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GLOBAL SOLVABILITY OF NONLINEAR WAVE EQUATION WITH A VISCOELASTIC BOUNDARY CONDITION

QIN TIEHU*

Abstract

The paper deals with the global solvability of nonlinear wave equation with a viscoelastic boundary condition. The problem is a mathematical mode for nonlinare one-dimensional motion of an elastic bar connected with a viscoelastic element at one end of the bar. Under some physically reasonable assumptions, the boundary condition is dissipative and the existence of global smooth solution of the problem is proved for small data.

Keywords Nonlinear wave equation, Viscoelastic boundary condition, Global solvability

1991 MR Subject Classification 35L05, 75F15

§1. Introduction

The paper deals with the following boundary value problem of nonlinear wave equation:

$$(\mathrm{P})\begin{cases} u_{tt}-\sigma(u_x)_x=0, \quad 0<x<L, \quad t>0, & (1.1)\\ u=0, \quad x=0, & (1.2)\\ u+\displaystyle\int_0^t a(t-\tau)\sigma(u_x)(x,\tau)d\tau=g(t), \quad x=L, & (1.3)\\ u=u_0(x), \ u_t=u_1(x), \ t=0. & (1.4)\end{cases}$$

Problem (P) is a mathematical model for nonlinear one-dimensional motion of an elastic bar connected with a viscoelastic element at $x=L$. The integral equation (1.3), satisfied at the end $x=L$ by u, is a nonlinear and nonlocal boundary condition. Under some physically reasonable assumptions on kernel $a(t)$, (1.3) is a dissipative boundary condition as well and we shall show that Problem (P) admits a global smooth solution for small data.

When $a(t)$ is a positive constant, (1.3) reduces to a local nonlinear dissipative boundary condition $u_t+a\sigma(u_x)=g'(t)$. In this case, the viscoelastic element comprises only a dashpot. Greenberg and Li Ta-tsien discussed the problem with $g'(t)=0$ in [3]. If the viscoelastic element at the end $x=L$ of the elastic bar consists of a spring and a dashpot in parallel, i.e. the Kelvin-Voigt's model, then

$$a(t)=\frac{e^{-kt/r}}{r} \quad \text{and} \quad g(t)=\frac{-ku_0(L)e^{-kt/r}}{r}.$$

For this particular case, the problem was studied recently by Alben and Cooper in [1]. And the problems discussed in [3] and [1] are just both the typical examples of our results, corresponding to the cases $a(t)\in L^1(0,\infty)$ and $a(t)\notin L^1(0,\infty)$, repectively.

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*Deparatment of Mathematics, Fudan University, Shanghai 200433, China .

The rest of the paper is organized as follows. Section 2 contains a statement of our results and an outline of the proof of the main theorem, based on some priori estimates. In section 3 we give some energy estimates of the problem. And Section 4 is devoted to the proof of the dependence of the energy integrals on the boundary value of the solution.

§2. Statement of Results and Outline of Proof

We first list the hypotheses on the kernel $a(t)$. These are:

$$a(t)=a_\infty+A(t),\quad A(t)\in C^3[0,\infty),\ \text{where } a_\infty \text{ is a constant,} \tag{2.1}$$

$$a(t)>0,\ a_\infty\ge 0,\ A(t)\ge 0,\ A'(t)\le 0, \tag{2.2}$$

$$t^jA^{(m)}(t)\in L^1(0,\infty),\ j,m\le 3. \tag{2.3}$$

For σ we reguire:

$$\sigma\in C^3(I\!R),\ \sigma(0)=0,\ \sigma'(0)>0. \tag{2.4}$$

As the main results in the paper, we have

Theorem 2.1. *Suppose that hypothese* (2.1)-(2.4) *hold. Assume further that* $u_0\in C^3([0,L])$, $u_1\in C^2([0,L])$ *and* $g\in C^3([0,\infty))\cap H^3(0,\infty)$ *satisfy the following compatibilty conditions*

$$u_0(L)=g(0), \tag{2.5}$$

$$u_1(L)+a(0)\sigma(u_0'(L))=g'(0), \tag{2.6}$$

$$\sigma'(u_0'(L))u_0''(L)+a(0)\sigma'(u_0'(L))u_1'(L)+a'(0)\sigma(u_0'(L))=g''(0), \tag{2.7}$$

$$\begin{aligned}&\sigma''(u_0'(L))u_0'(L)u_0''(L)+\sigma'(u_0'(L))u_0''(L)\\&+a(0)\sigma''(u_0'(L))u_0'^2(L)+a(0)\sigma'(u_0'(L))\sigma''(u_0'(L))u_0''^2(L)\\&+a(0)\sigma''^2(u_0'(L))u_0'''(L)+a'(0)\sigma'(u_0'(L))u_1'(L)+a''(0)\sigma(u_0'(L))=g'''(t),\end{aligned} \tag{2.8}$$

$$u_0(0)=u_1(0)=u''(0)=u_1''(0)=0. \tag{2.9}$$

Then there exists a constant $\varepsilon>0$ *such that if*

$$\|u_0\|^2_{H^3(0,l)}+\|u_1\|^2_{H^2(0,l)}+\|g\|^2_{H^3(0,\infty)}<\varepsilon^3,$$

the Problem (P) *admits a solution* $u\in C^3([0,L]\times[0,\infty))$. *Here* H^s *denotes the Sobolev space.*

In order to prove Theorem 2.1, we first give a theorem on local existence of the solution to Problem (P).

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *there exists a* $T>0$ *such that Problem* (P) *has a unique solution* $u\in C^3([0,L]\times[0,T])$ *and* T *depends only on* $\|u_0\|_{C^2[0,L]}$ *and* $\|u_1\|_{C^1[0,L]}$.

Reducing equation (1.1) to a first order system, then we can obtain Theorem 2.1 from [1, Theorem 2.3].

Without loss of generality, we can assume that

$$\sigma'(s)\ge\sigma_1>0,\ \forall s\in I\!R,$$

where σ' is a constant. There is no harm in making this assumption because we shall show a posteriori that

$$|u_x(x,t)| \le \delta,\ \forall x \in [0,L],\ t \ge 0 \tag{2.10}$$

for a sufficiently small $\delta > 0$.

We set

$$\begin{aligned} E(t) = & \max_{s\in[0,l]} \int_0^L (u^2 + u_x^2 + u_t^2 + u_{xx}^2 + \cdots + u_{ttt}^2)(x,s)dx \\ & + \int_0^t \int_0^L (u^2 + u_x^2 + u_t^2 + u_{xx}^2 + \cdots + u_{ttt}^2)(x,s)dxds, \\ \mu(t) = & \max_{\substack{s\in[0,t] \\ x\in[0,L]}} (u_x^2 + u_{xx}^2 + u_{xt}^2)(x,s), \\ G(g) = & \int_0^\infty (g^2(t) + g'^2(t) + g''^2(t) + g'''^2(t))dt \end{aligned}$$

and

$$U_0(u_0, u_1) = \|u_0\|^2_{H^3(0,L)} + \|u_1\|^2_{H^2(0,L)}.$$

The main work of the paper is to establish the following estimates.

Theorem 2.3. *Let* (2.1)–(2.4) *hold and* $u \in C^3([0,L]\times[0,L])$ *be a solution of Problem* (P). *Then there exists a constant* C, *independent of* u_0, u_1, g *and* T, *such that*

$$E(t) \le C(U_0(u_0,u_1) + G(g)) + C(\mu(t) + \mu^2(t))E(t) \tag{2.11}$$

for $4T^* < t \le T$, *where*

$$T^* = \sigma_1^{1/2} L. \tag{2.12}$$

Now we give a proof of Theorem 2.1 on the basis of Theorems 2.2 and 2.3.

Proof of Theorem 2.1. According to Theorem 2.2 and Sobolev embedding theorem, we can get a local solution $u \in C^3([0,L]\times[0,L])$ and T depends on ε. We first choose an $E^* > 0$ satisfying

$$C(L^{1/2}E^{*1/2} + LE^*) \le 1/4,$$

then take $\varepsilon > 0$ so small that

$$C\varepsilon^2 \le E^*/4, \tag{2.13}$$

$$T > 4T^* \tag{2.14}$$

and

$$E(t) \le E^*/2,\ \forall 0 \le t \le 4T^*. \tag{2.15}$$

For such an ε, the local solution u must satisfy

$$E(t) \le E^*/2,\ \forall 0 \le t \le T. \tag{2.16}$$

It is easy to see that

$$\mu(t) \le L^{1/2}E(t)^{1/2}.$$

So we have from (2.11)

$$E(t) \le C(U_0(u_0,u_1) + G(g)) + C(L^{1/2}E^{3/2}(t) + LE^2(t)),\ \forall 4T^* \le t \le T. \tag{2.17}$$

If $E(t) \le E^*$ for $4T^* \le t \le \tilde{T}$ $(< T)$, the estimate (2.17) gives

$$E(t) \le C\varepsilon^2 + C(L^{1/2}E^{*3/2} + LE^{*2}) \le E^*/2,\ 4T^* \le t \le \tilde{T}.$$

Consequently, by continuity, we have (2.16) from (2.15). Thus we complete the proof of Theorem 2.1 and get the boundness of $E(t)$ on $[0, \infty)$. Therefore, it is justified to make assumption (2.10).

The proof of Theorem 2.3 relies on the estimates give in Propositions 2.1 and 2.2.

Setting

$$\begin{aligned} E_1(t) &= \int_0^L \left(\frac{1}{2}u_t^2 + \int_0^{u_x} \sigma(\eta)d\eta\right) dx + \frac{1}{2}k_\infty u^2(L,t),\\ E_2(t) &= \frac{1}{2}\int_0^L (u_{tt}^2 + \sigma'(u_x)u_{xt}^2)dx + \frac{1}{2}k_\infty u_t^2(L,t),\\ E_3(t) &= \frac{1}{2}\int_0^L (u_{tt}^2 + \sigma'(u_x)u_{xtt}^2)dx + \frac{1}{2}k_\infty u_{tt}^2(L,t),\\ \tilde{E}(t) &= \int_0^L (u_x^2 + u_t^2 + \cdots + u_{ttt}^2)(x,t)dx, \end{aligned}$$

where k_∞ is defined in Lemma 3.2, we can get the following energy estimate.

Proposition 2.1. *Let $u \in C^3([0,L] \times [0,L])$ be a solution of Problem* (P). *Then*

$$\begin{aligned} &E_1(t) + E_2(t) + E_3(t) + \alpha_1 \int_0^t (u_t^2 + u_{tt}^2 + u_{ttt}^2)(L,s)ds\\ \le &C_1U_0(u_0,u_1) + C_1\|g\|_{H^3(0,t)}^2\\ &+ C_1(\mu(t) + \mu^2(t)) \int_0^t \tilde{E}(s)ds,\ 0 \le t \le T. \end{aligned} \tag{2.18}$$

Here and throughout the paper, we use $C_1, \alpha_1,\ C_2, \alpha_2, \cdots$ to denote various positive constants independent of u_0, u_1, g and T.

For deriving the estimate (2.11) from (2.18), we need a further estimate which gives the dependence of the solution on the boundary values.

Proposition 2.2. *Let $u \in C^3([0,L] \times [0,T])$ be a solution of Problem* (P). *Then*

$$\begin{aligned} \int_0^t \tilde{E}(s)ds \le &C_2U_0(u_0,u_1) + C_2 \int_0^t (u_t^2 + U_{tt}^2 + u_{ttt}^2)(L,s)ds\\ &+ C_2\|g\|_{H^3(0,t)}^2 + C_2(\mu(t)\\ &+ \mu^2(t)) \int_0^t \tilde{E}(s)ds,\ \forall t \in [4T^*, T]. \end{aligned} \tag{2.19}$$

We can prove Theorem 2.3 immediately from Propositions 2.1 and 2.2 and Poincare's inequality. The proofs of Propositions 2.1 and 2.2 are given in §3 and §4 respectively.

We close this section by giving another form of boundary condition (1.3). Differentiating both sides of equation (1.3), we get

$$u_t + a(0)\sigma(u_x) + \int_0^t a'(t-\tau)\sigma(u_x)(x,\tau)d\tau = g'(t),\ x = L. \tag{2.20}$$

The kernel $k(t)$ obtained by solving integral equation

$$a(0)k(t) + a'(t)/a(0) + \int_0^t a'(t-\tau)k(\tau)d\tau = 0, \tag{2.21}$$

can be used to express $\sigma(u_x)$ in terms u_t and g', that is,

$$\sigma(u_x) + u_t/a(0) + \int_0^t k(t-\tau)u_t(x,\tau)d\tau$$
$$= g'(t)/a(0) + \int_0^t k(t-\tau)g'(\tau)d\tau, \ x = L. \tag{2.22}$$

In the course of the proof of Theorem 2.3, instead of (1.3) we use boundary condition (2.22).

§3. Proof of Proposition 2.1

We fisrt give some lemmas.

Lemma 3.1. *Let $a(t)$ satisfy (2.1)-(2.3). Then there exists a positive constant α such that*

i) *if $a(t) \in L^1(0,\infty)$, then*

$$-\frac{\mathrm{Im}\hat{a}(i\xi)}{\xi|\hat{a}(i\xi)|^2} \geq \alpha, \quad \forall \xi \in \mathbb{R}, \tag{3.1}$$

ii) *if $a(t) \notin L^1(0,\infty)$, then*

$$\frac{a_\infty - \xi \mathrm{Im}\hat{A}(i\xi)}{|a_\infty + i\xi|\hat{A}(i\xi)|^2} \geq \alpha, \quad \forall \xi \in \mathbb{R}, \tag{3.2}$$

Proof. Suppose first $a(t) \in L^1(0,\infty)$. It is easily seen that conditions (2.1)-(2.3) imply

$$\xi \int_0^t a(t)\sin(\xi t)dt > 0, \quad \forall \xi \neq 0.$$

Thus

$$-\frac{\mathrm{Im}\hat{a}(i\xi)}{\xi|\hat{a}(i\xi)|^2} \geq 0, \quad \forall \xi \neq 0. \tag{3.3}$$

It is not difficult to verify

$$\lim_{\xi\to 0}\left(-\frac{\mathrm{Im}\hat{a}(i\xi)}{\xi|\hat{a}(i\xi)|^2}\right) = \int_0^\infty ta(t)dt/|\hat{a}(0)|^2 > 0 \tag{3.4}$$

and

$$\lim_{\xi\to \infty}\left(-\frac{\mathrm{Im}\hat{a}(i\xi)}{\xi|\hat{a}(i\xi)|^2}\right) = 1/a(0) > 0. \tag{3.5}$$

(3.3),(3.4) and (3.5) imply (3.1). In a similar way, we can prove (3.2). The proof of the lemma is completed.

Lemma 3.2. *Under the assumptions of Lemma 3.1, the kernel $k(t)$ in (2.22) satisfies*

i) $k(t) \in C^2([0,\infty))$,

ii) $k(t) = k_\infty + K(t)$, *where* $K^{(m)}(t) \in L^1(0,\infty)$, $m \leq 2$, *and*

$$k_\infty = \begin{cases} \hat{a}^{-1}(0), & if\ a(t) \in L^1(0,\infty), \\ 0, & if\ a(t) \notin L^1(0,\infty), \end{cases}$$

iii)

$$\int_0^T u(t)\int_0^t K(t-\tau)u(\tau)d\tau dt \geq (\alpha - a^{-1}(0))\int_0^t u^2(t)dt,$$
$$\forall T > 0 \ \ and \ \ u \in C([0,T]).$$

Proof. For the proof of assertions i) and ii), we refer to [2,4,5]. We note that our assumptions (2.2) and (2.3) are weaker than the corresponding assumptions in [2,4,5], by reason that their results are used only in obtaining the assertion i) and ii). Now we prove assertion iii). Setting

$$u_T(t) = \begin{cases} u(t), & t \in [0,T], \\ 0, & t < 0 \text{ or } t > T, \end{cases}$$

$$K^*(t) = \begin{cases} K(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

we have

$$\int_0^T u(t) \int_0^t K(t-\tau)u(\tau)d\tau dt = \int_{\infty}^{\infty} \tilde{u}_T(\xi)\overline{\tilde{K}^*(\xi)\tilde{u}_T(\xi)}d\xi$$
$$= \frac{1}{2\pi}\int_{\infty}^{\infty} \hat{K}(i\xi)\tilde{u}_t(\xi)|^2 d\xi, \tag{3.6}$$

where

$$\tilde{K}^*(\xi) = \int_{-\infty}^{\infty} e^{-it\xi}K^*(t)dt = \hat{K}(i\xi),$$

$$\tilde{u}_T(\xi) = \int_{-\infty}^{\infty} e^{-it\xi}u_T(t)dt.$$

From (2.21) and assertion ii) of the lemms, we find that

$$\hat{K}(i\xi) = \begin{cases} -\dfrac{i\overline{\hat{a}(i\xi)}}{\xi|\hat{a}(i\xi)|^2} - \dfrac{1}{a(0)} - \dfrac{1}{ia(0)\xi}, & \text{if } a \in L^1(0,\infty), \\ \dfrac{1}{a_\infty + i\xi\hat{A}(i\xi)} - \dfrac{1}{a(0)}, & \text{if } a \notin L^1(0,\infty). \end{cases}$$

Thus we have

$$\mathrm{Re}\hat{K}(i\xi) = \begin{cases} -\dfrac{\mathrm{Im}\hat{a}(i\xi)}{\xi|\hat{a}(i\xi)|^2} - \dfrac{1}{a(0)}, & \text{if } a \in L^1(0,\infty), \\ \dfrac{a_\infty - \xi\mathrm{Im}\hat{A}(i\xi)}{|a_\infty + i\xi\hat{A}(i\xi)|^2} - \dfrac{1}{a(0)}, & \text{if } a \notin L^1(0,\infty). \end{cases}$$

Substituting (3.6) into (3.5) and using Lemma 3.1, we conclude that

$$\int_0^T u(t)\int_0^t K(t-\tau)u(\tau)d\tau \geq (\alpha - \frac{1}{a(0)})\frac{1}{2\pi}\int_0^T |\tilde{u}_T(\xi)|^2 d\xi$$
$$= (\alpha - \frac{1}{a(0)})\int_0^T |u(t)|^2 dt.$$

This completes the proof of the lemma.

Proof of Proposition 2.1. Taking the compatibility condition (2.5) and assertion ii) of Lemma 3.2 into account, we can write the boundary condition (2.22) as

$$\sigma(u_x) + \frac{1}{a(0)}u_t + \int_0^t K(t-\tau)u_t(x,\tau)d\tau + k_\infty u$$
$$= k_\infty g(t) + \frac{1}{a(0)}g'(t) + \int_0^t K(t-\tau)g'(\tau)d\tau, \quad x = L. \tag{3.7}$$

We multiply equation (1.1) by u and integrate over $[0, L] \times [0, t]$, using integration by parts, (1.2) and (3.7) to obtain

$$\begin{aligned} &E_1(t) + \int_0^t u_t(L,s)(\frac{1}{a(0)}u_t(L,s) + \int_0^s K(t-\tau)u_t(L,\tau)d\tau)ds \\ =&E_1(0) + k_\infty \int_0^t u_t(L,s)g(s)ds + \frac{1}{a(0)}\int_0^t u(L,s)g'(s)ds \\ &+ \int_0^t u_t(L,s)\int_0^s K(s-\tau)g'(\tau)d\tau ds. \end{aligned} \tag{3.8}$$

Using the assertion iii) of Lemma 3.2 and inequality

$$|a_1a_2| \le \nu|a_1|^2 + \frac{1}{4\nu}|a_2|^2, \quad \forall \nu > 0$$

from (3.8) we obtain

$$\begin{aligned} &E_1(t) + \alpha \int_0^t u_t^2(L,s)ds \\ \le&E_1(0) + \nu\left(k_\infty + \frac{1}{a(0)} + 1\right)\int_0^t u_t^2(L,s)ds \\ &+ \frac{1}{4\nu}\left(k_\infty + \frac{1}{a(0)} + 1\right)\int_0^t \left(g^2(s) + g'^2(s) + \left(\int_0^s K(s-\tau)g'(\tau)d\tau\right)^2\right)ds. \end{aligned} \tag{3.9}$$

Noticing

$$\int_0^t \left(\int_0^s K(s-\tau)g'(\tau)d\tau\right)^2 ds \le \|K\|^2_{L^1(0,\infty)}\int_0^t g'^2(s)ds$$

and taking ν sufficiently small, we find that

$$E_1(t) + \alpha_3\int_0^t u_t^2(L,s)ds \le E_1(0) + C_3\int_0^t (g^2(s) + g'^2(s))ds \tag{3.10}$$

for some constants α_3 and C_3.

Differentiating (1.1) and (3.7) with respect to t, we can find that

$$u_{ttt} - \sigma(u_x)_{xt} = 0, \tag{3.11}$$

$$\begin{aligned} &\sigma(u_x)_t + \frac{1}{a(0)}u_{tt} + \int_0^t K(t-\tau)u_{tt}(x,\tau)d\tau + k_\infty u_t + K(t)u_t(x,0) \\ =&k_\infty g'(t) + \frac{1}{a(0)}g''(t) + \int_0^t K(t-\tau)g''(\tau)d\tau + K(t)g'(0), \quad x = L. \end{aligned} \tag{3.12}$$

We multiply (3.11) by u_{tt} and integrate over $[0, L] \times [0, t]$, using integration by parts, (1.2) and (3.12) to obtain

$$\begin{aligned} &E_2(t) + \int_0^t u_{tt}(L.s)\left(\frac{1}{a(0)}u_{tt}(L,s) + \int_0^s K(s-\tau)u_{tt}(L,\tau)d\tau\right)ds \\ =&E_2(0) + \frac{1}{2}\int_0^t\int_0^L \sigma''(u_x)u_{xt}^3(x,s)dxds + (g'(0) - u_t(L,0))\int_0^t K(s)u_{tt}(L,s)ds \\ &+ \int_0^t \left(k_\infty g'(s) + \frac{1}{a(0)}g''(s) + \int_0^s K(s-\tau)g''(\tau)d\tau\right)u_{tt}(L,s)ds, \end{aligned} \tag{3.13}$$

Following the procedure used to derive (3.10) from (3.8) and noticing

$$\left|\frac{1}{2}\int_0^t\int_0^L \sigma''(u_x)u_{xt}^3(x,s)dxds\right| \le C_4\mu(t)\int_0^t\int_0^L u_{xt}^3(x,s)dxds,$$

we conclude from (3.13) that

$$E_2(t)+\alpha_5\int_0^t u_{tt}^2(L,s)ds \le C_5(E_2(0)+u_t^2(L,0)+g'^2(0))+C_5\int_0^t(g'^2+g''^2)(s)ds$$
$$+C_5\mu(t)\int_0^t\int_0^L u_{xt}^2(x,s)dxds. \tag{3.14}$$

Differentiating (3.12) with respect to t, we have

$$\sigma(u_x)_{tt}+\frac{1}{a(0)}u_{ttt}+\int_0^t K(t-\tau)u_{ttt}(x,\tau)d\tau+K(t)u_{tt}(L,0)+K'(t)u_t(L,0)+k_\infty u_{tt}$$
$$=k_\infty g''(t)+\frac{1}{a(0)}g'''(t)+\int_0^t K(t-\tau)g'''(\tau)d\tau+K(t)g''(0)+K'(t)g'(0),\quad x=L. \tag{3.15}$$

For $h>0$, we apply the difference operator Δ_h defined by

$$(\Delta_h w)(x,t)=w(x,t+h)-w(x,t)$$

to (3.11), multiply the result by $\Delta_h u_{tt}$ and integrate over $[0,L]\times[0,t]$. After appropriate integrations by parts we divide the resulting equation by h, let $h\downarrow 0$, and then use (1.2) and (3.15). The result of this computation is

$$E_3(t)+\int_0^t u_{ttt}(L,s)\Big(\frac{1}{a(0)}u_{ttt}(L,s)+\int_0^t K(s-\tau)u_{ttt}(x,\tau)d\tau\Big)ds$$
$$=E_3(0)-\int_0^t(u_{tt}(L,0)K(s)+u_t(L,0)K'(s))u_{ttt}(L,s)ds$$
$$+\int_0^t(g''(0)K(s)+g'(0)K'(s))u_{ttt}(L,s)ds$$
$$+\int_0^t\left(k_\infty g''(s)+\frac{1}{a(0)}g'''(s)+\int_0^s K(s-\tau)g'''(\tau)d\tau\right)u_{ttt}(L,s)ds$$
$$+\int_0^t\int_0^L\left(\frac{1}{2}\sigma''(u_x)u_{xt}u_{xtt}^2+\sigma'''(u_x)u_{xx}u_{xt}^2u_{ttt}+2\sigma''(u_x)u_{xt}u_{xxt}u_{ttt}\right)(x,s)dxds$$
$$-\int_0^t\sigma''(u_x)u_{xt}^2u_{ttt}(L,s)ds. \tag{3.16}$$

A similar argument used to derive (3.14) from (3.13) yields

$$E_3(t)+\alpha_6\int_0^t u_{ttt}^2(L,s)ds$$
$$\le C_6(E_3(0)+u_t^3(L,0)+u_{tt}^2(L,0)+g'^2(0)+g''^2(0))+C_6\int_0^t(g''^2(s)+g'''^2(s))ds$$
$$+C_6(\mu(t)+\mu^2(t))\int_0^t\int_0^L(u_{xx}^2+u_{xt}^2+u_{xtt}^2+u_{xxt}^2+u_{ttt}^2)(x,s)dxds. \tag{3.17}$$

From the compatibility conditions (2.6) and (2.7), we have

$$g'^2(0)+g''^2\le C_7U_0(u_0,u_1). \tag{3.18}$$

Now we obtain the energy estimate (2.18) from (3.9), (3.13), (3.14) amd (3.17). The proof of Proposition 2.1 is completed.

§4. Proof of Proposition 2.2

Let u be a solution of Problem (P) in $C^3([0,L]\times[0,L])$, and $T>4T^*$.

Lemma 4.1.

$$\int_0^L (u_x^2+u_t^2)(x,t)dx \le C_8\int_{t-2T^*}^t (u_x^2+u_t^2)(L,\tau)d\tau + C_8\mu(t)\int_{t-2T^*}^t\int_0^L (u_x^2+u_t^2)(x,\tau)dxd\tau,\quad \forall t\in[2T^*,T], \tag{4.1}$$

$$\int_0^L (u_x^2+u_t^2)(x,t)dx \le C_8\int_t^{t+2T^*} (u_x^2+u_t^2)(L,\tau)d\tau + C_8\mu(t)\int_t^{t+2T^*}\int_0^L (u_x^2+u_t^2)(x,\tau)dxd\tau,\quad \forall t\in[0,2T^*], \tag{4.2}$$

Proof. Introducing the Riemann invariants

$$R=\frac{1}{2}\left(u_t-\int_0^{u_x}\sqrt{\sigma'(\eta)}d\eta\right),$$
$$S=\frac{1}{2}\left(u_t+\int_0^{u_x}u_x\sqrt{\sigma'(\eta)}d\eta\right),$$

equation (1.1) becomes

$$R_t+\lambda(S-R)R_x=0,\quad S_t-\lambda(S-R)S_x=0, \tag{4.3}$$

where $\lambda(S-R)=\sqrt{\sigma(F^{-1}(S-R))}$ and function F is defined by $F(\alpha)=\int_0^\alpha\sqrt{\sigma'(\eta)}d\eta$.

For $t\ge 2T^*$, let the backward 1-characteristic curve $l_1$through (L,t) meet line $x=0$ at $(0,\tau_1)$ and the backward 2-characteristic curve l_2 through $(0,\tau_1)$ meet line $x=L$ at (L,τ_2). We denote by Ω_1 the domain bounded by lines $\tau=t$, $x=0$ and curve l_1, and by Ω_2 the domain bounded by line $x=0$, curve l_1 and l_2 .

From (4.3), we have

$$\iint_{\Omega_1}((R_t+\lambda R_x)R+(S_t-\lambda S_x)S)dxd\tau=0. \tag{4.4}$$

Using integrations by parts and noticing that $R^2-S^2=0$ at line $x=0$, we obtain

$$\frac{1}{2}\int_0^L (R^2+S^2)(x,s)dx+\frac{1}{2}\int_{l_1}((R^2+S^2)\cos(\vec{n},\tau) + \lambda(R^2-S^2)\cos(\vec{n},x))dl-\frac{1}{2}\iint_{\Omega_1}\lambda'(R_x-S_x)(R^2-S^2)dxd\tau=0. \tag{4.5}$$

It is not difficult to verify that

$$\frac{1}{2}\int_{l_1}((R^2+S^2)\cos(\vec{n},\tau)+\lambda(R^2-S^2)\cos(\vec{n},x))dl=-\int_{l_1}\frac{\lambda}{\sqrt{1+\lambda^2}}S^2dl \tag{4.6}$$

and

$$\left|\iint_{\Omega_1}\lambda'(R_x-S_x)(R^2-S^2)dxd\tau\right|\le C_9\mu(t)\iint_{\Omega_1}(R^2+S^2)(x,\tau)dxd\tau. \tag{4.7}$$

Combining (4.5),(4.6) and (4.7), we have the estimate

$$\int_0^L (R^2+S^2)(x,t)dx \le 2\int_{l_1}\frac{\lambda}{\sqrt{1+\lambda^2}}Sdl + C_9\mu(t)\iint_{\Omega_1}(R^2+S^2)(x,\tau)dxd\tau. \tag{4.8}$$

Similarly, from

$$\iint_{\Omega_2}(S_t-\lambda S_x)Sdxd\tau = 0 \tag{4.9}$$

we can obtain estimate

$$\int_{l_1}\frac{\lambda}{\sqrt{1+\lambda^2}}Sdl \le \frac{1}{2}\int_{\tau_2}^t \lambda S^2(L,\tau)d\tau + C_{10}\mu(t)\iint_{\Omega_2}S^2dxd\tau. \tag{4.10}$$

Now (4.1) follows immediately from (4.8) and (4.10).

Instead of backward characteristic curves taking forward characteristic curves and proceeding as in the derivation of (4.1), we can obtain (4.2). The proof is completed.

Applying a similar argument, as in the proof of Lemma 4.1, to the differentiated equations

$$\begin{aligned} R_{tt}+\lambda R_{xt}+\lambda R_x &= 0,\\ S_{tt}-\lambda S_{xt}-\lambda_t S_x &= 0,\\ R_{ttt}+\lambda R_{xtt}+2\lambda_t R_{xt}+\lambda_{tt}R_x &= 0 \end{aligned}$$

and

$$S_{ttt}-\lambda S_{xtt}-2\lambda_t S_{xt}-\lambda_{tt}S_x = 0,$$

we can prove the following lemmas.

Lemma 4.2.

$$\begin{aligned} &\int_0^L (u_{xt}^2+u_{tt}^2)(x,t)dx\\ \le& C_{11}\int_{t-2T^*}^t (u_{xt}^2+u_{tt}^2)(L,\tau)d\tau + C_{11}\mu(t)\int_{t-2T^*}^t\int_0^L (u_{xx}^2+u_{xt}^2+u_{tt}^2)(x,\tau)dxd\tau,\\ &\forall t\in[2T^*,T]. \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\int_0^L (u_{xt}^2+u_{tt}^2)(x,t)dx\\ \le& C_{11}\int_t^{t+2T^*} (u_{xt}^2+u_{tt}^2)(L,\tau)d\tau + C_{11}\mu(t)\int_t^{t+2T^*}\int_0^L (u_{xx}^2+u_{xt}^2+u_{tt}^2)(x,\tau)dxd\tau,\\ &\forall t\in[0,2T^*]. \end{aligned} \tag{4.12}$$

Lemma 4.3.

$$\begin{aligned} &\int_0^L (u_{xtt}^2+u_{ttt}^2)(x,t)dx \le C_{12}\int_{t-2T^*}^t (u_{xtt}^2+u_{ttt}^2)(L,\tau)d\tau\\ &+C_{12}(\mu(t)+\mu^2(t))\int_{t-2T^*}^t\int_0^L (u_{xt}^2+u_{tt}^2+u_{xxt}^2+u_{xtt}^2+u_{ttt}^2)(x,\tau)dxd\tau,\\ &\forall t\in[2T^*,T], \end{aligned} \tag{4.13}$$

$$\int_0^L (u_{xtt}^2 + u_{ttt}^2)(x,t)dx \le C_{12}\int_t^{t+2T^*} (u_{xtt}^2 + u_{ttt}^2)(L,\tau)d\tau$$
$$+ C_{12}(\mu(t)+\mu^2(t))\int_t^{t+2T^*}\int_0^L (u_{xt}^2 + u_{tt}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2)(x,\tau)dxd\tau,$$
$$\forall t \in [0, 2T^*]. \tag{4.14}$$

Proof of Proposition 2.2. From Lemmas 4.2, 4.3 and (1.1), we find that

$$\int_{2T^*}^t \int_0^L (u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2)(x,\tau)dxd\tau$$
$$\le C_{13}\int_{2T^*}^t \int_{s-2T^*}^s (u_{xt}^2 + u_{tt}^2 + u_{xtt}^2 + u_{ttt}^2)(L,\tau)d\tau ds$$
$$+ C_{13}(\mu(t)+\mu^2(t))\int_{2T^*}^t \int_{s-2T^*}^s \int_0^L (u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2)(x,\tau)dxd\tau ds$$
$$\le 2T^*C_{13}\int_0^t (u_{xt}^2 + u_{tt}^2 + u_{xtt}^2 + u_{ttt}^2)(L,\tau)d\tau$$
$$+ 2T^*C_{13}(\mu(t)+\mu^2(t))\int_0^t\int_0^L (u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2)(x,\tau)dxd\tau$$

and

$$\int_0^{2T^*}\int_0^t (u_{xx}^2 + \cdots + u_{ttt}^2)(x,\tau)dxd\tau \le 2T^*C_{13}\int_0^t (u_{xt}^2 + u_{tt}^2 + u_{xtt}^2 + u_{ttt}^2)(L,\tau)d\tau$$
$$+ 2T^*C_{13}(\mu(t)+\mu^2(t))\int_0^t\int_0^L (u_{xx}^2 + \cdots + u_{ttt}^2)(x,\tau)dxd\tau.$$

Now we get

$$\int_0^t\int_0^L (u_{xx}^2 + \cdots + u_{ttt}^2)(x,\tau)dxd\tau \le C_{14}\int_0^t (u_{xt}^2 + u_{tt}^2 + u_{xtt}^2 + u_{ttt}^2)(L,\tau)d\tau$$
$$+ C_{14}(\mu(t)+\mu^2(t))\int_0^t\int_0^L (u_{xx}^2 + \cdots + u_{ttt}^2)(x,\tau)dxd\tau. \tag{4.15}$$

From (3.12) and (3.15), we find that

$$\int_0^t u_{xt}^2(L,\tau)d\tau \le C_{15}\int_0^t (u_t^2 + u_{tt}^2)(L,\tau)d\tau$$
$$+ C_{15}\int_0^t (g'^2 + g''^2)(\tau)d\tau + C_{15}(u_t^2(L,0) + g'^2(0)) \tag{4.16}$$

and

$$\int_0^t u_{xtt}^2(L,\tau)d\tau \le C_{15}\mu^2(t)\int_0^t u_{xt}^2(L,\tau)d\tau$$
$$+ C_{15}\int_0^t (u_{tt}^2 + u_{ttt}^2)(L,\tau)d\tau + C_{15}\int_0^t (g''^2 + g'''^2)(\tau)d\tau$$
$$+ C_{15}(u_t^2(L,0) + u_{tt}^2(L,0) + g'^2(0) + g''^2(0)). \tag{4.17}$$

The compatibility conditions (2.6) and (2.7) yield

$$g'^2(0) + g''^2(0) \le C_{16}(u_x^2(L,0) + u_t^2(L,0) + u_{xt}^2(L,0) + u_{tt}^2(L,0)). \tag{4.18}$$

Using Sobolev embedding theorem, we have

$$u_x^2(L,0)+u_t^2(L,0)+u_{xt}^2(L,0)+u_{tt}^2(L,0)\le C_{17}U_0(u_0,u_1). \tag{4.19}$$

From (4.15)-(4.19), we obtain

$$\begin{aligned}&\int_0^t\int_0^L(u_{xx}^2+\cdots++u_{ttt}^2)(x,\tau)dxd\tau\\ \le& C_{18}U_0(u_0,u_1)+C_{18}\int_0^t(u_t^2+u_{tt}^2+u_{ttt}^2)(L,\tau)d\tau+C_{18}\int_0^t(g'^2+g''^2+g'''^2)(s)ds\\ &+C_{18}(\mu(t)+\mu^2(t))\int_0^t\int_0^L(u_{xx}^2+\cdots+u_{ttt}^2)(x,s)dxds.\end{aligned} \tag{4.20}$$

The boundary condition (3.7) can be written as

$$\begin{aligned}&\sigma(u_x)_t+k_\infty Lu_x+\frac{1}{a(0)}u_t+\int_0^tK(t-\tau)u_t(x,\tau)d\tau-\int_0^L\int_x^Lu_{xx}(\xi,t)d\xi dx\\ =&k_\infty g'(t)+\frac{1}{a(0)}g'(t)+\int_0^tK(t-\tau)g'(\tau)d\tau,\quad x=L.\end{aligned} \tag{4.21}$$

Noticing

$$|\sigma(u_x)+k_\infty Lu_x|\le(\sigma_1+k_\infty L)|u_x|,$$

from (4.21) we obtain

$$\int_0^tu_x^2(L,t)dt\le C_{19}\int_0^t(u_t^2(L,t)+g^2(t)+g'^2(t))dt+C_{19}\int_0^t\int_0^Lu_{xx}^2(x,t)dxdt. \tag{4.22}$$

Combining (4.20), Lemma 4.1 and (4.22), we conclude (2.19). The proof of Proposition 2.2 is completed.

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REFERENCES

[1] Alber, H. D. & Cooper, J., A free boundary value problem for a quasilinear hyperbolic system (to appear in *J. fur die reine und augew. Math.*).

[2] Dafermos, C. M. & Nohel, J. A., Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, *Comm. Partial Diff. Equa.* , 4 (1979), 219-278.

[3] Greeberg, J. M. & Li Ta-tsien, The effect of boundary damping for the quasilinear wave equation, *J. Diff. Equa.* , **52** (1984), 66-75.

[4] MaCcamy, R. C., An integro-differential equation with application in heat flow, *Quart. Appl. Math.* , **35** (1977), 1-19.

[5] Macamy, R. C., A model for one-dimensional, nonlinear viscoelasticity, *Quart. Appl. Math.*, **35** (1977), 21-33.