the Ricci tensor and the scalar curvature of M, respectively.

Let H^α and Δ denote the $(n \times n)$-matrix (h^α_{ij}) and the Laplacian on M, respectively. The following formula can be found in [1] or [9]:

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \|\nabla\sigma\|^2 + (1+a)\sum h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{il}R_{lkjk})$$
$$+\frac{1}{2}(1-a)\sum \operatorname{tr}\,(H^\alpha H^\beta - H^\beta H^\alpha)^2 + a\sum(\operatorname{tr}\,H^\alpha H^\beta)^2 - \frac{c}{4}(na-1)\|\sigma\|^2 \tag{1.10}$$

for any real number a.

§2. Scalar Curvature

In this section, we improve Theorem A as follows.

Theorem 2.1. *Let M be an n-dimensional compact totally real minimal submanifold in the quaternion projective space $QP^n(c)$. If*

$\rho \geq (n-2)(5n^2+4n+1)c/4(5n+2)$ *or equivalently* $\|\sigma\|^2 \leq (n+1)(3n+2)c/4(5n+2)$,

then either (i) *M is totally geodesic in $QP^n(c)$ or* (ii) *$n = 2$ and M is a flat surface in $QP^2(c)$ with the parallel second fundamental form and a parallel normal subbundle of fiber dimension* 4.

For proving it we establish firstly the following

Lemma 2.1. *Let M be a compact totally real minimal surface in $QP^2(c)$ with nonnegative Gauss curvature. If M is not totally geodesic, then M is a flat surface in $QP^2(c)$ with the parallel second fundamental form and a parallel normal subbundle of fiber dimension 4. Moreover, with respect to an adapted dual orthonormal frame field (ω^A), the connection form (ω^A_B) of $QP^2(c)$, restricted to M, is given by*

$$\left[\begin{array}{cccc:c} 0 & \omega^1_2 & -a\omega^2 & -a\omega^1 & \\ \omega^2_1 & 0 & -a\omega^1 & a\omega^2 & 0 \\ a\omega^2 & a\omega^1 & 0 & 2\omega^2_1 & \\ a\omega^1 & -a\omega^2 & 2\omega^1_2 & 0 & \\ \hdashline & & 0 & & (\omega^\mu_\nu) \end{array}\right], \qquad \begin{array}{l} a = \sqrt{2c}/4, \\ \mu,\nu = J(1), J(2), K(1), K(2). \end{array}$$

Proof. By the proof of Theorem B in [1] (cf. (3.5) below), we have for $n = 2$

$$\frac{1}{2}\Delta(\|\sigma\|^2) \geq \|\nabla\sigma\|^2 + 3K_M\|\sigma\|^2,$$

where K_M stands for the Gauss curvature of M. Thus, if $K_M \geq 0$ and $\|\sigma\|^2 \neq 0$, then we obtain immediately

$$\nabla\sigma = 0, \qquad K_M = 0. \tag{2.1}$$

Moreover, at most two of matrices $\{H^\alpha\}$ are nonzero and, with respect to suitable frames, these two matrices have the following forms (cf. [2], Lemma 1):

$$A = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where a and b are scalar factors. Without loss of generality, we may assume that $H^{I(1)} = A$, i.e.,

$$h_{11}^{I(1)} = h_{22}^{I(1)} = 0, \qquad h_{12}^{I(1)} = a. \tag{2.2}$$

By using (1.3) and the minimality, one can easily see that

$$h_{11}^{I(2)} = a, \qquad h_{22}^{I(2)} = -a, \qquad h_{12}^{I(2)} = 0. \tag{2.3}$$

Therefore, we should have $H^{I(2)} = B$ with $b = a$ and $H^{J(1)} = H^{J(2)} = H^{K(1)} = H^{K(2)} = 0$.

Since $K_M = 0$, it follows from (1.9) that $a^2 = c/8$. Thus, we may assume that

$$a = \sqrt{2c}/4.$$

Since $\nabla\sigma = 0$, i.e., $h_{ijk}^{\alpha} = 0$, we have

$$dh_{ij}^{\alpha} = \sum_k h_{kj}^{\alpha}\omega_i^k + \sum_k h_{ik}^{\alpha}\omega_j^k - \sum_\beta h_{ij}^{\beta}\omega_\beta^{\alpha}.$$

Setting $\alpha = \varphi(k)$ where $\varphi = J$ or K, $i = j = 1$, we see that $\omega_{I(2)}^{\varphi(k)} = 0$. Similarly, setting $\alpha = \varphi(k)$, $i = 1, j = 2$, we see that $\omega_{I(1)}^{\varphi(k)} = 0$. Hence,

$$\omega_{I(i)}^{J(k)} = \omega_{I(i)}^{K(k)} = 0, \tag{2.4}$$

which implies that the normal subbundle with the fiber spanned by $e_{J(1)}$, $e_{J(2)}$, $e_{K(1)}$ and $e_{K(2)}$ is parallel in the normal bundle over M.

Setting again $\alpha = I(1)$, $i = j = 2$, we see that

$$\omega_{I(2)}^{I(1)} = 2\omega_1^2. \tag{2.5}$$

Moreover, from (2.2) and (2.3) it follow that

$$\omega_1^{I(1)} = h_{12}^{I(1)}\omega^2 = a\omega^2, \qquad \omega_2^{I(1)} = h_{21}^{I(1)}\omega^1 = a\omega^1,$$
$$\omega_1^{I(2)} = h_{11}^{I(2)}\omega^1 = a\omega^1, \qquad \omega_2^{I(2)} = h_{22}^{I(2)}\omega^2 = -a\omega^2.$$

These with (2.4) and (2.5) prove the lemma.

Remark 2.1. From the proof of lemma we see that the minimal flat surface M in $QP^2(c)$ is totally geodesic with respect to the parallel normal subbundle of fiber dimension 4.

Remark 2.2. The flat torus minimally embedding in $CP^2(c)$ with the parallel second fundamental form [9] provides an example of such totally real minimal surfaces in $QP^2(c)$ as in Lemma 2.1.

Proof of Theorem 2.1. We use the method in [5]. Let $UM \to M$ be the unit tangent bundle over M. Define a function $f : UM \to R$ by

$$f(u) = \|\sigma(u,u)\|^2 \quad \text{for } u \in UM. \tag{2.6}$$

Since UM is compact, f attains its maximum at some vector in UM. Suppose that this vector is $v \in UM_{x_0}$ for some point $x_0 \in M$. As is seen in [5], we can choose tangent vectors $e_1, \cdots, e_n$ at x_0 such that $v = e_1$ and the matrix $(\sum\limits_\alpha h_{11}^{\alpha}h_{ij}^{\alpha})_{n\times n}$ is diagonalized at x_0.

Setting

$$b_{ij} = \sum_\alpha h_{11}^{\alpha}h_{ij}^{\alpha}, \tag{2.7}$$

we have from the maximum condition [5] at x_0

$$f(v)=b_{11}=\max_{u\in UM} f(u), \tag{2.8}$$

$$b_{ij}=0 \qquad (i\neq j), \tag{2.9}$$

$$2\sum_\alpha (h^\alpha_{1k})^2+b_{kk}-f(v)\le 0 \qquad (k\neq 1), \tag{2.10}$$

$$\sum_\alpha h^\alpha_{11}h^\alpha_{11ii}\le 0. \tag{2.11}$$

Summing up for i in (2.11) and using (2.9) and the Ricci identity, we get

$$0\ge \sum_i (b_{ii}R_{i11i}+b_{11}R_{1i1i})+\sum_{\alpha,\beta,i} h^\alpha_{11}h^\beta_{1i}R_{\beta\alpha 1i}. \tag{2.12}$$

By (1.1), (1.3), (1.5) and (1.7), we find

$$\sum_i (b_{ii}R_{i11i}+b_{11}R_{1i1i})=\frac{c}{4}nf(v)+\sum_{i,\alpha} b_{ii}(h^\alpha_{1i})^2-\sum_i (b_{ii})^2-f(v)\sum_{i,\alpha}(h^\alpha_{1i})^2,$$

$$\sum_{\alpha,\beta,i} h^\alpha_{11}h^\beta_{1i}R_{\beta\alpha 1i}=\frac{c}{4}f(v)+\sum_{i,\alpha} b_{ii}(h^\alpha_{1i})^2-f(v)\sum_{i,\alpha}(h^\alpha_{1i})^2.$$

Introducing these into (2.12), we obtain

$$0\ge \frac{c}{4}(n+1)f(v)+2\sum_{\substack{\alpha\\ k\neq 1}} b_{kk}(h^\alpha_{1k})^2-2f(v)\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2-\sum_{k\neq 1}(b_{kk})^2-f(v)b_{11}. \tag{2.13}$$

By the following inequalities

$$2b_{kk}\ge -f(v)-\sum_\alpha (h^\alpha_{kk})^2 \quad\text{and}\quad (b_{kk})^2\le f(v)\sum_\alpha (h^\alpha_{kk})^2,$$

(2.13) can be written as

$$0\ge f(v)\Big\{\frac{c}{4}(n+1)-2\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2-\sum_{\alpha,i}(h^\alpha_{ii})^2\Big\}-f(v)\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2-\sum_{k\neq 1}\Big(\sum_\alpha (h^\alpha_{kk})^2\Big)\Big(\sum_\beta (h^\beta_{1k})^2\Big). \tag{2.14}$$

Summing up for $k(\neq 1)$ in (2.10) and using the minimality, we have

$$2\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2\le nf(v),$$

from which it follows that

$$f(v)\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2\le f(v)\Big\{\frac{n}{2}a\sum_\alpha (h^\alpha_{11})^2+(1-a)\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2\Big\} \tag{2.15}$$

for any real number $0\le a\le \frac{2}{n}$.

On the other hand, by (2.8) and (2.10), one can easily see that

$$\sum_\alpha (h^\alpha_{1k})^2\le f(v),$$

which together with (2.8) yields

$$\sum_{k\neq 1}\Big(\sum_\alpha (h^\alpha_{kk})^2\Big)\Big(\sum_\beta (h^\beta_{1k})^2\Big)\le f(v)\Big\{\frac{n}{2}a\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{kk})^2+\Big(1-\frac{n}{2}a\Big)\sum_{\substack{\alpha\\ k\neq 1}}(h^\alpha_{1k})^2\Big\}. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14), we get

$$0 \geq f(v)\Big\{\frac{c}{4}(n+1) - \Big(4 - a - \frac{n}{2}a\Big)\sum_{\substack{\alpha \\ k\neq 1}}(h_{1k}^{\alpha})^2 - \Big(1 + \frac{n}{2}a\Big)\sum_{\alpha,i}(h_{ii}^{\alpha})^2\Big\}. \tag{2.17}$$

Taking $a = 4/(3n+2)$ in (2.17), we obtain immediately

$$0 \geq f(v)\Big\{\frac{c}{4}(n+1) - \frac{5n+2}{3n+2}\|\sigma\|^2\Big\} \geq 0 \tag{2.18}$$

according to the hypothesis of the theorem.

As the same as the proof of Theorem 1.1 in [5], it can be concluded from (2.18) that either M is totally geodesic or $n = \dim M = 2$. In the later case, the condition of the theorem becomes $\rho \geq 0$, i.e., the Gauss curvature of M is nonnegative. Thus, by the above Lemma 2.1, Theorem 2.1 is proved.

Remark 2.3. Obviously, our pinching constant is better than that of Theorem A. A result analogous to Theorem 2.1 for compact minimal submanifolds in a sphere has been obtained by the author[7]. As a direct consequence of Theorem 2.1, we can improve the result of [6] for totally real minimal submanifolds in $CP^n(c)$ as follows.

Corollary 2.1. *Let M be an n-dimensional compact totally real minimal submanifold in the complex projective space $CP^n(c)$. If $\rho \geq (n-2)(5n^2+4n+1)c/4(5n+2)$ or equivalently $\|\sigma\|^2 \leq (n+1)(3n+2)c/4(5n+2)$, where ρ and $\|\sigma\|^2$ denote the scalar curvature and the length square of the second fundamental form of M, respectively, then either M is totally geodesic or $n = 2$ and $M = S^1 \times S^1$.*

§3. Sectional Curvature and Ricci Curvature

We now shall use (1.10) to give some pinching conditions for totally real minimal submanifolds in $QP^n(c)$ in terms of the sectional curvature and the Ricci curvature. First of all, we prove the following

Theorem 3.1. *Let M be an n-dimensional compact totally real minimal submanifold in $QP^n(c)$. If the sectional curvature K_M of M satisfies*

$$K_M \geq (n-2)c/8n, \tag{3.1}$$

then either (i) *M is totally geodesic or* (ii) *$n = 2$ and M is a flat surface in $QP^2(c)$ with the parallel second fundamental form and a parallel normal subbundle of fiber dimension* 4.

Proof. Let K_M denote the function which assigns to each point of M the infimum of the sectional curvature of M at that point. By the minimality it is easy to see that (cf. [9])

$$\sum h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{il}^{\alpha}R_{lkjk}) \geq nK_M\|\sigma\|^2. \tag{3.2}$$

On the other hand, by virtue of Proposition 1 in [3] we have

$$\sum(\text{tr } H^{\alpha}H^{\beta})^2 \geq -\frac{1}{n}\sum \text{tr } (H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha})^2. \tag{3.3}$$

Introducing (3.2) and (3.3) into (1.10), we find

$$\begin{aligned}\frac{1}{2}\Delta(\|\sigma\|^2) \geq & \|\nabla\sigma\|^2 + (1+a)nK_M\|\sigma\|^2 + \Big(\frac{1-a}{2} - \frac{a}{n}\Big)\sum \text{tr } (H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha})^2 \\ & - \frac{c}{4}(na-1)\|\sigma\|^2 \end{aligned} \tag{3.4}$$

for $1 \geq a \geq 0$.

By taking $a = n/(n+2)$ in (3.4), we have

$$\frac{1}{2}\Delta(\|\sigma\|^2) \geq \|\nabla\sigma\|^2 + \frac{2n(n+1)}{n+2}\|\sigma\|^2\Big(K_M - \frac{n-2}{8n}c\Big). \tag{3.5}$$

Thus, (3.1) implies that the right hand side of (3.5) is nonnegative. Since M is compact, we have $\nabla\sigma = 0$, i.e., M has parallel second fundamental form. Moreover, we have either $\|\sigma\|^2 = 0$ or $K_M = (n-2)c/8n$ and all equalities in (3.2)–(3.5) hold. The latter case may occur only if $n = 2$. In fact, if $n \geq 3$, for the same reason as in the proof of Theorem 1 in [3], one easily see that M would have constant sectional curvature K_M. Since the second fundamental form of M is parallel, the constant K_M would be either $c/4$ or zero according to Theorem 10 of [1]. It contradicts the equality of (3.1) for $n \geq 3$.

Hence, either M is totally geodesic or M is a surface with nonnegative Gauss curvature. Combining with Lemma in §2, we complete the proof of Theorem 3.1.

Remark 3.1. The proof of Theorem 6 in [1] is incorrect because $\sum(\text{tr } A_\alpha^2)^2 \geq \|\sigma\|^4/3n$ for M in $QP^n(c)$. So, the pinching constant produced by using the method in [1] should be $(3n-4)c/4(6n-1)$ but not $(n-2)c/4(2n-1)$. Anyhow, our pinching constant $(n-2)c/8n$ is always better.

Now we consider the pinching problem for the Ricci curvature.

Theorem 3.2. *Let M be an n-dimensional compact totally real minimal submanifold in $QP^n(c)$ with $n \geq 4$. If the Ricci curvature of M satisfies*

$$\text{Ric }(M) \geq \Big(n - 2 - \frac{1}{n}\Big)c/4, \tag{3.6}$$

then either (i) *M is totally geodesic in $QP^n(c)$ or* (ii) *$n = 4$ and M is a locally symmetric Einstein space which is not of constant curvature.*

Proof. The formula (1.10) with $a = -1$ gives

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \|\nabla\sigma\|^2 + \sum_{\alpha,\beta}\text{tr }(H^\alpha H^\beta - H^\beta H^\alpha)^2 - \sum_{\alpha,\beta}\text{tr }(H^\alpha H^\beta)^2 + \frac{c}{4}(n+1)\|\sigma\|^2. \tag{3.7}$$

Let Q be the function which assigns to each point of M the infimum of the Ricci curvature of M at that point. From (1.9) we get

$$-\|\sigma\|^2 \geq n\Big[Q - \frac{c}{4}(n-1)\Big]. \tag{3.8}$$

Let α_i be the eigenvalues of the matrix H^α. For a fixed α, we see from (1.8) that

$$\begin{aligned}\sum_\beta \text{tr }(H^\alpha H^\beta - H^\beta H^\alpha)^2 &= -\sum_{\substack{i,j\\ \beta\neq\alpha}}(h_{ij}^\beta)^2(\alpha_i - \alpha_j)^2 \geq -4\sum_{\substack{i,j\\ \beta\neq\alpha}}(h_{ij}^\beta)^2(\alpha_i)^2\\ &\geq -4\sum_i\Big[\frac{c}{4}(n-1) - Q - (\alpha_i)^2\Big](\alpha_i)^2\\ &\geq [4Q - (n-1)c]\text{tr }(H^\alpha)^2 + \frac{4}{n}[\text{tr }(H^\alpha)^2]^2,\end{aligned} \tag{3.9}$$

from which it follows that

$$\sum_{\alpha,\beta}\text{tr }(H^\alpha H^\beta - H^\beta H^\alpha) \geq [4Q - (n-1)c]\|\sigma\|^2 + \frac{4}{n}\sum_\alpha[\text{tr }(H^\alpha)^2]^2. \tag{3.10}$$

For a suitable choice of $\{e_\alpha\}$, we may assume that $\text{tr }(H^\alpha H^\beta) = 0$ for $\alpha \neq \beta$. Then, introducing (3.10) into (3.7) and using (3.8) and the fact that

$$\sum_\alpha [\mathrm{tr}\,(H^\alpha)^2]^2 \le \|\sigma\|^4, \tag{3.11}$$

we see easily that

$$\begin{aligned}\frac{1}{2}\Delta(\|\sigma\|^2) &\ge \|\nabla\sigma\|^2 - \frac{n-4}{n}\sum_\alpha [\mathrm{tr}\,(H^\alpha)^2]^2 + \Big[4Q - \frac{c}{4}(3n-5)\Big]\|\sigma\|^2 \\ &\ge \|\nabla\sigma\|^2 + n\|\sigma\|^2\Big[Q - \frac{c}{4}\Big(n^2 - 2 - \frac{1}{n}\Big)\Big].\end{aligned} \tag{3.12}$$

Thus, if $n \ge 5$, (3.12) with (3.6) yields the equality (3.11), which implies that at most one of matrices $\{H^\alpha\}$ is nonzero. Furthermore, by means of (1.3) and the minimality we conclude that all of $\{H^\alpha\}$ are zero, i.e., M is totally geodesic in $QP^n(c)$.

If $n = 4$, (3.12) with (3.6) yields that either $\|\sigma\|^2 = 0$ or $\|\nabla\sigma\|^2 = 0$ and equalities (3.8)–(3.10) hold. In latter case, M is an Einstein space with parallel second fundamental form. By (3.6) and Theorem 10 of [1], M is not of constant curvature except that M is totally geodesic. Since $\nabla\sigma = 0$, M is locally symmetric. Hence the theorem is proved completely.

In the next section, we shall use a different way to prove a similar theorem for $n = 3$.

§4. 3-dimensional Submanifolds

In this section, we consider the case that $n = 3$. Firstly, we show that Theorem 3.2 is valid for $n = 3$. Precisely, we prove the following

Theorem 4.1. *Let M be a 3-dimensional compact totally real minimal submanifold in $QP^3(c)$. If the Ricci curvature of M is larger than $c/6$, then M is totally geodesic in $QP^3(c)$.*

Proof. We return to the formula (2.13) and restrict ourselves to the point $x_0 \in M$ where the function f defined by (2.6) attains its maximum. By (2.7) and (2.8) it is easy to see that

$$(b_{kk})^2 \le \Big(\sum_\alpha (h_{11}^\alpha)^2\Big)\Big(\sum_\alpha (h_{kk}^\alpha)^2\Big) \le (b_{11})^2, \tag{4.1}$$

from which and the minimality of dimension 3 it follows that

$$b_{22} \le 0, \qquad b_{33} \le 0, \qquad \sum_{k\ne1} (b_{kk})^2 \le \Big(\sum_{k\ne1} b_{kk}\Big)^2 = (b_{11})^2. \tag{4.2}$$

By virtue of (2.7) and (1.8) we have

$$-\sum_{\substack{\alpha\\ k\ne1}} (h_{1k}^\alpha)^2 = R_{11} - \frac{c}{2} + b_{11}. \tag{4.3}$$

Substituting (4.3) into (2.13) and using (4.2), we find

$$0 \ge 2f(v)R_{11} + 2\sum_{\substack{\alpha\\ k\ne1}} b_{kk}(h_{1k}^\alpha)^2. \tag{4.4}$$

By (4.1), (4.2) and (2.10) we have respectively

$$\sum_{\substack{\alpha\\ k\ne1}} b_{kk}(h_{1k}^\alpha)^2 \ge \frac{1}{2}\sum_{k\ne1} b_{kk}(b_{11} - b_{kk}) = -\frac{1}{2}\sum_i (b_{ii})^2 \tag{4.5}$$

and

$$\sum_{\substack{\alpha \\ k\neq 1}} b_{kk}(h_{1k}^{\alpha})^2 \geq -b_{11}\sum_{\substack{\alpha \\ k\neq 1}}(h_{1k}^{\alpha})^2 = f(v)(R_{11} - \frac{c}{2} + b_{11}). \tag{4.6}$$

Introducing these inequalities into (4.4) respectively, we can obtain

$$0 \geq 2f(v)R_{11} - \frac{1}{2}\sum_i (b_{ii})^2 + f(v)(R_{11} - \frac{c}{2} + b_{11}) = 3f(v)(R_{11} - \frac{c}{6}). \tag{4.7}$$

Thus, if Ric $(M) > c/6$, then (4.7) implies that $f = 0$, i.e., M is totally geodesic.

Remark 4.1. As a direct consequence, a 3-dimensional compact totally real minimal submanifold in $CP^3(1)$ with Ric $(M) > 1/6$ must be totally geodesic in $CP^3(1)$. This improves the pinching constant $3/16$ in Theorem 1 of [4] for $n = 3$.

Now, in the similar manner, we may improve slightly Theorem 2.1 for $n = 3$ as follows.

Theorem 4.2. *Let M be a 3-dimensional compact totally real minimal submanifold in $QP^3(c)$. If $\rho > 5c/6$ or equivalently $\|\sigma\|^2 < 2c/3$, then M is totally geodesic in $QP^3(c)$.*

Proof. As the same as the proof of Theorem 4.1 above, it follows from (2.13), (4.1), (4.5) and (4.6) that

$$\begin{aligned} 0 &\geq cf(v) + \Big[\sum_{\substack{\alpha \\ k\neq 1}} b_{kk}(h_{1k}^{\alpha})^2 - 2f(v)\sum_{\substack{\alpha \\ k\neq 1}}(h_{1k}^{\alpha})^2\Big] + \Big[\sum_{\substack{\alpha \\ k\neq 1}} b_{kk}(h_{1k}^{\alpha})^2 - \sum_{k\neq 1}(b_{kk})^2 - (b_{11})^2\Big] \\ &\geq cf(v) - 3f(v)\sum_{\substack{\alpha \\ k\neq 1}}(h_{1k}^{\alpha})^2 - \frac{3}{2}f(v)b_{11} - \frac{3}{2}\sum_{k\neq 1}(b_{kk})^2 \\ &\geq \frac{3}{2}f(v)\Big[\frac{2}{3}c - \sum_{\alpha,i}(h_{ii}^{\alpha})^2 - 2\sum_{\substack{\alpha \\ k\neq 1}}(h_{1k}^{\alpha})^2\Big] \geq \frac{3}{2}f(v)\Big[\frac{2}{3}c - \|\sigma\|^2(x_0)\Big]. \end{aligned} \tag{4.8}$$

Thus, under the hypothesis of the theorem, (4.8) implies that $f = 0$ identically, i.e., M is totally geodesic. The theorem is proved.

Remark 4.2. Some results analogous to Theorems 4.1 and 4.2 for compact minimal submanifolds in a sphere have been obtained (cf. [8]).

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