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THE ESTIMATION OF σ^2 IN LINEAR REGRESSION WITH CENSORED DATA**

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Abstract

This paper proposes a new estimator of the parameter σ^2 in simple linear regression model when the observations are randomly censored on the right. This estimator is explicitly defined and easily computable. The strong consistency and the asymptotic normality are also established under certain conditions.

Keywords Censored data, Strong consistency
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Consider the linear regression model

$$y_i = \alpha + \beta x_i + \varepsilon_i \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where ε_i are independent random variables with mean zero and finite variance σ^2. Sometimes the y_i may be censored and therefore not completely observable. We can only observe

$$z_i = \min(y_i, u_i), \qquad \delta_i = I_{(y_i \le u_i)}, \tag{2}$$

where u_i are censoring random variables which are supposed to be independent (also independent of $\{y_i\}$). The problem is to estimate the parameters of α, β, σ^2 based on (z_1, δ_1), (z_2, δ_2), $\cdots$, (z_n, δ_n). In recent years some authors have studied the estimations of α, β with the pioneering effort made by Miller[4], Buckley and James[1]. But there is not an ideal method for σ^2. In the present work we will give an estimate of σ^2 which is asymptotic unbiasedness, strong consistency and asymptotic normality.

In sequel, we discuss model (1), (2) and always assume ε_i i.i.d. with continuous distribution function F, $E\varepsilon_i = 0$, var $\varepsilon_i = \sigma^2 < \infty$. Also we suppose that censoring random variables u_i are i.i.d. and positive with continuous distribution function G. The main idea of this paper is the generalization of the Class K method due to Zheng Zukang[6,7]. Recall the Class K method: If y_i is censored we add something to it to make up for the censored part and if y_i is uncensored we also modify it appropriately to ensure unbiasedness in the sense that the modification of y_i has the same expectation as y_i. For known G case, we give

Definition 1. *Let ϕ_1, ϕ_2 be continuous functions such that*

$$\begin{cases} \text{(i)} \quad [1 - G(y)]\phi_1(y) + \int_0^y \phi_2(t)dG(t) = y, \\ \text{(ii)} \quad \phi_1, \phi_2 \textit{ are independent of } F_i \textit{ (but may depend on } G), \end{cases} \tag{3}$$

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where F_i is the distribution function of y_i.

The class of all such pairs (ϕ_1, ϕ_2) is called the Class K.

Let

$$y_i^* = \delta_i\phi_1(z_i) + (1-\delta_i)\phi_2(z_i). \tag{4}$$

It is easy to verify

$$Ey_i^* = Ey_i. \tag{5}$$

Here are the examples:

$$\phi_1(z) = \frac{z}{1-G(z)}, \qquad \phi_2(z) = 0;$$
$$\phi_1(z) = \frac{z}{1-G(z)} - \int_0^z udG(u), \quad \phi_2(z) = z(1-G(z)) - \int_0^z udG(u);$$
$$\phi_1(z) = \int_0^z \frac{du}{1-G(u)}, \qquad \phi_2(z) = \int_0^z \frac{du}{1-G(u)};$$
$$\phi_1(z) = \int_0^z \frac{du}{1-G(u)} - G(z), \qquad \phi_2(z) = \int_0^z \frac{du}{1-G(u)} - 2G(z) + 1.$$

In order to construct the estimates of σ^2, the same technique will be used to the second moment of y_i. We need

Definition 2. *Let $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ be continuous functions such that*

$$\begin{cases} \text{(i)} \quad [1-G(y)]\widetilde{\phi}_1(y) + \int_0^y \widetilde{\phi}_2(t)dG(t) = y^2, \\ \text{(ii)} \quad \widetilde{\phi}_1, \widetilde{\phi}_2 \textit{ are independent of } F_i \textit{ (but may depend on } G). \end{cases} \tag{6}$$

We denote the class of all pairs $(\widetilde{\phi}_1, \widetilde{\phi}_2)$ by the Class $\widetilde{K}$. Similarly, let

$$\widetilde{y}_i = \delta_i\widetilde{\phi}_1(z_i) + (1-\delta_i)\widetilde{\phi}_2(z_i). \tag{7}$$

We have

$$E\widetilde{y}_i = Ey_i^2. \tag{8}$$

Some elements of the Class $\widetilde{K}$ are:

$$\widetilde{\phi}_1(z) = \frac{z^2}{1-G(z)}, \qquad \widetilde{\phi}_2(z) = 0;$$
$$\widetilde{\phi}_1(z) = 2\Big[\int_0^z \frac{udu}{1-G(u)}\Big], \quad \widetilde{\phi}_2(z) = 2\Big[\int_0^z \frac{udu}{1-G(u)}\Big].$$

Now we suggest

$$\hat{\sigma}^2 = \frac{1}{n}\sum \widetilde{y}_i - [\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2}] \tag{9}$$

with $\widetilde{y}_i$ of (7) and

$$\begin{cases} \hat{\beta} = \dfrac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}, \\ \hat{\alpha} = \overline{y^*} - \hat{\beta}\overline{x}, \end{cases} \tag{10}$$

where $\overline{x} = \frac{1}{n}\sum x_i$, $\overline{x^2} = \frac{1}{n}\sum x_i^2$, $\overline{y^*} = \frac{1}{n}\sum y_i^*$ ($\sum$ always from 1 to n).

Remark. By Zheng Zukang[6], var $y_i^* > \sigma^2$. We write

$$\text{var } y_i^* = \sigma^2 + T_i \tag{11}$$

with $T_i = T_i(F_i, G) > 0$. So the usual estimates of σ^2,

$$\frac{1}{n-2}\sum(y_i^* - \hat{\alpha} - \hat{\beta}x_i)^2,$$

can not be available any longer.

Theorem 1. *If* $\sup_{i\le n} T_i = o(n)$, *then*

$$\lim E\hat\sigma_n^2 = \sigma^2. \tag{12}$$

Proof. Since

$$\begin{aligned}\text{var }\hat\alpha &= \text{var }\overline{y^*} + \overline{x}^2\text{var }\hat\beta - 2\text{cov }(\overline{y^*}, \hat\beta\overline{x})\\ &= \frac{1}{n^2}\sum \text{var } y_i^* + \overline{x}^2\frac{\sum(x_i-\overline{x})^2\text{var } y_i^*}{[\sum(x_i-\overline{x})^2]^2} - 2\frac{\overline{x}}{n}\frac{\sum(x_i-\overline{x})\text{var } y_i^*}{\sum(x_i-\overline{x})^2}\\ &= \sum \text{var } y_i^*\Big\{\frac{1}{n^2} + \frac{\overline{x}^2(x_i-\overline{x})^2}{[\sum(x_i-\overline{x})^2]^2} - \frac{2\overline{x}(x_i-\overline{x})}{n\sum(x_i-\overline{x})^2}\Big\},\\ \text{var }\hat\beta &= \sum \text{var } y_i^*\frac{(x_i-\overline{x})^2}{[\sum(x_i-\overline{x})^2]^2},\end{aligned}$$

and

$$\text{cov }(\hat\alpha,\hat\beta) = \frac{1}{n}\frac{\sum(x_i-\overline{x})\text{var } y_i^*}{\sum(x_i-\overline{x})^2} - \overline{x}\frac{\sum(x_i-\overline{x})^2\text{var } y_i^*}{[\sum(x_i-\overline{x})^2]^2},$$

we have

$$\begin{aligned}E\hat\sigma^2 &= \frac{1}{n}\sum E\widetilde{y}_i - [E\hat\alpha^2 + 2\overline{x}E\hat\alpha\hat\beta + \overline{x^2}E\hat\beta^2]\\ &= \frac{1}{n}\sum E\widetilde{y}_i - [\text{var }\hat\alpha + \alpha^2 + 2\overline{x}\text{cov }(\hat\alpha,\hat\beta) + 2\overline{x}\alpha\beta + \overline{x^2}\text{var }\hat\beta + \overline{x^2}\beta^2]\\ &= \sigma^2 + \frac{1}{n}\sum(\alpha+\beta x_i)^2 - [\alpha^2 + 2\overline{x}\alpha\beta + \overline{x^2}\beta^2] - \Big[\sum \text{var } y_i^*\\ &\quad\cdot\Big(\frac{1}{n^2} + \frac{\overline{x}^2(x_i-\overline{x})^2}{[\sum(x_i-\overline{x})^2]^2} - \frac{2\overline{x}(x_i-\overline{x})}{n\sum(x_i-\overline{x})^2}\Big) + \sum \text{var } y_i^*\frac{2\overline{x}(x_i-\overline{x})}{n\sum(x_i-\overline{x})^2}\\ &\quad - \sum \text{var } y_i^*\frac{(x_i-\overline{x})^2\cdot 2\overline{x}^2}{[\sum(x_i-\overline{x})^2]^2} + \sum \text{var } y_i^*\overline{x^2}\frac{(x_i-\overline{x})^2}{[\sum(x_i-\overline{x})^2]^2}\Big]\\ &= \sigma^2 - \sum \text{var } y_i^*\Big\{\frac{1}{n^2} + \frac{(x_i-\overline{x})^2}{n\sum(x_i-\overline{x})^2}\Big\},\end{aligned}$$

or

$$\begin{aligned}|E\hat\sigma^2 - \sigma^2| &= \sum \text{var } y_i^*\Big\{\frac{1}{n^2} + \frac{(x_i-\overline{x})^2}{n\sum(x_i-\overline{x})^2}\Big\}\\ &= o(1).\end{aligned}$$

Thus we conclude.

Remark 1. In the proof we can see that $E\hat\sigma^2$ is less than σ^2 strictly. It means that $\hat\sigma^2$ is underestimate in average.

Furthermore, denoting

$$\tau_{F_i} = \inf\{t : F_i(t) = 1\} \tag{13}$$

and

$$\tau = \sup \tau_{F_i},$$

we obtain

Theorem 2. *Suppose that* $\sup T_i = R < \infty$, $\sup|x_i| = M < \infty$ *and* $S_n^2 = \sum(x_i-\overline{x})^2 \to$

∞. *If*

$$\int_0^\infty \sup P(|\widetilde{\phi}_j(z_j)| \geq t)dt \qquad (j=1,2), \tag{14}$$

then

$$\hat{\sigma}^2 - \sigma^2 \to 0 \qquad a.s. \tag{15}$$

In particular, if for $j = 1,2$, $\widetilde{\phi}_j$ is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$ with $\widetilde{\phi}_j(0) = 0$ and $E\widetilde{\phi}_j(z_0) < \infty$, $E\widetilde{\phi}_j(z_0') < \infty$, where z_0, z_0' are the random variables with distribution functions $1-[1-G(t)][1-F(t-\alpha-|\beta|M)]$ and $1-[1-G(t)][1-F(t-\alpha+|\beta|M)]$ respectively, then (14) holds.

Remark. (i) A simple but more strict assumption on $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ of strong consistency is

$$\max_{j=1,2} \sup_{t\leq\tau} |\widetilde{\phi}_j(t)| \leq L < \infty \tag{16}$$

for some constant L. In practice, (16) fails even in the case that $\tau=\infty$ and $G(x)>0$ for $x>0$.

(ii) The assumption on $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ in Theorem 2 can be satisfied in many situations. The above examples of the Class $\widetilde{K}$ are the illustrations of the particular part of Theorem 2 under some suitable G and F.

Proof. By the theorem of strong consistency of least square estimates due to Lai and Wei[2], we get

$$\hat{\alpha} \to \alpha \qquad \text{a.s.},$$
$$\hat{\beta} \to \beta \qquad \text{a.s.}$$

of (10) under our assumptions (cf. [7]).

On the other hand, $\hat{\sigma}^2 - \sigma^2$ can be decomposited as follows:

$$\begin{aligned}\hat{\sigma}^2 - \sigma^2 &= \frac{1}{n}\sum(\widetilde{y}_i - E\widetilde{y}_i) + \frac{1}{n}\sum Ey_i^2 - \sigma^2 - (\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2})\\ &= \frac{1}{n}\sum(\widetilde{y}_i - E\widetilde{y}_i) + \frac{1}{n}\sum(\alpha+\beta x_i)^2 - (\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2})\\ &= \frac{1}{n}\sum(\widetilde{y}_i - E\widetilde{y}_i) + (\alpha^2-\hat{\alpha}^2) + \overline{x^2}(\beta^2-\hat{\beta}^2) + 2\overline{x}(\alpha\beta - \hat{\alpha}\hat{\beta}).\end{aligned}$$

We only need to show that the first term tends to zero almost surely. We will use the following fact (cf. [3]): Let V_i $(i=1,2,\cdots)$ be the independent random variables such that there exists a random variable W with $P(|W|\geq t) \geq P(|V_i|\geq t)$ for all $t\geq 0$ and $E|W|<\infty$. Then

$$\frac{1}{n}\sum_{i=1}^n (V_i - EV_i) \to 0 \qquad \text{a.s.}$$

It is also clear that the condition can be changed by

$$\int_0^\infty \sup P(|V_i|\geq t)dt < \infty.$$

In our case

$$\begin{aligned}\int_0^\infty \sup P(|\widetilde{y}_i|\geq t)dt &\leq \int_0^\infty \sup P(|\widetilde{\phi}_1(z_i)| + |\widetilde{\phi}_2(z_i)| \geq t)dt\\ &\leq \int_0^\infty \sup P(|\widetilde{\phi}_1(z_i)| \geq \frac{t}{2})dt + \int_0^\infty \sup P(|\widetilde{\phi}_2(z_i)| \geq \frac{t}{2})dt < \infty.\end{aligned}$$

Therefore we conclude.

For the particular case, denoting the roots of $\widetilde{\phi}_j(u) = t$ for $t \geq 0$ by A_{jt} and B_{jt} with $-\infty < B_{jt} \leq 0 \leq A_{jt} < \infty$, we obtain

$$P(\widetilde{\phi}_j(z_i) \geq t) = P(z_i \geq A_{jt}) + P(z_i \leq B_{jt}).$$

If there does not exist such A_{jt} or B_{jt}, the corresponding probability turns zero. It is clear that for any $t \geq 0$,

$$\begin{aligned} P(z_i \geq t) &= [1 - G(t)]P(y_i \geq t) = [1 - G(t)]P(\alpha + \beta x_i + \varepsilon_i \geq t) \\ &\leq [1 - G(t)]P(\alpha + |\beta|M + \varepsilon_i \geq t) \leq [1 - G(t)][1 - F(t - \alpha - |\beta|M)] \\ &= P(z_0 \geq t), \\ P(z_i \leq -t) &= 1 - P(z_i > -t) = 1 - [1 - G(-t)]P(\alpha + \beta x_i + \varepsilon_i > -t) \\ &= 1 - P(\alpha + \beta x_i + \varepsilon_i > -t) = P(\alpha + \beta x_i + \varepsilon_i \leq -t) \\ &\leq P(\alpha - |\beta|M + \varepsilon_i \leq -t) = 1 - P(\alpha - |\beta|M + \varepsilon_i > -t) \\ &= 1 - [1 - G(-t)]P(\alpha - |\beta|M + \varepsilon_i > -t) \\ &= 1 - [1 - G(-t)][1 - F(-t - \alpha + |\beta|M)] = P(z_0' \leq -t). \end{aligned}$$

Thus,

$$\begin{aligned} P(\widetilde{\phi}_j(z_i) \geq t) &= P(z_i \geq A_{jt}) + P(z_i \leq B_{jt}) \\ &\leq P(z_0 \geq A_{jt}) + P(z_0' \leq B_{jt}) \\ &\leq P(\widetilde{\phi}_j(z_0) \geq t) + P(\widetilde{\phi}_j(z_0') \geq t) \end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \sup P(\widetilde{\phi}_j(z_i) \geq t)dt \\ &\leq \int_0^\infty P(\widetilde{\phi}_j(z_0) \geq t)dt + \int_0^\infty P(\widetilde{\phi}_j(z_0') \geq t)dt \\ &= E\widetilde{\phi}_j(z_0) + E\widetilde{\phi}_j(z_0') \\ &< \infty. \end{aligned}$$

It completes the proof.

For the asymptotic behaviour of $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$, we notice that

$$\begin{aligned} &\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2} \\ =&(\overline{y^*} - \hat{\beta}\overline{x})^2 + 2(\overline{y^*} - \hat{\beta}\overline{x})\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2} \\ =&\overline{y^*}^2 + \hat{\beta}^2(\overline{x^2} - \overline{x}^2) \\ =&\overline{y^*}^2 + \frac{[\sum(x_i - \overline{x})y_i^*]^2}{n[\sum(x_i - \overline{x})^2]}. \end{aligned}$$

It turns out that

$$\begin{aligned} \hat{\sigma}^2 - \sigma^2 &= \frac{1}{n}\sum \widetilde{y}_i - \sigma^2 - (\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta}\overline{x} + \hat{\beta}^2\overline{x^2}) \\ &= \frac{1}{n}\sum \widetilde{y}_i - \sigma^2 - \overline{y^*}^2 - \frac{[\sum(x_i - \overline{x})y_i^*]^2}{n[\sum(x_i - x)^2]} \\ &= \frac{1}{n}\sum [\widetilde{y}_i - \sigma^2 - (\alpha + \beta x_i)^2] + \frac{1}{n}\sum(\alpha + \beta x_i)^2 - (\overline{y^*} - \alpha - \beta\overline{x})^2 \end{aligned}$$

$$
\begin{aligned}
&-(\alpha+\beta\overline{x})^2-2(\alpha+\beta\overline{x})(\overline{y^*}-\alpha-\beta\overline{x})-\frac{1}{n}\sum(x_i-\overline{x})^2\\
&\cdot\Big\{\Big[\frac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}-\beta\Big]^2+\beta^2+2\beta\Big[\frac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}-\beta\Big]\Big\}\\
=&\frac{1}{n}\sum[\widetilde{y}_i-\sigma^2-(\alpha+\beta x_i)^2]-2(\alpha+\beta\overline{x})(\overline{y^*}-\alpha-\beta\overline{x})\\
&-\frac{1}{n}\sum(x_i-\overline{x})^2 2\beta\Big[\frac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}-\beta\Big]-(\overline{y^*}-\alpha-\beta\overline{x})^2\\
&-\frac{1}{n}\sum(x_i-\overline{x})^2\Big[\frac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}-\beta\Big]^2.
\end{aligned}
$$

Using this expression, we can easily prove

Theorem 3. *Suppose that* $\sup T_i=R<\infty$, $\sup|x_i|=M<\infty$ *and there exist positive constants* C_1 *and* C_2 *such that*

$$C_1 n\geq\sum(x_i-\overline{x})^2\geq C_2 n^{\frac{1+d}{2}}$$

for some $d>0$ *and large* n. *Denote* $U_i=\widetilde{y}_i-2y_i^*(\alpha+\beta x_i)$ *and assume that*

$$\sigma^{*^2}=\lim_{n\to\infty}\frac{1}{n}\sum\operatorname{var}U_i^2 \tag{17}$$

exists with finite positive value. If

(i) $\{U_i\}$ *obey the Linderberg condition;*

or

(ii) $\sup E|U_i-EU_i|^{2+\Delta}<\infty$, *for some* $\Delta>0$,

then

$$\sqrt{n}(\hat{\sigma}^2-\sigma^2)\xrightarrow{\mathfrak{D}}N(0,\sigma^{*^2}).$$

Proof. Since

$$
\begin{aligned}
\sqrt{n}E\Big[\frac{\sum(x_i-\overline{x})y_i^*}{\sum(x_i-\overline{x})^2}-\beta\Big]^2&=\sqrt{n}\frac{\sum(x_i-\overline{x})^2\operatorname{var}y_i^*}{[\sum(x_i-\overline{x})^2]^2}\\
&\leq\sqrt{n}\sup\operatorname{var}y_i^*\frac{1}{\sum(x_i-\overline{x})^2}=o(1)
\end{aligned}
$$

and

$$\sqrt{n}E(\overline{y^*}-\alpha-\beta\overline{x})^2=\sqrt{n}\operatorname{var}\overline{y^*}=o(1),$$

we obtain

$$
\begin{aligned}
\sqrt{n}(\hat{\sigma}^2-\sigma^2)=&\frac{1}{\sqrt{n}}\sum[\widetilde{y}_i-\sigma^2-(\alpha+\beta x_i)^2]-\frac{2}{\sqrt{n}}(\alpha+\beta\overline{x})\\
&\cdot\sum(y_i^*-\alpha-\beta x_i)-\frac{2\beta}{\sqrt{n}}[\sum(x_i-\overline{x})y_i^*-\sum(x_i-\overline{x})\\
&\cdot(\alpha+\beta x_i)]+o_p(1)\\
=&\frac{1}{\sqrt{n}}\sum\{[\widetilde{y}_i-2(\alpha+\beta\overline{x})y_i^*-2\beta(x_i-\overline{x})y_i^*]\\
&-[\sigma^2+(\alpha+\beta x_i)^2-2(\alpha+\beta\overline{x})(\alpha+\beta x_i)\\
&-2\beta(x_i-\overline{x})(\alpha+\beta x_i)]\}+o_p(1)\\
=&\frac{1}{\sqrt{n}}\sum\{[\widetilde{y}_i-2(\alpha+\beta x_i)y_i^*]-[\sigma^2-(\alpha+\beta x_i)^2]\}+o_p(1)
\end{aligned}
$$

$$= \frac{1}{\sqrt{n}}\sum \widetilde{U}_i + o_p(1),$$

where $\widetilde{U}_i = [\widetilde{y}_i - 2(\alpha+\beta x_i)y_i^*] - [\sigma^2 - (\alpha+\beta x_i)^2]$ with $E\widetilde{U}_i = 0$. By the CLT we conclude with the assumptions.

In the remains of this paper we will discuss the unknown G case. We assume in the sequel that

$$\overline{F}(t) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}F_i(t) \quad \text{exists for all } t, \tag{18}$$

$$\tau < \infty, \tag{19}$$

and for some known positive $\gamma < 1$,

$$G(\tau_{\overline{F}}) \le \gamma,$$

where $\tau_{\overline{F}} = \inf\{t : \overline{F}(t) = 1\}$. It leads

$$\tau_{\overline{F}} < \tau_G. \tag{20}$$

It is natural to substitute $G(t)$ by an estimator of it, regarding u_i censored by y_i. Here we suggest the modified Kaplan-Meier estimator

$$\hat{G}_n(t) = \min\Big\{1 - \prod_{z_{(i)}\le t}\Big(1 - \frac{1}{n-i+1}\Big)^{1-\delta_{(i)}}, \gamma\Big\}. \tag{21}$$

Using the similar technique of Peterson[2], the uniform strong consistency of $\hat{G}_n(t)$ can be established, i.e.,

$$\sup_{t\le\gamma}|\hat{G}_n(t) - G(t)| \to 0 \qquad \text{a.s.}$$

To indicate the dependence of $\hat{G}_n(t)$, we denote $\phi_1(\hat{G}_n)$, $\phi_2(\hat{G}_n)$,$\widetilde{\phi}_1(\hat{G}_n)$, $\widetilde{\phi}_2(\hat{G}_n)$ instead of ϕ_1, ϕ_2, $\widetilde{\phi}_1$, $\widetilde{\phi}_2$. For example,

$$\phi_1 = \frac{\delta z}{1-G(z)} \to \phi_1(\hat{G}_n) = \frac{\delta z}{1-\hat{G}_n(z)}.$$

Also the notations $y_i^*(\hat{G}_n)$, $\widetilde{y}_i(\hat{G}_n)$, $\hat{\beta}(\hat{G}_n)$,$\hat{\alpha}(\hat{G}_n)$, and $\hat{\sigma}^2(\hat{G}_n)$ are used. Thus (9) becomes

$$\hat{\sigma}^2(\hat{G}_n) = \frac{1}{n}\sum \widetilde{y}_i(\hat{G}_n) - [\hat{\alpha}^2(\hat{G}_n) + 2\hat{\alpha}(\hat{G}_n)\hat{\beta}(\hat{G}_n)\overline{x} + \hat{\beta}(\hat{G}_n)\overline{x^2}]. \tag{22}$$

For keeping the consistency of $\hat{\sigma}^2(\hat{G}_n)$, we should restrict the selections of (ϕ_1,ϕ_2) and $(\widetilde{\phi}_1,\widetilde{\phi}_2)$. We need the following

Definition 3. *Let K_c ($\widetilde{K}_c$) be the class of all pairs $(\phi_1,\phi_2)\in K((\widetilde{\phi}_1,\widetilde{\phi}_2)\in\widetilde{K})$ with the following properties (at the censoring distribution G):*

(i) *For every d with $1 > d > 0$ and every s, there exists a constant C such that*

$$\max_{\substack{j=1,2\\ t\le s}}|\phi_j(\widetilde{G})| \le C$$

$$(\max_{\substack{j=1,2\\ t\le s}}|\widetilde{\phi}_j(\widetilde{G})| \le C)$$

for all distribution functions $\widetilde{G}$ with $\widetilde{G}(s) < 1$.

(ii) *For every* $\varepsilon > 0$ *and every* s *with* $G(s) < 1$, *there exists* $\eta > 0$, *such that*

$$\max_{\substack{j=1,2\\ t\le s}} |\phi_j(\widetilde{G}) - \phi_j(G)| \le \varepsilon$$

$$(\max_{\substack{j=1,2\\ t\le s}} |\widetilde{\phi}_j(\widetilde{G}) - \widetilde{\phi}_j(G)| \le \varepsilon)$$

for all distribution functions $\widetilde{G}$ *with* $\sup_{t\le s} |\widetilde{G}(t) - G(t)| \le \eta$.

Theorem 4. *Suppose that* $(\phi_1, \phi_2) \in K_c$, $(\widetilde{\phi}_1, \widetilde{\phi}_2) \in \widetilde{K}_c$, *and the design constants* x_i *satisfy*

$$\sup|x_i| < \infty, \qquad \liminf_{n\to\infty} \frac{\sum(x_i - \overline{x})^2}{n} > 0.$$

Then

$$\hat{\sigma}^2(\hat{G}_n) - \hat{\sigma}^2 \to 0 \qquad \text{a.s.}$$

The proof is very similar to that of $\hat{\beta}(\hat{G}_n) - \hat{\beta} \to 0$ a.s. (cf. [7]), and we omit the details here.

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