Chin. Ann. of Math.
14B: 3(1993), 307-318.

ON THE LIMITING BEHAVIORS OF INCREMENTS OF SUMS OF RANDOM VARIABLES WITHOUT MOMENT CONDITIONS**

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Abstract

The a.s. limiting behaviors of big increments of partial sums of a sequence of random variables are obtained without moment conditions.The theorems sharpen the results of Lin[5].

Keywords Random variables, Moment conditions, Limiting behaviors
1991 MR Subject Classifications 60F15

§1. Introduction and Results

The a.s. limiting behaviors of increments of partial sums of a sequence of random variables are profound and elegant results in the probability theory. Csörgő and Révész[1] have obtained many fine results of this kind for i.i.d. random variables. Lin[3,4] and Shao[6] generalized their results to more general cases. But all require existence of either moment generating functions or $2+\delta$-order moments ($\delta > 0$). Recently, Lin[5] first considered increment problem without moment hypotheses according to an idea that strong limit theorems depend (in principle) on probabilities rather than moments. Lin's theorem imply the existing results with moment conditions; however, more complicated conditions are imposed. The purpose of the present paper is to give a group of simpler conditions, with the results that we sharpen Lin's theorems.

Let $\{X_n, n \geq 1\}$ be a sequence of independent but not necessarily identical distributed random variables and $\{a_n, n \geq 1\}$ a non-decreasing sequence of positive integers. Denote $S_n = \sum_{i=1}^{n} X_i$. Furthermore, let $\{B_{nN}, n = 0, 1, \cdots, N; N = 1, 2, \cdots\}$ be a double sequence of positive numbers, which is non-decreasing on N for fixed n and tends to infinite as $N \to \infty$ uniformly in n. Denote

$$B_N = B_{0N},$$
$$b_N^2 = 2\{\log(B_{N+a_N}^2/B_{a_N}^2) + \log\log B_{a_N}^2\}.$$

For every N, $n + a_n \leq N < (n+1) + a_{N+1}$, define

$$B_N' = B_{n,n+a_n} \ (\text{so } B_{N+a_N}' = B_{N,N+a_N}),$$
$$b_N'^2 = 2\{\log(B_{n+a_n}^2/B_N'^2) + \log\log B_N'^2\}.$$

Manuscript received January 12, 1991.
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** Project supported by the National Natural Science Foundation of China,the Zhejiang Provincial Science Foundation and the Fok Yingting Education Foundation.

And define for $\varepsilon > 0$

$$X_{j\varepsilon} = (X_j \vee (-\varepsilon B_j' b_j'^{-1})) \wedge (\varepsilon B_j' b_j'^{-1}),$$

$$T_N(\varepsilon) = B_{N+a_N}^{-2} \sum_{j=1}^{N+a_N} \operatorname{Var}(X_{j\varepsilon}), \quad T_{nN}(\varepsilon) = B_{n,n+a_N}^{-2} \sum_{j=n+1}^{n+a_N} \operatorname{Var}(X_{j\varepsilon}),$$

$$T_-^2 = \lim_{\varepsilon\downarrow 0} \liminf_{N\to\infty} T_N(\varepsilon) \wedge T_{N,N}(\varepsilon),$$

$$T_+^2 = \lim_{\varepsilon\downarrow 0} \limsup_{N\to\infty} \max_{0\le n\le N} T_{nN}(\varepsilon).$$

Assume $T_+ < \infty$.

Theorem 1.1. *Let $\{X_n\}$ be an independent sequence. Suppose that there exist sequences $\{a_n\}$ and $\{B_{nN}\}$ of above-mentioned kinds satisfying, for any $\varepsilon > 0$,*

(i) $\sum_{n=1}^{\infty} P\{|X_n| \ge \varepsilon B_n' b_n'\} < \infty$;

(ii) $\lim_{N\to\infty} \max_{0\le n\le N} \max_{1\le k\le a_N} (B_{a_N} b_N)^{-1} \Big| \sum_{j=n+1}^{n+k} E\{X_j I(|X_j| \le \varepsilon B_j' b_j')\}\Big| = 0$;

(iii) $\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \max_{0\le n\le N} \sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/(B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2) \log B_{a_N}^2)^{-\beta}) < \infty$ *for some* $\beta > 0$;

(iv) $\overline{\lim_{N\to\infty}} (\max_{0\le n\le N} B_{n,n+a_N})/(\min_{0\le n\le N} B_{n,n+a_N}) < \infty$;

(v) $B_{N+a_N} \le AB_{N-1+a_{N-1}}$ *and* $B_{a_N} \le AB_{a_{N-1}}$ *for some* $A > 0$ *and every* $N \ge 2$.

Then

$$T_- \le \limsup_{N\to\infty} \frac{|S_{N+a_N} - S_N|}{B_{N,N+a_N} b_N} \le \limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \frac{|S_{n+k} - S_n|}{B_{n,n+a_N} b_N} \le T_+ \quad a.s.$$

Remark 1.1. This theorem greatly simplifies the conditions of Lin's Theorem 1 in [5]. Hence, of course, our theorem also implies the existing results about increments of partial sums of a sequence with moment conditions, e.g., conditions relative to a non-negative and non-decreasing continuous function $H(x)$, the condition "$EX_n^2 \log^p(1+|x_n|) < \infty$ for some $p > 0$" and the condition "$E\exp\{\alpha \log^r(1+|X_n|)\} < \infty$ for some $\alpha > 0$ and $r > 1$" (see [5] Remarks 1 and 2, also see [3], [4] and [6]). Furthermore, it is interesting to say that the condition "$(l_2)'$ $\lim_{n\to\infty} a_{n+1}/a_n = 1$" imposed in [5] can be relaxed, according to our theorem, as condition

$$(l_2)'' \qquad a_N \le Aa_{N-1} \text{ for some } A > 0 \text{ and every } N \ge 2.$$

In order that a theorem without moment hypotheses implies that for the case when there exist r-order moment generating functions ($0 < r \le 1$), we rewrite Theorem 1.1 as follows (cf. [5] Remark 3):

Theorem 1.2. *Suppose that the following conditions are satisfied:*

(i)' $\sum_{n=1}^{\infty} P\{|X_n| \ge a\log^\alpha n\} < \infty$ *for some* $a > 0$ *and* $\alpha \ge 1$;

(ii)' $\lim_{N\to\infty} \max_{0\le n\le N} \max_{1\le k\le a_N} (B_{a_N} b_N)^{-1} \Big| \sum_{j=n+1}^{n+k} E\{X_j I(|X_j| \le a\log^\alpha j)\}\Big| = 0$;

(iii)' *there exists a* $t_0 > 0$ *such that*

$$E\exp(t|X_n|^{1/\alpha}) I(\tau \log^{\alpha-1} n < |X_n| \le a\log^\alpha n) \le M < \infty$$

for every n, any $|t| \le t_0$ *and some* $\tau > 0$ *as* $\alpha > 1$ *and* $\tau = 0$ *as* $\alpha = 1$;

(iv)' $a_N \le N$ *and* $a_N/(\log N)^{2\alpha-1} \to \infty$;

(v)' $a_N \le Aa_{N-1}$ *for some* $A > 0$ *and every* $N \ge 2$.

Define

$$X_j = \begin{cases} (X_j \vee (-\varepsilon \log^{\alpha-1} j)) \wedge (\varepsilon \log^{\alpha-1} j) & as\ \alpha > 1, \\ X_j I\Big(|X_j| < \dfrac{1}{\varepsilon}\Big) & as\ \alpha = 1. \end{cases}$$

Let T_+ *and* T_- *be the same as in Theorem* 1.1. *Then the conclusion in Theorem* 1.1 *remains true.*

§2. Proof of Theorems.

In order to prove our theorems, the following lemmas are employed.

Lemma 2.1[5]. *Let* $\{g_n(\varepsilon)\}$ *be a sequence of non-negative functions, defined for all* $\varepsilon > 0$. *Define* $g_* = \lim_{\varepsilon\downarrow 0} \varliminf_{n\to\infty} g_n(\varepsilon)$ *and* $g^* = \varlimsup_{\varepsilon\downarrow 0} \varlimsup_{n\to\infty} g_n(\varepsilon)$. *Then there exist sequences* $\{\varepsilon_n\}$ *and* $\{\varepsilon'_n\}$ *such that* $\varepsilon_n \downarrow 0$, $\varepsilon'_n \downarrow 0$ *and* $\varliminf_{n\to\infty} g_n(\varepsilon_n) \ge g_*$, $\varlimsup_{n\to\infty} g_n(\varepsilon'_n) \le g^*$. *Moreover, if* $g_n(\varepsilon)$ *is a non-decreasing function of* ε *for every* $n \ge 1$, *then* $\varliminf_{n\to\infty} g_n(\varepsilon_n) = g_*$.

The following conclusion is well-known.

Lemma 2.2. *Let* $\{X_n\}$ *be a sequence of independent random variables with means zero. Suppose that there exists some* $d > 0$ *such that*

$$|X_k| \le ds_n \qquad a.s.$$

for $1 \le k \le n$, *where* $s_n^2 = \sum_{k=1}^{n} EX_k^2$. *If* $\gamma > 0$, *then there exist constants* $\varepsilon(\gamma)$ *and* $\pi(\gamma)$ *such that, when* $\varepsilon \ge \varepsilon(\gamma)$ *and* $\varepsilon d \le \pi(\gamma)$, *we have*

$$P\{S_n \ge \varepsilon s_n\} \ge \exp\{-(1+\gamma)\varepsilon^2/2\}. \tag{2.1}$$

Lemma 2.3. *Let* $\{\xi_n, n \ge 1\}$ *be a sequence of independent random variables with* $E\xi_n = 0$. *And let* $\{a_n, n \ge 1\}$ *be a sequence of positive integers tending to infinity. Suppose that there exists a double sequence* $\{\sigma_{nN}, n = 0, 1, \cdots, N; N = 1, 2, \cdots\}$ *of positive numbers, which is non-decreasing on* N *for fixed* n *and tends to infinity as* $N \to \infty$ *uniformly in* n. *Put*

$$\sigma_N^2 = \min_{0\le n\le N} \sigma_{na_N}^2,$$

$$\beta_{nN} = \{2\sigma_{na_N}^2 \log(\sigma_{0,N+a_N}^2/\sigma_{nN}^2) + \log\log \sigma_{na_N}^2\}^{1/2}.$$

If

(a) $\sigma_{0a_N} \le A\sigma_{0a_{N-1}}$ *for some* $A > 0$;

(b) $\varlimsup_{N\to\infty} \max_{1\le n\le N} \sum_{i=n+1}^{n+a_N} E\xi_i^2/\sigma_{na_N}^2 \le 1$;

(c) *there exists an* $\varepsilon > 0$ *such that*

$$|\xi_N| \le \varepsilon\{\sigma_N^2/(\log(\sigma_{0,N+a_N}^2/\sigma_N^2) + \log\log\sigma_N^2)\}^{1/2}, \qquad a.s.$$

then there is a $c(\varepsilon)$ *with* $c(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *such that*

$$\varlimsup_{N\to\infty} \max_{0\le n\le N} \max_{1\le k\le a_N} \Big|\sum_{i=n+1}^{n+k} \xi_i\Big| / \beta_{nN} \le 1 + c(\varepsilon) \qquad a.s.$$

The proof of this Lemma can be found in [7].

Proof of Theorem 1.1. Without loss of generality, we assume $T_- > 0$. The proof of the theorem is based on the following facts.

Fact 1.

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{N+a_N}^2 = 0.$$

Proof. Take such an a_n for N that

$$a_{n-1} < N + a_N \le a_n. \tag{2.2}$$

Then, by condition (v),

$$\begin{aligned}&\sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{N+a_N}^2\\ \le&\sum_{j=1}^{a_n} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{a_{n-1}}^2\\ \le& A^2\sum_{j=1}^{a_n} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{a_n}^2.\end{aligned}$$

Hence

$$\begin{aligned}&\overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{N+a_N}^2\\ \le& A^2\ \overline{\lim_{N\to\infty}} \max_{0\le n\le N} \sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{a_N}^2.\end{aligned}$$

And further, by condition (iii),

$$\begin{aligned}&\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')/B_{N+a_N}^2\\ \le& A^2\ \overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \max_{0\le n\le N} \{\sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \le \varepsilon B_j' b_j')\\ &\times [(B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2]^{-\beta}\}/\{B_{a_N}^2[(B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2]^{-\beta}\} = 0.\end{aligned}$$

Fact 2.

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} \mathrm{Var}\,(X_{j\varepsilon})/B_{N+a_N}^2 \le HA^2T_+^2 \text{ for some } H > 0.$$

Proof. Define n to be the same as in Fact 1. Then

$$\begin{aligned}\sum_{j=1}^{N+a_N} \mathrm{Var}\,(X_{j\varepsilon})/B_{N+a_N}^2 &\le \sum_{j=1}^{a_n} \mathrm{Var}\,(X_{j\varepsilon})/B_{a_{n-1}}^2\\ &\le A^2\sum_{j=1}^{a_n} \mathrm{Var}\,(X_{j\varepsilon})/B_{a_n}^2.\end{aligned}$$

Hence, by condition (iv) there exists an $H > 0$ such that

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} \operatorname{Var}(X_{j\varepsilon})/B_{N+a_N}^2$$
$$\leq HA^2 \overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \max_{0\leq n\leq N} T_{nN}(\varepsilon) \leq HA^2T_+^2.$$

Fact 3. There exists a subsequence $\{N_j\}$ of integers such that for $j \geq 1$

$$B_{a_{N_{j+1}}} b_{N_{j+1}} / B_{a_{N_j}} b_{N_j} \leq 2AC \tag{2.3}$$

for some $C > 0$, and for any $\varepsilon > 0$

$$\sum_{j=1}^{\infty} (B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon} / \log^{1+\varepsilon} B_{a_{N_j}} < \infty. \tag{2.4}$$

Proof. Define N_j by

$$N_j = \min\{n : B_{a_n} b_n \geq 2^j\},$$

which implies that

$$B_{a_{N_j}} b_{N_j} \geq 2^j, \quad B_{a_{N_j-1}} b_{N_j-1} < 2^j.$$

From condition (v) we can find a constant $C > 0$ such that for every $N \geq 2$

$$b_N \leq C b_{N-1}.$$

Therefore

$$B_{a_{N_j}} b_{N_j} \leq ACB_{a_{N_j-1}} b_{N_j-1}$$

and

$$AC2^j \geq B_{a_{N_j}} b_{N_j} \geq 2^j.$$

The latter implies (3), and further we have either

$$B_{a_{N_j}} \geq 2^{j/2} \tag{2.5}$$

or

$$B_{a_{N_j}} < 2^{j/2}, \qquad b_{N_j} \geq 2^{j/2}. \tag{2.6}$$

If (2.5) is true, then

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon} / \log^{1+\varepsilon} B_{a_{N_j}}$$
$$\leq (\log B_{a_{N_j}})^{-(1+\varepsilon)} \leq (\frac{1}{2} j \log 2)^{-(1+\varepsilon)}.$$

If (2.6) is true, then

$$\log(B_{N_j+a_{N_j}}^2 / B_{a_{N_j}}^2) + \log\log B_{a_{N_j}}^2 \geq 2^{j-1}$$

and

$$\log(B_{N_j+a_{N_j}}^2 / B_{a_{N_j}}^2) \geq 2^{j-1} - \log\log 2^j \geq 2^{j-2}.$$

Hence, for all large j,

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon} \leq e^{-\frac{\varepsilon}{2} 2^{j-2}} \leq j^{-(1+\varepsilon)}.$$

In any way, we have for all large j

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon} / \log^{1+\varepsilon} B_{a_{N_j}} \leq j^{-(1+\varepsilon)},$$

which implies (2.4).

Fact 4. For some $\delta > 0$ and $C > 0$

$$\overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} \text{Var}\,(X_{j\varepsilon}) \Big/ \sum_{j=1}^{N-1+a_{N-1}} \text{Var}\,(X_{j\varepsilon}) \le C$$

uniformly in $0 < \varepsilon < \delta$.

Proof. By condition (v) and Fact 2,

$$\sum_{j=1}^{N+a_N} \text{Var}\,(X_{j\varepsilon}) \Big/ \sum_{j=1}^{N-1+a_{N-1}} \text{Var}\,(X_{j\varepsilon})$$

$$=(B^2_{N+a_N}/B^2_{N-1+a_{N-1}})\Big(\sum_{j=1}^{N+a_N} \text{Var}\,(X_{j\varepsilon})/B^2_{N+a_N}\Big) \Big/ \Big(\sum_{j=1}^{N-1+a_{N-1}} \text{Var}\,(X_{j\varepsilon})/B^2_{N-1+a_{N-1}}\Big)$$

$$\le A^2 \cdot 2HA^2T_+^2/\frac{1}{2}T_-^2 = 4HA^4T_+^2/T_-^2$$

provided that N is large enough and ε is small enough.

Using these facts, we proceed to prove our conclusion.

For a given $\delta > 0$, let $\varepsilon = \varepsilon(\delta)$ be indicated later. Define $c_n = \varepsilon B'_n {b'_n}^{-1}$, $d_n = \varepsilon B'_n b'_n$ and

$$\begin{aligned} X'_n &= X_{n\varepsilon}, \\ Y_n &= (X_n - c_n \text{sign}\, X_n) I(c_n < |X_n| \le d_n), \\ Z_n &= X'_n + Y_n, \\ S'_n &= \sum_{k=1}^{n} (X'_k - EX'_k), \\ U_n &= \sum_{k=1}^{n} (Y_k - EY_k), \\ V_n &= \sum_{k=1}^{n} (Z_k - EZ_k). \end{aligned}$$

Then

$$Z_n = X_n I(|X_n| \le d_n) + c_n \text{sign}\, X_n I(|X_n| > d_n),$$

$$|X_n - Z_n| \le |X_n| I(|X_n| > d_n).$$

So, by dint of condition (i),

$$P\{X_n \ne Z_n,\ \text{i.o.}\} = 0. \tag{2.7}$$

Hence we may only consider Z_n instead of X_n. From conditions (i) and (ii) and the definition of c_n, we have

$$\lim_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \frac{1}{B_{n,n+a_N} b_N} \Big| \sum_{j=n+1}^{n+k} EZ_j \Big| = 0. \tag{2.8}$$

Combining it with (2.7) implies that the conclusion of the theorem is equivalent to

$$T_- \le \limsup_{N\to\infty} \frac{|V_{N+a_N} - V_N|}{B_{N,N+a_N} b_N} \le \limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \frac{|V_{n+k} - V_n|}{B_{n,n+a_N} b_N} \le T_+ \quad \text{a.s.} \tag{2.9}$$

As the first step, we prove

$$\limsup_{N\to\infty}\max_{1\le n\le N}\max_{1\le k\le a_N}\frac{|U_{n+k}-U_n|}{B_{n,n+a_N}b_N}\le\delta\quad\text{a.s.}\tag{2.10}$$

Let $r=r(\delta)$ be a positive integer indicated later on. Put $Y_0=0$. Define a function of n, N and r as follows:

$$\begin{aligned}n_r=\max\Big\{k:&\sum_{j=1}^{k}\text{Var }(Y_j)\le\frac{i}{r}B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta},\text{where } i \text{satisfies}\\&\frac{i}{r}B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}\le\sum_{j=1}^{n}\text{Var }(Y_j)\\&<\frac{i+1}{r}B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}\Big\}.\end{aligned}$$

Put $U_0=0$. Write

$$|U_{n+k}-U_n|\le|U_{n+k}-U_{(n+k)_r}|+|U_{(n+k)_r}-U_{n_r}|+|U_n-U_{n_r}|.\tag{2.11}$$

Consider the first term of the right hand side. We have

$$\begin{aligned}&P\Big\{\max_{1\le n\le N}\max_{1\le k\le a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\ge\frac{\delta}{2ACH}\Big\}\\&\le\{r\sum_{j=1}^{N+a_N}\text{Var }(Y_j)/B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}+1\}\\&\quad\times\max_{1\le n\le N+a_N}P\{|U_n-U_{n_r}|\ge\frac{\delta}{2ACH^2}B_{a_N}b_N\}.\end{aligned}\tag{2.12}$$

From Fact 1, there exists a constant $M>0$ such that

$$\begin{aligned}\sum_{j=1}^{N+a_N}\text{Var }(Y_j)\le&2\sum_{j=1}^{N+a_N}\{EX_j^2I(c_j<|X_j|\le d_j)\\&+c_j^2P(c_j<|X_j|\le d_j)\}\le 2MB_{N+a_N}^2.\end{aligned}\tag{2.13}$$

Estimate the probability in the right hand side of (2.12). Let $\varepsilon=\dfrac{\beta\delta}{16ACH^3}$, $t=\dfrac{\beta b_N}{4\varepsilon H}=\dfrac{4ACH^2}{\delta}b_N$. Then, for $j\le N+a_N$, we have $d_j\le\varepsilon HB_{a_N}b_N$ and

$$\begin{aligned}&E\exp\Big\{\frac{t}{B_{a_N}}(Y_j-EY_j)\Big\}\\&\le1+\frac{t^2}{2B_{a_N}^2}\text{Var }(Y_j)\Big\{1+\frac{1}{3}\Big(\frac{2td_j}{B_{a_N}}\Big)+\frac{1}{12}\Big(\frac{2td_j}{B_{a_N}}\Big)^2+\cdots\Big\}\\&\le1+\frac{t^2}{2B_{a_N}^2}\text{Var }(Y_j)\exp\Big\{\frac{\beta}{4}b_N^2\Big\}\\&\le\exp\Big\{\frac{8A^2C^2H^4}{\varepsilon^2B_{a_N}^2}b_N^2\Big(\frac{B_{N+a_N}^2}{B_{a_N}^2}\log B_{a_N}^2\Big)^{\beta/2}\text{Var }(Y_j)\Big\}.\end{aligned}\tag{2.14}$$

Put $l_n = \max\{m : m_r = n_r\}$. Using Lévy's maximum inequality, we obtain

$$\begin{aligned}
&P\{\max_{m_r=n_r} |U_m - U_{n_r}| \ge \frac{\delta}{2ACH^2} B_{a_N} b_N\} \\
\le& 2P\{|U_{l_n} - U_{n_r}| \ge \frac{\delta}{2ACH^2} B_{a_N} b_N - 2\sqrt{\operatorname{Var}(U_{l_N} - U_{n_r})}\} \\
\le& 2P\Big\{|U_{l_n} - U_{n_r}| \ge \frac{\delta B_{a_N} b_N}{3ACH^2}\Big\} \\
\le& 2\exp\Big(-\frac{\delta t b_n}{3ACH^2}\Big) \prod_{j=n_r+1}^{l_n} E\exp\Big\{\frac{t}{B_{a_N}}(Y_j - EY_j)\Big\} \\
\le& 2\exp\Big\{-\frac{4}{3} b_N^2 + o(b_N^2)\Big\} \\
\le& ((B_{N+a_N}^2 / B_{a_N}^2) \log B_{a_N}^2)^{-2}
\end{aligned} \tag{2.15}$$

for every large N, where the definition of n_r and condition (iv) are employed for the last but two inequality. Combining (2.12), (2.13) with (2.15) yields

$$\begin{aligned}
&P\Big\{\max_{1\le n\le N} \max_{1\le k\le a_N} \frac{1}{B_{n,n+a_N} b_N} |U_{n+k} - U_{(n+k)_r}| \ge \frac{\delta}{2ACH}\Big\} \\
\le& 5rM(B_{a_N}^2 / B_{N+a_N}^2)^{1-\beta} / (\log B_{a_N}^2)^{2-\beta}.
\end{aligned} \tag{2.16}$$

By (2.4), we have

$$P\Big\{\max_{1\le n\le N_j} \max_{1\le k\le a_{N_j}} \frac{1}{B_{n,n+a_{N_j}} b_{N_j}} |U_{n+k} - U_{(n+k)_r}| \ge \frac{\delta}{2ACH}\Big\} < \infty,$$

which implies that

$$\limsup_{j\to\infty} \max_{1\le n\le N_j} \max_{1\le k\le a_{N_j}} \frac{1}{B_{n,n+a_{N_j}} b_{N_j}} |U_{n+k} - U_{(n+k)_r}| \ge \frac{\delta}{2ACH} \quad \text{a.s.} \tag{2.17}$$

Furthermore, noting that the ranges in two max's in (2.17) enlarge as j increases and using (2.3), (2.17) implies that

$$\limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \frac{1}{B_{n,n+a_N} b_N} |U_{n+k} - U_{(n+k)_r}| \le \delta \quad \text{a.s.} \tag{2.18}$$

The second and the third terms of the right hand side of (2.11) can be treated by the same procedure except that condition (iii) is applied for the second term, and we have the similar inequalities. (2.10) is proved. Thus inequality (2.9) is equivalent to

$$\begin{aligned}
(1-2\delta)T_- \le& \limsup_{N\to\infty} \frac{|S'_{N+a_N} - S'_N|}{B_{N,N+a_N} b_N} \\
\le& \limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \frac{|S'_{n+k} - S'_n|}{B_{n,n+a_N} b_N} \le (1+3\delta)T_+ \quad \text{a.s.}
\end{aligned} \tag{2.19}$$

for any $0 < \delta < 1/2$.

It is easy to verify that the condition of Lemma 2.3 is satisfied for $\{X'_n\}$. Consequently we have the right inequality of (2.19) from the condition of T_+.

Next, we prove the left of (19). Let $v_{n,n+a_N}^2 = \sum_{j=n+1}^{n+a_N} \operatorname{Var} X'_j$, $v_{a_N}^2 = \max_{1\le n\le N} v_{n,n+a_N}^2$. For $j \le N + a_N$,

$$\begin{aligned}
|X'_j - EX'_j| &\le 2c_j \le 2c_{N+a_N} \\
&= 2\varepsilon {b'}^{-1}_{N+a_N} (B_{N,N+a_N} / v_{N,N+a_N}) v_{N,N+a_N}.
\end{aligned}$$

Noting $0 < T_- < \infty$, we have $B_{N,N+a_N} b_N / v_{N,N+a_N} \to \infty$ as $N \to \infty$ and $(\varepsilon b'^{-1}_{N+a_N}(B_{N,N+a_N} v^{-1}_{N,N+a_N})(B_{N,N+a_N} b_N v^{-1}_{N,N+a_N}) \le Q\varepsilon$ for some constant $Q > 0$ and every $N \ge 1$. Hence, if let $\varepsilon = \varepsilon(\delta)$ be small enough, we use Lemma 2.2 and obtain

$$\begin{aligned} &P\Big\{\frac{1}{B_{N,N+a_N} b_N}|S'_{N+a_N} - S'_N| \ge (1-\delta)T_-\Big\} \\ &\ge \exp\Big\{-\frac{(1+\delta)(1-\delta)^2 T_-^2 B^2_{N,N+a_N} b_N^2}{2v^2_{N,N+a_N}}\Big\} \\ &\ge \Big(\frac{B^2_{a_N}}{B^2_{N+a_N}\log B^2_{a_N}}\Big)^{1-\delta} \ge \frac{B^2_{a_N}}{B^2_{N+a_N}\log B^2_{N+a_N}}. \end{aligned} \tag{2.20}$$

According to the definition of T_- and T_+, we can choose an ε such that the right hand side of (2.20) is larger than

$$R v^2_{N,N+a_N}/(v^2_{0,N+a_N}\log v^2_{0,N+a_N}) \tag{2.21}$$

for some $R > 0$. Let $0 < \eta < \delta \wedge (\delta^2 T_-^2 / 4T_+^2)$, $N_1 = 1$. Define N_{k+1} by

$$N_{k+1} = \min\{n : v^2_{0,n} + \eta v^2_{n,n+a_n} \ge v^2_{0,N_k+a_{N_k}}\}. \tag{2.22}$$

Then, we have

$$v^2_{0,N_{k+1}} + \eta v^2_{N_{k+1},N_{k+1}+a_{N_{k+1}}} \ge v^2_{0,N_k+a_{N_k}} \quad \text{for every } k, \tag{2.23}$$

$$v^2_{0,n} + \eta v^2_{n,n+a_n} < v^2_{0,N_k+a_{N_k}} \quad \text{for every } n < N_{k+1}. \tag{2.24}$$

Hence we find that $N_{k+1} > N_k$ and $N_{k+1} + a_{N_{k+1}} > N_k + a_{N_k}$ for every $k \ge 1$. At first, we prove that

$$\sum_{k=1}^{\infty} v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}}) = \infty. \tag{2.25}$$

In terms of (2.24), we get

$$\begin{aligned} &v^2_{0,N_{k-1}+a_{N_{k-1}}} \ge v^2_{0,N_k-1} = v^2_{0,N_k} - \mathrm{Var}\,(X'_{N_k}) \\ &\ge v^2_{0,N_k} - \varepsilon^2 B'^2_{N_k} b'^{-2}_{N_k} \ge v^2_{0,N_k} - H\varepsilon^2 B^2_{N_k,N_k+a_{N_k}} b'^{-2}_{N_k} \\ &\ge v^2_{0,N_k} - v^2_{N_k,N_k+a_{N_k}} = v^2_{0,N_k+a_{N_k}} - 2v^2_{N_k,N_k+a_{N_k}}, \end{aligned} \tag{2.26}$$

when k is large enough. And

$$v^2_{0,N_{k-1}+a_{N_{k-1}}} \ge \eta v^2_{0,N_k-1+a_{N_k-1}} \ge \frac{\eta}{C} v^2_{0,N_k+a_{N_k}} \tag{2.27}$$

by (2.24) and Fact 4. Now using (2.26) and (2.27), we have

$$\begin{aligned} &\sum_{k=2}^{\infty} v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}}) \\ &\ge \frac{1}{2}\sum_{k=2}^{\infty}(v^2_{0,N_k+a_{N_k}} - v^2_{0,N_{k-1}+a_{N_{k-1}}})/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}}) \\ &\ge \frac{\eta}{2C}\sum_{k=2}^{\infty}(v^2_{0,N_k+a_{N_k}} - v^2_{0,N_{k-1}+a_{N_{k-1}}})/(v^2_{0,N_{k-1}+a_{N_{k-1}}}\log(\frac{C}{\eta}v^2_{0,N_{k-1}+a_{N_{k-1}}})) \\ &\ge \frac{\eta}{4C}\sum_{k=2}^{\infty}\int_{v^2_{0,N_{k-1}+a_{N_{k-1}}}}^{v^2_{0,N_k+a_{N_k}}} \frac{1}{x\log x}dx = \infty, \end{aligned}$$

which proves that (2.25) holds true.

Put

$$G = \{k : N_k \geq N_{k-1} + a_{N_{k-1}}\},$$
$$K = \{k : N_k < N_{k-1} + a_{N_{k-1}}\}$$

. To prove the left of (2.19), we consider two cases.

Case 1. Suppose that

$$\sum_{k \in G} v^2_{N_k, N_k + a_{N_k}} / (v^2_{0, N_k + a_{N_k}} \log v^2_{0, N_k + a_{N_k}}) = \infty. \tag{2.28}$$

Then, by (2.21)

$$\sum_{k \in G} P\Big\{\frac{1}{B_{N_k, N_k + a_{N_k}} b_{N_k}} |S'_{N_k + a_{N_k}} - S'_{N_k}| \geq (1-\delta) T_-\Big\} = \infty. \tag{2.29}$$

By noting that $\{S'_{N_k + a_{N_k}} - S'_{N_k}, k \in G\}$ is an independent sequence, (2.29) implies that

$$\overline{\lim_{\substack{k \to \infty \\ k \in G}}} \frac{|S'_{N_k + a_{N_k}} - S'_{N_k}|}{B_{N_k, N_k + a_{N_k}} b_{N_k}} \geq (1-\delta) T_- \qquad \text{a.s.},$$

and hence

$$\overline{\lim_{N \to \infty}} \frac{|S'_{N + a_N} - S'_N|}{B_{N, N + a_N} b_N} \geq (1-\delta) T_- \qquad \text{a.s.}$$

i.e., the left of (2.19) holds true.

Case 2. Suppose that

$$\sum_{k \in G} v^2_{N_k, N_k + a_{N_k}} / (v^2_{0, N_k + a_{N_k}} \log v^2_{0, N_k + a_{N_k}}) < \infty. \tag{2.30}$$

Then, by (2.25),

$$\sum_{k \in K} v^2_{N_k, N_k + a_{N_k}} / (v^2_{0, N_k + a_{N_k}} \log v^2_{0, N_k + a_{N_k}}) = \infty. \tag{2.31}$$

For every $k \in K$, by (2.23)

$$0 \leq v^2_{0, N_{k-1} + a_{N_{k-1}}} - v^2_{0, N_k} \leq \eta v^2_{N_k, N_k + a_{N_k}}. \tag{2.32}$$

Hence

$$(1-\eta) v^2_{N_k, N_k + a_{N_k}} \leq v^2_{0, N_k + a_{N_k}} - v^2_{0, N_{k-1} + a_{N_{k-1}}} \leq v^2_{N_k, N_k + a_{N_k}}. \tag{2.33}$$

Write

$$|S'_{N_k + a_{N_k}} - S'_{N_k}| \geq |S'_{N_k + a_{N_k}} - S'_{N_{k-1} + a_{N_{k-1}}}| - |S'_{N_{k-1} + a_{N_{k-1}}} - S'_{N_k}|.$$

Consider the first term of the right hand side. Noting (2.33), we can use Lemma 2.2 again like (2.20) and obtain

$$\begin{aligned}
&\sum_{k \in K} P\Big\{\frac{1}{B_{N_k, N_k + a_{N_k}} b_{N_k}} |S'_{N_k + a_{N_k}} - S'_{N_{k-1} + a_{N_{k-1}}}| \geq (1-\delta) T_-\Big\} \\
&\geq \sum_{k \in K} (B^2_{a_{N_k}} / (B^2_{N_k + a_{N_k}} \log B^2_{N_k + a_{N_k}}))^{(1-\delta)/(1-\eta)} \\
&\geq c \sum_{k \in K} v^2_{N_k, N_k + a_{N_k}} / (v^2_{0, N_k + a_{N_k}} \log v^2_{0, N_k + a_{N_k}}) = \infty
\end{aligned}$$

for some $c > 0$, which implies that

$$\overline{\lim_{\substack{k\to\infty\\ k\in K}}} \frac{|S'_{N_k+a_{N_k}} - S'_{N_{k-1}+a_{N_{k-1}}}|}{B_{N_k,N_k+a_{N_k}} b_{N_k}} \geq (1-\delta)T_-. \tag{2.34}$$

By using (2.32) and noting $\eta \leq \delta^2 T_-^2/4T_+^2$,

$$\begin{aligned}&\prod_{j=N_k+1}^{N_{k-1}+a_{N_{k-1}}} E\exp\Big\{\frac{3b_{N_k}}{\delta T_- B_{N_k,N_k+a_{N_k}}}(X'_j - EX'_j)\Big\}\\ &\leq \exp\Big\{\frac{3b_{N_k}^2}{\delta^2 T_-^2 B_{N_k,N_k+a_{N_k}}^2}\sum_{j=N_k+1}^{N_{k-1}+a_{N_{k-1}}} \operatorname{Var}(X'_j)\Big\}\\ &\leq \exp\{3\eta b_{N_k}^2 v_{N_k,N_k+a_{N_k}}^2/\delta^2 T_-^2 B_{N_k,N_k+a_{N_k}}^2\}\\ &\leq \exp\{4\eta T_+^2 b_{N_k}^2/\delta^2 T_-^2\} \leq \exp(b_{N_k}^2)\end{aligned}$$

for large k and small ε. Hence, using the definitions of T_+ and T_-, for large k and small ε, we have

$$\begin{aligned}&P\Big\{\frac{1}{B_{N_k,N_k+a_{N_k}} b_{N_k}}|S'_{N_{k-1}+a_{N_{k-1}}} - S'_{N_k}| \geq \delta T_-\Big\}\\ &\leq \exp\{-3b_{N_k}^2 + b_{N_k}^2\} = B_{a_{N_k}}^4/(B_{N_k+a_{N_k}}^4 \log^2 B_{a_{N_k}}^2)\\ &\leq v_{N_k,N_k+a_{N_k}}^3/(v_{0,N_k+a_{N_k}}^3 \log^2 v_{N_k,N_k+a_{N_k}}^2)\\ &\leq v_{N_k,N_k+a_{N_k}}^2/(v_{0,N_k+a_{N_k}}^2 \log^2 v_{0,N_k+a_{N_k}}^2).\end{aligned} \tag{2.35}$$

By (2.23)

$$\begin{aligned}v_{N_k,N_k+a_{N_k}}^2 &= v_{0,N_k+a_{N_k}}^2 - v_{0,N_k}^2\\ &\leq v_{0,N_k+a_{N_k}}^2 - v_{0,N_{k-1}+a_{N_{k-1}}}^2 + \eta v_{N_k,N_k+a_{N_k}}^2.\end{aligned}$$

So, if $\eta < 1/2$,

$$v_{N_k,N_k+a_{N_k}}^2 \leq 2(v_{0,N_k+a_{N_k}}^2 - v_{0,N_{k-1}+a_{N_{k-1}}}^2).$$

Using this inequality and imitating the proof of (2.25), we can get

$$\sum_{k=1}^{\infty} v_{N_k,N_k+a_{N_k}}^2/(v_{0,N_k+a_{N_k}}^2 \log^2 v_{0,N_k+a_{N_k}}^2) < \infty.$$

Therefore

$$\sum_{k\in H} v_{N_k,N_k+a_{N_k}}^2/(v_{0,N_k+a_{N_k}}^2 \log^2 v_{0,N_k+a_{N_k}}^2) < \infty.$$

Thus, from (2.35) we have

$$\overline{\lim_{\substack{k\to\infty\\ k\in K}}} \frac{|S'_{N_{k-1}+a_{N_{k-1}}} - S'_{N_k}|}{B_{N_k,N_k+a_{N_k}} b_{N_k}} \leq \delta T_- \qquad a.s. \tag{2.36}$$

Combining (2.36) with (2.34) yields

$$\overline{\lim_{\substack{k\to\infty\\ k\in K}}} \frac{|S'_{N_k+a_{N_k}} - S'_{N_k}|}{B_{N_k,N_k+a_{N_k}} b_{N_k}} \geq (1-2\delta)T_- \qquad a.s.,$$

and hence

$$\overline{\lim_{N\to\infty}} \frac{|S'_{N+a_N} - S'_N|}{B_{N,N+a_N} b_N} \geq (1-2\delta)T_- \qquad a.s.,$$

i.e., the left of (2.19) holds true. The theorem is proved.

The route of proof of Theorem 1.2 is the same as that for Theorem 1.1. We omit it.

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