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GENERALIZATION OF THE NOTION OF FUNDAMENTAL GROUP**

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Abstract

The classical definition of fundamental group for a topological space is based on the pathwise connectedness. A space with less path will not be able to be described effectively by its fundamental group. The author introduces a definition of generalized fundamental group for a given topological space by means of its own connectedness. For a well-behaved space, e.g., a locally pathwise and semilocally simply connected compact metric space, the generalized fundamental group coincides with the classical one.

Keywords Fundamental group, Topological space, Pathwise connectedness
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The usual definition of fundamental group [1,5,6] is based on the pathwise connectedness, i.e., the connectedness of an interval in reals R is taken to be the standard one. A space with less pathwise connectedness will not be able to be described effectively by its fundamental group. For instance, letting X be a totally pathwise disconnected space, x an arbitrary point of X, we always have $\pi(X,x)=1$. But maybe the space is quite complicated, and the fundamental group gives no information.

In this paper we introduce a definition of generalized fundamental group for a given space by means of its own connectedness. For some locally pathwise connected and semilocally simply connected space the generalized fundamental group coincides with the classical one. In a subsequent paper[3] we establish the theory of covering space associated with the sheaf of generalized fundamental group for a space not necessarily being locally pathwise connected.

Recently, John F. Kennison gave a definition of the fundamental group in [7, Definition 4] which uses a Čech-type approach. His idea is similar to ours in some sense. It might be interesting to compare the definitions of Kennison's and ours. The author wishes to thank the referee for his helpful comments and for pointing out the reference[7] related to this paper.

§1. Generalization of Fundamental Group

Let X be a topological space. Let $\mathfrak{U}_X$ be the collection of all open neighbourhoods of the diagonal Δ in $X\times X$. Let $U\in\mathfrak{U}_X$. A subset A of X is said to be of order U if $A\times A\subset U$. A is said to be a U-set if there is an open neighbourhood of A which is

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of order U. Let x and y be two points of X. A U-path λ from x to y or with origin x and terminal y in X consists of a connected subset $S(\lambda)$, called the support of λ, and an ordered sequence $D_1(\lambda), \cdots, D_k(\lambda)$ of nonempty connected U-sets, called a representation of λ, such that $x \in D_1(\lambda)$, $y \in D_k(\lambda)$ and $D_i(\lambda) \cap D_{i+1}(\lambda) \neq \emptyset$ for $i = 1, \cdots, k-1$ and $S(\lambda) = \bigcup_{i=1}^{k} D_i(\lambda)$. Each $D_i(\lambda)$ is called a U-segment of λ, $i = 1, \cdots, k$, and we can take an arbitrary point $x_i \in D_i(\lambda) \cap D_{i+1}(\lambda)$, $i = 1, \cdots, k-1$, as the terminal of $D_i(\lambda)$ and the origin of $D_{i+1}(\lambda)$ and regard $x(= x_0)$ to be the origin of $D_1(\lambda)$ and $y(= x_k)$ to be the terminal of $D_k(\lambda)$. If we extend the representation of λ by repeating some terms and keeping the order we can get a new representation $D'_1(\lambda), \cdots, D'_{k'}(\lambda)$ such that $D'_1(\lambda) = \cdots = D'_{i_1}(\lambda) = D_1(\lambda), \cdots, D'_{i_1+\cdots+i_{k-1}+1}(\lambda) = \cdots = D'_{i_1+\cdots+i_k}(\lambda) = D_k(\lambda)$, $i_1 + \cdots + i_k = k'$. We call the sequence $D'_1(\lambda), \cdots, D'_{k'}(\lambda)$ an extension of the sequence $D_1(\lambda), \cdots, D_k(\lambda)$ or the sequence $D_1(\lambda), \cdots, D_k(\lambda)$ a contraction of the sequence $D'_1(\lambda), \cdots, D'_{k'}(\lambda)$ and regard them as representing the same U-path λ.

Two U-path λ and λ' with the same origin and terminal are said to be U-contiguous if λ and λ' can be represented into sequences with the same length $D_1(\lambda), \cdots, D_k(\lambda)$ and $D_1(\lambda'), \cdots, D_k(\lambda')$ such that $D_i(\lambda) \cap D_{i+1}(\lambda') \cap D_i(\lambda') \cap D_{i+1}(\lambda') \neq \emptyset$, $i = 1, \cdots, k-1$, and $D_i(\lambda) \cup D_i(\lambda')$ is a connected U-set for each $i = 1, \cdots, k$. Two U-paths λ and λ' with the same origin and terminal are said to be U-homotopic, $\lambda \underset{U}{\simeq} \lambda'$, if there is a sequence $\lambda_1, \cdots, \lambda_l$ of U-paths such that $\lambda_1 = \lambda$ and $\lambda_l = \lambda'$, and λ_i is U-contiguous to λ_{i+1} for all $i = 1, \cdots, l-1$.

Let λ be a U-path from x to y and λ' another U-path from y to z. The product $\lambda\lambda'$ of λ and λ' is a U-path from x to z defined as follows: if $D_1(\lambda), \cdots, D_k(\lambda)$ and $D_1(\lambda'), \cdots, D_{k'}(\lambda')$ are representations of λ and λ' respectively, we take

$$S(\lambda\lambda') = S(\lambda) \cup S(\lambda')$$

and

$$D_i(\lambda\lambda') = D_i(\lambda), \quad i \leq k$$
$$D_i(\lambda\lambda') = D_{i-k}(\lambda'), \quad k < i \leq k + k'.$$

Let λ be a U-path from x to y. The inverse λ^{-1} of λ is a U-path from y to x defined as: $S(\lambda^{-1}) = S(\lambda)$ and if $D_1(\lambda), \cdots, D_k(\lambda)$ is a representation of λ, we take

$$D_i(\lambda^{-1}) = D_{k-i+1}(\lambda), \qquad i = 1, \cdots, k.$$

U-homotopy is an equivalence relation in the set of all U-paths with the same origin and terminal. The U-homotopy class of U-path λ is denoted by $[\lambda]$. If $\lambda \underset{U}{\simeq} \lambda'$ and $\mu \underset{U}{\simeq} \mu'$, and $\lambda\mu$ and $\lambda'\mu'$ can be defined, we have

$$\lambda\mu \underset{U}{\simeq} \lambda'\mu'.$$

Set $[\lambda][\mu] = [\lambda\mu]$. It is easy to see that if x is the origin of λ then $[\lambda\lambda^{-1}] = [x]$, where x denotes the U-path with $S(x) = x$.

Theorem 1.1. *Let X be a topological space and $x \in X$. Let U be in $\mathfrak{U}_X$. The set of U-homotopy equivalence classes of U-paths with origin and terminal x forms a group under the operations of multiplication and inverse as defined above. This group is denoted* *by $P(X,x;U)$ and is called the generalized fundamental group of order U for pair (X,x). If V is in $\mathfrak{U}_X$ with $V \subset U$, there is a canonical homomorphism $h_U^V : P(X,x;V) \to P(X,x;U)$ satisfying*

$$h_U^V h_V^W = h_U^W, \qquad \textit{for } W \subset V \subset U \textit{ in } \mathfrak{U}_X.$$

Proof. The class $[x]$ will be the identity of $P(X,x;U)$. We must show that for $[\lambda] \in P(X,x;U)$

$$[x][\lambda] = [\lambda] = [\lambda][x],$$
$$[\lambda][\lambda^{-1}] = [x]$$

and

$$([\lambda][\mu])[\nu] = [\lambda]([\mu][\nu]).$$

The proof is straightforward.

For $V \in \mathfrak{U}_X$ with $V \subset U$, every V-path is a U-path and V-homotopy implies U-homotopy. Thus the last statement follows.

Let $\mathfrak{U}_X$ be directed by inclusion. We then obtain an inverse system $\{P(X,x;U),h\}$ over the directed set $\mathfrak{U}_X$. Take the inverse limit[2].

$$P(X,x) = \varprojlim_{\mathfrak{U}_X} P(X,x;U)$$

with the projection $h_U : P(X,x) \to P(X,x;U)$.

Definition 1.1. *We call $P(X,x)$ the generalized fundamental group for pair (X,x), and x the base point.*

We introduce a uniformity for $P(X,x)$. For $U \in \mathfrak{U}_X$ the collection $[U]$ of couples of classes in $P(X,x)$ mapped into the same U-homotopy class by $h_U : P(X,x) \to P(X,x;U)$ is called a vicinity associated with U. Clearly we have

$$[U] = [\overset{2}{U}] = [U]^{-1}$$

and

$$[V] \subset [U] \quad \text{for } V \subset U \text{ in } \mathfrak{U}_X.$$

All vicinities associated with the members of $\mathfrak{U}_X$ form a base for a uniformity[4] for $P(X,x)$. This gives a uniform topology of $P(X,x)$, so that $P(X,x)$ is a topological group. Thus we have

Theorem 1.1. *With the uniform topology associated with $\mathfrak{U}_X$ defined above the generalized fundamental group $P(X,x)$ becomes a topological group. The neighbourhood of identity of $P(X,x)$ associated with $U \in \mathfrak{U}_X$ is an invariant subgroup of $P(X,x)$ and is the kernel of the projection h_U.*

§2. Generalized Path Class Space

Given two base points x_0 and x_1 in X, we cannot in general expect any relationship between $P(X,x_0)$ and $P(X,x_1)$. For example, if x_0 and x_1 do not lie in a common connected component of X, there can be no relationship. But for a connected and locally connected space, the situation is better.

Let X be a topological space, and let $x_0, x_1 \in X$. Let U be in $\mathfrak{U}_X$. The collection of U-homotopy classes of U-paths from x_0 to x_1 is denoted by $P(X, x_0, x_1; U)$, and for $V \subset U$ in $\mathfrak{U}_X$ there is a canonical map $h_U^V : P(X, x_0, x_1; V) \to P(X, x_0, x_1; U)$ such that for $W \subset V \subset U$ in $\mathfrak{U}_X$

$$h_U^V h_V^W = h_U^W.$$

It can be given a uniform topology just as for $P(X, x)$.

The first thing that we are interested in is whether the generalized path class space is nonempty.

Lemma 2.1. *Let X be connected, and $x, y \in X$. Then for any open covering $\{U_\alpha\}$ of X there is a finite number of elements $U_1, \cdots, U_k$ of $\{U_\alpha\}$ such that $U_i \cap U_{i+1} \neq \emptyset$ and $x \in U_1$ and $y \in U_k$.*

Proof. Set

$$\wedge = \{\beta : \text{there is a finite number of elements } U_{\alpha_1}, \cdots, U_{\alpha_k} \text{ of } \{U_\alpha\} \text{ such that } U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset \text{ and } U_{\alpha_k} = U_\beta \text{ and } x \in U_{\alpha_1}\}.$$

Let $A = \bigcup_{\alpha \in \wedge} U_\alpha$ and $B = \bigcup_{\alpha \notin \wedge} U_\alpha$. It is easy to see that A and B are open, $A \cup B = X$ and $A \cap B = \emptyset$. Clearly $A \neq \emptyset$; thus $A = X$. This implies the conclusion.

Lemma 2.2. *Let X be locally connected, and let x and y be in the same component of X. Let U be in $\mathfrak{U}_X$. Then we have $P(X, x, y; U) \neq \emptyset$.*

Proof. By local connectedness we can take an open covering $\{U_\alpha\}$ for X consisting of connected U-sets. Lemma 2.2 follows from Lemma 2.1.

Lemma 2.3. *Let X be locally connected, and let x and y be in the same component of X. Let U and V be in $\mathfrak{U}_X$ with $V \subset U$. Then the canonical map $h_U^V : P(X, x, y; V) \to P(X, x, y; U)$ is surjective.*

Proof. Let λ be a U-path from x to y with support $S(\lambda)$ an open connected subset of X containing x and y and a representation consisting of open U-sets $D_i(\lambda)$, $i = 1, \cdots, k$. Take an open covering for $S(\lambda)$ of V-sets which is a refinement of $\{D_i(\lambda)\}_{i=1,\cdots,k}$. By Lemma 2.2 $P(S(\lambda), x, y; V) \neq \emptyset$, i.e., there is a V-path λ' from x to y in $S(\lambda)$, then in X. λ' determines a V-homotopy class $[\lambda'] \in P(X, x, y; V)$ and obviously $h_U^V[\lambda'] = [\lambda]$.

Theorem 2.1. *Let X be connected and locally connected, and x and y in X. Then $P(X, x, y) \neq \emptyset$.*

Proof. Lemma 2.2 and Lemma 2.3 complete the proof of Theorem 2.1.

§3. Changing Base Point

We can define an isomorphism $\gamma_\# : P(X, x) \to P(X, y)$ for each $\gamma \in P(X, x, y)$ as follows.

Lemma 3.1. *Let $U \in \mathfrak{U}_X$ and $[\gamma_U] \in P(X, x, y; U)$. There is an isomorphism*

$$[\gamma_U]_\# : P(X, x; U) \to P(X, y; U)$$

defined by

$$[\gamma_U]_\#([\alpha]) = [\gamma_U]^{-1}[\alpha][\gamma_U] = [\gamma_U^{-1}\alpha\gamma_U] \quad \textit{for } [\alpha] \in P(X, x; U).$$

And the following commutative diagram

$$\begin{array}{ccc} P(X,x;V) & \xrightarrow{h_U^V} & P(X,x;U) \\ {\scriptstyle [\gamma_V]_\#}\downarrow & & {\scriptstyle [\gamma_U]_\#}\downarrow \\ P(X,y;V) & \xrightarrow{h_U^V} & P(X,y;U) \end{array}$$

holds provided $h_U^V([\gamma_V]) = [\gamma_U]$ *for* $U, V \in \mathfrak{U}_X$ *with* $V \subset U$.

Proof. For $[\alpha], [\beta] \in P(X,x;U)$ one obtains

$$\begin{aligned}[\gamma_U]_\#([\alpha])[\gamma_U]_\#([\beta]) &= [\gamma_U]^{-1}[\alpha][\gamma_U][\gamma_U]^{-1}[\beta][\gamma_U] \\ &= [\gamma_U]^{-1}[\alpha][\beta][\gamma_U] = [\gamma_U]_\#([\alpha][\beta]).\end{aligned}$$

So $[\gamma_U]_\#$ is a homomorphism. In fact, $[\gamma_U]_\#$ is an isomorphism because it has an inverse, namely, $[\gamma_U]_\#^{-1} = [\gamma_U^{-1}]_\#$. The commutativity of the diagram is straightforward.

Theorem 3.1. *Let* γ *be in* $P(X,x,y)$. *There is an isomorphism* $\gamma_\# : P(X,x) \to P(X,y)$ *such that*

$$\begin{array}{ccc} P(X,x) & \xrightarrow{h_U} & P(X,x;U) \\ {\scriptstyle \gamma_\#}\downarrow & & {\scriptstyle [\gamma_U]_\#}\downarrow \\ P(X,y) & \xrightarrow{h_U} & P(X,y;U) \end{array}$$

is commutative, where $[\gamma_U] = h_U(\gamma) \in P(X,x,y;U)$ *and* $U \in \mathfrak{U}_X$.

Proof. We define $\gamma_\#$ by

$$\gamma_\#(\alpha) = \varprojlim [\gamma_U]_\#(h_U(\alpha)) \quad \text{for } \alpha \in P(X,x).$$

For $V \in \mathfrak{U}_X$ with $V \subset U$ we have

$$[\gamma_U]_\# h_U(\alpha) = [\gamma_U]_\# h_U^V h_V(\alpha) = h_U^V [\gamma_V]_\# h_V(\alpha).$$

That is an isomorphism from inverse system $\{P(X,x;U)\}$ onto inverse system $\{P(X,y;U)\}$. So $\gamma_\# : P(X,x) \to P(X,y)$ is a well defined isomorphism with inverse $\gamma_\#^{-1} = (\gamma^{-1})_\#$ and the diagram is commutative.

Remark 3.1. For the connected and locally connected space X and any x and y in X, Theorem 3.1 implies that all generalized fundamental groups are isomorphic, that is, associated with the space X is a certain abstract group $P(X)$, the generalized fundamental group of X.

Corollary 3.1. *Let* γ_1 *and* γ_2 *be in* $P(X,x,y)$. *Then*

$$(\gamma_2)_\# = (\gamma_1)_\# i_a,$$

where i_a *is the inner automorphism of* $P(X,x)$ *due to* $a = \gamma_1\gamma_2^{-1} \in P(X,x)$. *Particularly, each* $a \in P(X,x)$ *induces the inner automorphism* i_a.

Proof. For $[\alpha] \in P(X,x)$

$$\begin{aligned}(\gamma_1)_\#^{-1}(\gamma_2)_\#([\alpha]) &= \gamma_1\gamma_2^{-1}[\alpha]\gamma_2\gamma_1^{-1} \\ &= (\gamma_1\gamma_2^{-1})[\alpha](\gamma_1\gamma_2^{-1})^{-1} = i_a[\alpha],\end{aligned}$$

that is, $(\gamma_1)_\#^{-1}(\gamma_2)_\# = i_a$, or $(\gamma_2)_\# = (\gamma_1)_\# i_a$.

Corollary 3.2. *Let* $\sigma : I \to X$ *be a path from* x *to* y. *Then* σ *determines an isomorphism* $\sigma_\# : P(X,x) \to P(X,y)$.

§4. Comparison with the Classical One

Let X be a pathwise connected and locally pathwise connected space, and let x be in X. Let $\pi(X,x)$ be the classical fundamental group for (X,x). We will compare this with the generalized fundamental group $P(X,x)$. For $U \in \mathfrak{U}_X$ each loop λ at x can be decomposed into pieces by a partition $I_1=[0,t], \cdots, I_k=[t_{k-1},1]$ for $I=[0,1]$ and determines a U-path and it is easy to show that the homotopy class $[\lambda]$ determines a U-homotopy class. So we have a canonical map

$$\eta_U : \pi(X,x) \to P(X,x;U),$$

which is clearly a homomorphism. This induces a canonical homomorphism

$$\eta : \pi(X,x) \to P(X,x).$$

Theorem 4.1. *Suppose that X is locally pathwise connected. Then η is surjective.*

Lemma 4.1. *Let X be locally pathwise connected. Let A be a connected subset of X. Let x and y be two points of A. Let B be an open neighbourhood of A. Then there is a path in B from x to y.*

Proof. All points in A which can be connected by a path in B with origin x form a relative open subset in A which is nonempty. The connectedness of A implies that the subset is the whole A.

Proof of Theorem 4.1. For $U \in \mathfrak{U}_X$ let λ_U be a U-path with a representation $D_1(\lambda_U), \cdots, D_k(\lambda_U)$ such that $[\lambda_U]=h_U([\lambda]) \in P(X,x;U)$. By the definition $D_1(\lambda_U), \cdots, D_k(\lambda_U)$ are connected U-sets and $D_i(\lambda_U) \cap D_{i+1}(\lambda_U) \neq \emptyset$ for $i=1,\cdots,k-1$ and $x \in D_1(\lambda_U) \cap D_k(\lambda_U)$.

Without loss of generality we can assume that all $D_i(\lambda_U)$'s are open, for if it is not the case we can modify λ_U in its U-homotopy class as follows. Let $x_i \in D_i(\lambda_U) \cap D_{i+1}(\lambda_U)$ for $i=1,\cdots,k-1$ and $x_k=x_0=x \in D_k(\lambda_U) \cap D_1(\lambda_U)$. For each $i=1,\cdots,k$, $D_i(\lambda_U)$ is a connected subset of order U. We can get an open connected covering $\mathcal{C}_i$ of order U for $D_i(\lambda_U)$. By Lemma 2.1 we can choose a finite number of elements $D'_{i,1},\cdots,D'_{i,j_i}$ of $\mathcal{C}_i$ such that $D'_{i,l} \cap D'_{i,l+1} \neq \emptyset$ for $l=1,\cdots,j_i-1$ and $x_{i-1} \in D'_{i,1}$ and $x_i \in D'_{i,j_i}$. So the sequence

$$D'_l = D'_{i,l-(j_1+\cdots+j_{i-1})}, \qquad j_1+\cdots+j_{i-1} < l \leq j_1+\cdots+j_i, \quad i=1,\cdots,k$$

determines a U-path λ'_U which is U-homotopic to λ_U.

By Lemma 4.1 there is a loop l at x such that with some decomposition $I_1,\cdots,I_k$ of $I=[0,1]$ each $l(I_i)$ is contained in $D_i(\lambda_U)$. Thus l determines a U-path with origin and terminal x which is U-homotopic to λ_U. This completes the proof.

Theorem 4.1. *Let X be a locally pathwise connected and semilocally simply connected space satisfying*

(H) *For any open covering $\{N_\alpha\}$ of X there is a $U \in \mathfrak{U}_X$ such that if $A \subset X$ is of order U, then $A \subset N_\alpha$ for some α.*

Then η is an isomorphism and the topology of $\pi(X,x) \cong P(X,x)$ is discrete.

Proof. Because X is locally pathwise and semilocally simply connected, for each point of X there is an open pathwise connected neighbourhood in which any two paths with the same origin and terminal are homotopic in X. These open sets form an open covering $\{N_\alpha\}$

for X. Hypothesis (H) implies that there is a $U \in \mathfrak{U}_X$ such that if $A \subset X$ is of order U, then $A \subset N_\alpha$ for some α.

Let l and l' be two loops at x and be U-homotopic. First we assume that they are U-contiguous, that is, there is a decomposition $I_1 = [0, t_1], \cdots, I_k = [t_{k-1}, 1]$ for I such that $l(t_i) = l'(t_i)$, $i = 1, \cdots, k-1$ and $l(I_i) \cup l'(I_i)$ is of order U. Therefore $l|I_i$ and $l'|I_i$ are contained in some N_{α_i} and have the same origin and terminal for $i = 1, \cdots, k$. Thus $l|I_i$ and $l'|I_i$ are homotopic in X with origin and terminal fixed; that is to say, U-contiguous loops are homotopic. Therefore U-homotopic loops are homotopic. This proves the injectivity.

The last statement follows from the fact that each element of $\pi(X, x) \stackrel{\sim}{=} P(X, x)$ has a neighbourhood associated with U as above consisting only of itself.

Remark 4.1. We hoped to prove the conclusion of Theorem 4.1 without imposing hypothesis (H), but so far have not succeed.

Corollary 4.1. *For a compact metric space X which is locally pathwise and semilocally simply connected, η is an isomorphism and $\pi(X, x) \stackrel{\sim}{=} P(X, x)$ is discrete.*

Proof. The Lebesgue number of a given open covering guarantees that the condition (H) is satisfied.

Corollary 4.2. *For a CW complex X, η is an isomorphism and $\pi(X, x) \stackrel{\sim}{=} P(X, x)$ is discrete.*

Proof. The paracompactness and the local compactness imply that $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is compact and $X_i \subset \text{Int } X_{i+1}$, $i = 1, 2, \cdots$. Taking the Lebesgue numbers δ_i of $W_i = X_i - \text{Int } X_{i-1}$ and assuming $\delta_1 \geq \delta_2 \geq \cdots$, we can get a positive function δ on X by means of a partition of unity such that $\delta|W_i \leq \delta_i$. The function δ guarantees that the condition (H) is satisfied.

§5. An Example

The following is one of the simplest examples which has nontrivial generalized fundamental group and trivial classical fundamental group.

Let X be a countable space with finite complement topology. The space X is connected, locally connected and compact, but totally pathwise disconnected. Any open subset of X is connected.

Let x and y be two different points in X. Set $V_x = (X \backslash \{x\}) \times (X \backslash \{x\})$, $V_y = (X \backslash \{y\}) \times (X \backslash \{y\})$ and $V = V_x \coprod V_y$. Then V is an open neighbourhood of the diagonal Δ in $X \times X$ not containing (x, y) and (y, x). Let G be a connected open subset of X which contains x and y. Then $G \times G \not\subset V$, that is to say, G is not of order V, and every subset A of X which contains both x and y is thus not a V-set.

Let X' be a copy of X, and x' and y' in X' the copies of x and y, respectively. The space Y is constructed from X and X' by connecting x and x' by a copy of the unit segment $[0, 1]$ and y and y' by another copy of $[0, 1]$. X and X' are regarded as the subspaces of Y. The segment between x and x' is denoted by I_x and the other one between y and y' by I_y.

Let V' be the copy of V corresponding to X'. Let $V_x \in \mathfrak{U}_{I_x}$ and $V_y \in \mathfrak{U}_{I_y}$. Let U be the image of the disjoint union $V \coprod V' \coprod V_x \coprod V_y$ in $Y \times Y$. It is obvious that $U \in \mathfrak{U}_Y$.

A U-path λ with origin x and terminal y in Y has support in X if and only if λ is a V-path in X. And a U-path λ with origin x and terminal y in X is not U-homotopic to a U-path λ' with origin X and terminal y and support $S(\lambda') \subset Y\backslash(X\backslash\{x,y\})$. Thus $g = [\lambda\lambda'^{-1}]$ is not the identity of $P(Y,x;U)$. Therefore, $P(Y,x;U)$ is nontrivial. Moreover, $P(Y,x;U) \supset \{g^n; n \in \mathbb{Z}\} \cong C_\infty$, the infinite cyclic group.

For $W \in \mathfrak{U}_Y$ with $W \subset U$, one has $h_U^W : P(Y,x;W) \to P(Y,x;U)$. Since Y is locally connected, Lemma 2.3 implies that h_U^W is surjective. Passing to the inverse limit, one claims that the generalized fundamental group $P(Y,x)$ is nontrivial. However, the classical fundamental group $\pi(Y,x)$ is trivial.

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