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HOW BIG ARE THE LAG INCREMENTS OF A 2-PARAMETER WIENER PROCESS? (II)**

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Abstract

The author investigated how big the lag increments of a 2-parameter Wiener process is in [1]. In this paper the limit inferior results for the lag increments are discussed and the same results as the Wiener process are obtained. For example, if

$$\lim_{T\to\infty}\{\log T/a_T+\log(\log b_T/a_T^{\frac{1}{2}}+1)\}/\log\log T=r,\quad 0\le r\le\infty,$$

then

$$\varliminf_{T\to\infty}\sup_{a_T\le t\le T}\sup_{t\le s\le T}\sup_{R\in L_s^*(t)}|W(R)|/d(T,t)=\alpha_r\quad \text{a.s.},$$

$$\varliminf_{T\to\infty}\sup_{a_T\le t\le T}\sup_{R\in \widetilde{L}_T(t)}|W(R)|/d(T,t)=\alpha_r\quad \text{a.s.},$$

where $\alpha_r=(r/(r+1))^{\frac{1}{2}}$, $L_s^*(t)$ and $\widetilde{L}_T(t)$ are the sets of rectangles which satisfy some conditions. Moreover, the limit inferior results of another class of lag increments are discussed.

Keywords Wiener process, Lag increments, Inferior limit
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§1.

Let $\{W(x,y);0\le x,y<\infty\}$ be a 2-parameter Wiener process, and let $b_T(\ge T^{1/2})$ be a non-decreasing function of T. Put $R=[x_1,x_2]\times[y_1,y_2]$, $\lambda(R)=(x_2-x_1)(y_2-y_1)$,

$$D_T=\{(x,y):0\le x,y\le b_T,xy\le T\}.$$

For $0<t\le T$, denote

$$L_T^*(t)=L_T^*(t,b_T,T)=\{R:R\subset D_T,x_2y_2=T,\lambda(R)=t\},$$
$$L_T(t)=L_T(t,b_T,T)=\{R:R\subset D_T,x_2y_2=T,\lambda(R)\le t\},$$
$$\widetilde{L}_T(t)=\widetilde{L}_T(t,b_T,T)=\{R:R\subset D_T,x_2y_2=t',\lambda(R)\le t,\quad t,t'\le T\},$$
$$d(T,t)=\{2t(\log T/t+\log(1+\log b_T/t^{\frac{1}{2}})+\log\log t)\}^{1/2}.$$

We discussed how big the lag increments of a 2-parameter Wiener process are and proved the following theorem in [1].

Theorem 1. *Let* $\gamma_T=d^{-1}(T,T)=\{2T(\log(1+\log b_T/T^{1/2})+\log\log T)\}^{-1/2}$. *If*
(a) γ_T *is a non-increasing function of* T,

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(b) *for any* $\varepsilon > 0$ *there exists a* $\theta_0 = \theta_0(\varepsilon) > 1$ *such that*

$$\overline{\lim_{k\to\infty}}\, \gamma_{\theta^k}/\gamma_{\theta^{k+1}} \le 1 + \varepsilon$$

if $1 < \theta \le \theta_0$, *then*

$$\overline{\lim_{T\to\infty}} \sup_{0<t\le T} \sup_{R\in L_T^*(t)} |W(R)|/d(T,t) = 1 \quad a.s.,$$

$$\overline{\lim_{T\to\infty}} \sup_{0<t\le T} \sup_{R\in L_T(t)} |W(R)|/d(T,t) = 1 \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{R\in \widetilde{L}_T(t)} |W(R)|/d(T,t) = 1 \quad a.s.$$

From this theorem, we can immediately write the following

Theorem 1′. *Suppose that* γ_T *is defined as in Theorem* 1. *Then, for* $0 < a_T \le T$, *we have*

$$\overline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{R\in L_T^*(t)} |W(R)|/d(T,t) = 1 \quad a.s.,$$

$$\overline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{R\in L_T(t)} |W(R)|/d(T,t) = 1 \quad a.s.,$$

$$\overline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{R\in \widetilde{L}_T(t)} |W(R)|/d(T,t) = 1 \quad a.s.$$

In this article, we will discuss its inferior limit and get following two theorems that same as the results for the Wiener process in [2].

Theorem 1.1. *Let* $0 < a_T \le T$ *be a non-decreasing function of* T. *If*

(iv) $\lim_{T\to\infty} (\log T/a_T + \log(1 + \log b_T/a_T^{1/2}))/\log\log T = r,\ 0 \le r \le \infty,$

then

$$\underline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{t\le s\le T} \sup_{R\in L_s^*(t)} |W(R)|/d(T,t) = \alpha_r, \tag{1.1}$$

$$\underline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{R\in \widetilde{L}_T(t)} |W(R)|/d(T,t) = \alpha_r, \tag{1.2}$$

where

$$\alpha_r = \begin{cases} \sqrt{\dfrac{r}{r+1}} & \text{if } 0 \le r < \infty, \\ 1 & \text{if } r = \infty. \end{cases} \tag{1.3}$$

Theorem 1.2. *Let* $d_T (\ge \sqrt{T})$ *be a non-decreasing function of* T. *If*

(iv′) $\lim_{T\to\infty} (\log(T + a_T)/T + \log(1 + \log b_T/T^{1/2}))/\log\log T = r,\ 0 \le r \le \infty,$

then

$$\underline{\lim_{T\to\infty}} \sup_{0\le t\le d_T} \sup_{R\in L_{t+T}^*(T)} |W(R)|/d(t+T,T) = \alpha_r \quad a.s., \tag{1.4}$$

$$\underline{\lim_{T\to\infty}} \sup_{0\le t\le d_T} \sup_{R\in \widetilde{L}_{t+T}(T)} |W(R)|/d(t+T,T) = \alpha_r \quad a.s. \tag{1.5}$$

Proof of Theorem 1.1. (1) We prove

$$\underline{\lim_{T\to\infty}} \sup_{a_T\le t\le T} \sup_{t\le s\le T} \sup_{R\in L_s^*(t)} |W(R)|/d(T,t) \ge \alpha_r \quad \text{a.s.} \tag{1.6}$$

It is clear that (1.6) holds true for $r = 0$. Let us now consider the case of $0 < r < \infty$, the case of $r = \infty$ is similar. We take a real number $\theta > 1$ and for any $\varepsilon > 0$ denote $\alpha^2 = r/(r+1) - \varepsilon$, $T_n = \theta^n$. Let $\lim_{T\to\infty} a_T/T = \rho$ and let $L = L(T)$ be the largest integer for which we have

$$\frac{T^{L+1}}{(T-a_T)^L b_T} \le b_T \quad \text{if } \rho < 1,$$

$$a_T^{1/2} M^{L+1} = T^{1/2} M^{L+1} \le b_T \quad \text{if } \rho = 1,$$

where constant $M(= 1/\varepsilon) > 1$. Define the rectangles

$$S_i(T) = \begin{cases} \left[\left(\dfrac{T-a_T}{T}\right)^{i+1} b_T, \left(\dfrac{T-a_T}{T}\right)^{i} b_T\right] \times \left[0, \dfrac{T^{i+1}}{(T-a_T)^i b_T}\right] & \text{if } \rho < 1, \\ [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}] & \text{if } \rho = 1. \end{cases}$$

When $0 < \rho < 1$, it is easy to see that $L \sim c\frac{T}{a_T}\log\frac{b_T^2}{T}$ and $S_i \bigcap S_j = \emptyset$ if $i \ne j$, $S_i \in L_T^*(a_T)$. Then we have

$$\begin{aligned} I_T &:= P\left\{\max_{0\le i\le L} |W(S_i)|/\sqrt{a_T} \le \alpha(2\phi_T)^{1/2}\right\} \\ &\le \{1 - c\phi_T^{-1/2}(1-\phi_T^{-1})\exp(-\alpha^2\phi_T)\}^{L+1}, \end{aligned}$$

where $\phi_T = \log T/a_T + \log(1 + \log b_T/a_T^{1/2}) + \log\log a_T$. By condition (iv) we have

$$\begin{aligned} I_T &\le c\exp\{-(\log T)^{(r-\varepsilon)(1-\alpha^2)-\alpha^2}\} \\ &\le c\exp\{-(\log T)^{\varepsilon(r+\alpha^2)}\}. \end{aligned}$$

By using the Borel-Cantelli lemma, we obtain

$$\varliminf_{n\to\infty} \max_{0\le i\le L} |W(S_i(n))|/d(T_{n+1}, a_{T_{n+1}}) \ge \alpha.$$

Notice that for $T_n \le T \le T_{n+1}$

$$\begin{aligned} &\varliminf_{T\to\infty} \sup_{a_T\le t\le T} \sup_{t\le s\le T} \sup_{R\in L_s^*(t)} |W(R)|/d(T,t) \\ &\ge \varliminf_{n\to\infty} \sup_{R\in L_{T_n}^*(a_{T_n})} |W(R)|/d(T_{n+1}, a_{T_{n+1}}) \\ &\quad - \varlimsup_{n\to\infty} \sup_{R\in \widetilde{L}_{T_n}(a_T - a_{T_n})} |W(R)|/d(T_n, a_{T_n}) \\ &\ge r/(r+1) - 2\varepsilon. \end{aligned}$$

By the arbitrariness of ε, we gets (1.6).

When $\rho = 1$, we have

$$\begin{aligned} I_T &:= P\{\max_{0\le i\le L} W(S_i)/\sqrt{T(1-\varepsilon)} \le \frac{\alpha}{\sqrt{1-\varepsilon}}(2(\log(\log\frac{b_T}{\sqrt{T}} + 1) + \log\log T))^{\frac{1}{2}}\} \\ &\le c\exp\{-c(\log T)^{\varepsilon(\alpha^2+\varepsilon)/(1-\varepsilon)}\}. \end{aligned}$$

By the same argument as above, one also gets (1.6).

(2) In order to prove Theorem 1.1, now it is enough only to show that

$$\varlimsup_{T\to\infty} \sup_{a_T\le t\le T} \sup_{R\in \widetilde{L}_T(t)} |W(R)|/d(T,t) \le \alpha_r \quad \text{a.s.} \tag{1.7}$$

It is clear that (1.7) holds true for $r=\infty$. In the case of $0 \le r < \infty$, we need only to prove that

$$\overline{\lim_{n\to\infty}} \sup_{a_{T_n}\le t\le T_n} \sup_{R\in\widetilde{L}_{T_n}(t)} |W(R)|/d(T_n,t) \le \left(\frac{r}{r+1}\right)^{\frac12} \quad \text{a.s.} \tag{1.8}$$

for $T_n = e^{e^n}$. For any $\varepsilon > 0$, denote $\alpha^2 = \frac{r}{r+1} + 2\varepsilon$. We take real number $\theta > 1$ such that $\frac{2\alpha^2}{(2+\varepsilon)\theta} > \frac{r}{r+1} + \varepsilon$. Let $t_k = \theta^k a_{T_n}$, $k_n = [\log_\theta(T_n/a_{T_n})]$. We have

$$\begin{aligned}
&\sup_{a_{T_n}\le t\le T_n} \sup_{R\in\widetilde{L}_{T_n}(t)} |W(R)|/d(T_n,t)\\
&\le \max_{0\le k\le k_n} \sup_{t_k\le t\le t_{k+1}} \sup_{R\in\widetilde{L}_{T_n}(t\wedge T_n)} |W(R)|/d(T_n,t_k)\\
&\le \max_{0\le k\le k_n} \sup_{R\in\widetilde{L}_{T_n}(t_{k+1}\wedge T_n)} |W(R)|/d(T_n,t_k) =: \max_{0\le k\le k_n} A_{nk}.
\end{aligned}$$

An inspection of the proof of [3, Theorem 1.12.6] shows that for large n we have

$$\begin{aligned}
P\{A_{nk}\ge\alpha\} &\le c\frac{T_n}{t_{k+1}}\left(1+\log\frac{T_n}{t_{k+1}}\right)\left(1+\log\frac{b_{T_n}}{\sqrt{t_{k+1}}}\right)\\
&\quad\times\exp\left\{-\frac{2\alpha^2}{(2+\varepsilon)\theta}\left(\log\frac{T_n}{t_k}+\log\left(1+\log\frac{b_{T_n}}{\sqrt{t_k}}\right)+\log\log t_k\right)\right\}\\
&\le c(\log T_n)^{r+\varepsilon-(r-\varepsilon+1)\frac{2\alpha^2}{(2+\varepsilon)\theta}}\theta^{-k(1-\alpha^2)}\\
&\le ce^{-n\varepsilon(\frac{r^2}{1+r}-\varepsilon)}\theta^{-k(1-\alpha^2)},
\end{aligned}$$

where the following inequality has been used which follows from condiction (iv),

$$(\log T_n)^{r-\varepsilon} \le (T_n/a_{T_n})(1+\log(b_{T_n}/a_{T_n}^{1/2})) \le (\log T_n)^{r+\varepsilon}.$$

By using the Borel-Cantelli lemma, we can prove (1.7).

Proof of Theorem 1.2. (1) Let us first prove that

$$\underline{\lim}_{T\to\infty} \sup_{0\le t\le d_T} \sup_{R\in L^*_{t+T}(T)} |W(R)|/d(t+T,T) \ge \alpha_r \quad \text{a.s.} \tag{1.9}$$

It is clear that (1.9) holds true for $r=0$. We now consider the case of $0<r<\infty$, the case of $r=0$ is similar. For any $\varepsilon>0$, denote $\alpha^2 = r/(r+1) - 2\varepsilon$. We take real number $\theta > 1$ such that

$$\eta = \frac{2\varepsilon^2}{2+\varepsilon}\cdot\frac{\theta}{\theta-1} - 1 > 0, \quad \alpha^2\theta < \frac{r}{r+1} - \varepsilon.$$

Let $T_k = \theta^k$. If $T_k \le T \le T_{k+1}$, we have

$$\begin{aligned}
&\sup_{0\le t\le d_T} \sup_{R\in L^*_{t+T}(T)} |W(R)|/d(t+T,T)\\
&\ge \sup_{0\le t\le d_{T_k}} \sup_{R\in L^*_{t+T_k}(T_k)} |W(R)|/d(t+T_{k+1},T_{k+1})\\
&\quad - 4 \sup_{0\le t\le d_{T_k}} \sup_{R\in L^*_{t+T_{k+1}}(T_{k+1}-T_k)} |W(R)|/d(t+T_{k+1},T_{k+1})\\
&=: A_k - 4B_k.
\end{aligned}$$

For simplicity, we denote $\widetilde{b}_k = b_{T_k+d_{T_k}}$. If $\lim_{T\to\infty} T/(T+d_T) = \rho < 1$, let

$$L = \max\left\{j : \left(\frac{T_k+d_{T_k}}{d_{T_k}}\right)^{j+1} \le \widetilde{b}_k^2\right\} \sim c\frac{T_k+d_{T_k}}{T_k}\log\frac{\widetilde{b}_k^2}{T_k+d_{T_k}},$$

and

$$S_i(k)=\left[\left(\frac{d_{T_k}}{T_k+d_{T_k}}\right)^{i+1}\widetilde{b}_k,\left(\frac{d_{T_k}}{T_k+d_{T_k}}\right)^{i}\widetilde{b}_k\right]\times\left[0,\frac{(T_k+d_{T_k})^{i+1}}{\widetilde{b}_kT_k^i}\right],$$

$i=0,1,\cdots,L$. We have $S_i(k)\bigcap S_j(k)=\emptyset(i\neq j)$, $S_i(k)\in L^*_{T_k+d_{T_k}}(T_k)$. Denote $\phi_k=\log\frac{T_k+d_{T_k}}{T_k}+\log(1+\log\frac{\widetilde{b}_k}{\sqrt{T_k}})+\log\log T_k$. By (iv′) we have

$$\begin{aligned}I_k&:=P(A_k\le\alpha)\\&\le P\left\{\max_{0\le j\le L}|W(S_i(k))|/T_k^{1/2}\le\alpha(2\theta\phi_{k+1})^{1/2}\right\}\\&\le\{1-c\widetilde{\phi}_{k+1}^{1/2}(1-\phi_{k+1}^{-1})\exp(-\alpha^2\theta\phi_{k+1})\}^{L+1}\\&\le\exp\{-c(\log T_k)^{(r-\varepsilon)(1-\alpha^2\theta)-\alpha^2\theta}\\&\le c\exp(-k^{(r+\alpha^2)\varepsilon}).\end{aligned}$$

By using the Borel-Cantelli lemma, it implies

$$\varliminf_{k\to\infty}A_k\ge\alpha. \tag{1.10}$$

Let us now show that $\varlimsup_{k\to\infty}B_k\le\varepsilon$. Denote $t_k=2^nT_k$. Notice that

$$\begin{aligned}B_k\le&\max_{0\le n<\infty}\sup_{t_n\le t\le t_{n+1}}\sup_{R\in\widetilde{L}_{t+T_{k+1}}(T_{k+1}-T_k)}|W(R)|/d(t_n+T_{k+1},T_{k+1})\\&\vee\sup_{0\le t\le T_k}\sup_{R\in\widetilde{L}_{t+T_{k+1}}(T_{k+1}-T_k)}|W(R)|/d(T_{k+1},T_{k+1})\\&\triangleq B_{1k}\vee B_{2k}.\end{aligned}$$

It follows from [3, Theorem 1.12.6] and condition (iv′) that

$$P(B_{1k}\ge\varepsilon)\le\sum_{n=0}^{\infty}P\left\{\sup_{R\in\widetilde{L}_{t_{n+1}+T_{k+1}}(T_{k+1}-T_k)}\frac{|W(R)|}{\sqrt{T_{k+1}-T_k}}\ge\varepsilon(2\frac{\theta}{\theta-1}\phi_{n,k})^{1/2}\right\},$$

where $\phi_{n,k}=\log\frac{t_n+T_{k+1}}{T_{k+1}}+\log(1+\log b_{t_n+T_{k+1}}T_{k+1}^{-1/2})+\log\log T_{k+1}$,

$$\begin{aligned}P(B_{lk}\ge\varepsilon)\le&C\sum_{i=0}^{\infty}\frac{t_{n+1}+T_{k+1}}{T_{k+1}-T_k}(1+\log b_{t_{n+1}+T_{k+1}}T_{k+1}^{-1/2})\times\\&\times\left(1+\log\frac{t_{n+1}+T_{k+1}}{\sqrt{T_{k+1}-T_k}}\right)\exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{\theta}{\theta-1}\phi_{n,k}\right\}\\\le&C\sum_{n=1}^{\infty}2^nn(2^nk)^{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{\theta}{\theta-1}}\le C\sum_{n=1}^{\infty}2^{-n\eta_k-1-\eta}\\\le&Ck^{-1-\eta}.\end{aligned}$$

On the other hand, we also have

$$P(B_{2k} \geq \varepsilon) \leq P\left\{ \sup_{R\in\widetilde{L}_{T_k+T_{k+1}}(T_{k+1}-T_k)} \frac{|W(R)|}{\sqrt{T_{k+1}-T_k}} \geq \varepsilon\left(\frac{2\theta}{\theta-1}\left(\log\frac{1+\theta}{\theta}+\log(1+\log b_{T_k+T_{k+1}}T_{k+1}^{-1/2})+\log\log T_{k+1}\right)\right)^{1/2}\right\} \leq Ck^{-1-\eta}.$$

So, by using the Borel-Cantelli lemma, we have $\overline{\lim_{k\to\infty}} B_k \leq \varepsilon$; it combined with (1.10) proves that (1.9) holds true.

If $\rho = 1$, imitating the proof of Theorem 1.1 and the above, we obtain (1.9) again.

(2) In order to prove Theorem 6, we need only to prove that

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq d_T} \sup_{R\in\widetilde{L}_{t+T}(T)} |W(R)|/d(t+T,T) \leq \alpha_r \quad \text{a.s.} \tag{1.11}$$

It is clear that (1.11) holds true for $r=\infty$. For the case of $0 \leq r < \infty$, put $T_n = 2^{2^n}$. Let us show that

$$\overline{\lim_{n\to\infty}} \sup_{0\leq t\leq d_{T_n}} \sup_{R\in\widetilde{L}_{t+T_n}(T_n)} |W(R)|/d(t+T_n,T_n) \leq \alpha_r \quad \text{a.s.} \tag{1.12}$$

Denote $t_k = 2^k T_n$, $k_n = 1 + [\log_2 d_{T_n} T_n^{-1}]$. We have

$$\begin{aligned}
&\sup_{0\leq t\leq d_{T_n}} \sup_{R\in\widetilde{L}_{t+T_n}(T_n)} |W(R)|/d(t+T_n,T_n) \\
\leq &\max_{0\leq k\leq k_n} \max_{t_k\leq t\leq t_{k+1}} \sup_{R\in\widetilde{L}_{t+T_n}(T_n)} |W(R)|/d(t+T_n,T_n) \\
&\vee \sup_{0\leq t\leq T_n} \sup_{R\in\widetilde{L}_{t+T_n}(T_n)} |W(R)|/d(t+T_n,T_n) \\
\leq &\max_{0\leq k\leq k_n} \sup_{R\in\widetilde{L}_{(2^{k+1}+1)T_n}(T_n)} W(R)/d((2^k+1)T_n,T_n) \\
&\vee \sup_{R\in\widetilde{L}_{2T_n}(T_n)} |W(R)|/d(T_n,T_n) \triangleq D_n \vee E_n.
\end{aligned}$$

For any $\alpha \in (\sqrt{\frac{r}{r+1}},1)$, we take $\varepsilon > 0$ such that $\frac{2\alpha^2}{2+\varepsilon} > \frac{r}{r+1}+1$. It follows from [1, Theorem 1.12.6] and condition (iv$'$) that

$$\begin{aligned}
P(D_n \geq \alpha) \leq & Ck_n \cdot 2^{k_n(1-\frac{2\alpha^2}{2+\varepsilon})}(\log T_n)^{-\frac{2\alpha^2}{2+\varepsilon}} \\
\leq & Cn \cdot 2^{n((r+\varepsilon)(1-\frac{2\alpha^2}{2+\varepsilon})-\frac{2\alpha^2}{2+\varepsilon})} \leq Cn \cdot 2^{-n\varepsilon^2}, \\
P(E_n \geq \alpha) \leq & C(1+\log\frac{b_{2T_n}}{\sqrt{T_n}})\exp\left\{-\frac{2\alpha^2}{2+\varepsilon}\log((1+\log\frac{b_{2T_n}}{\sqrt{T_n}})\log T_n)\right\} \\
\leq & C2^{n((r+\varepsilon)(1-\frac{2\alpha^2}{2+\varepsilon})-\frac{2\alpha^2}{2+\varepsilon})} \leq C2^{-n\varepsilon^2}.
\end{aligned}$$

By using the Borel-Cantelli lemma, we prove that (1.12) is true.

§2.

From Theorems 3 and 4 in [1], we have the following result: Letting $0 < a_T \leq T$ be a non-decreasing function of T and a_T/T be a non-increasing function of T, if

(a) $\beta(t+a,a)$ is a non-increasing function of a,

(b) for any $\varepsilon>0$, there exists $\theta_0=\theta_0(\varepsilon)>1$ such that for any $1<\theta\le\theta_0$ and k

$$\sup_{t\ge 0}\beta(t+\theta^k,\theta^k)/\beta(t+\theta^{k+1},\theta^{k+1})\le\theta^{1/2}(1+\varepsilon),$$

$$\sup_{n\ge 1}\beta(\theta^{(n-1)k},\theta^k)/\beta(\theta^{nk},\theta^k)\le 1+\varepsilon,$$

and $\lambda(T,t)=\{2t((\log Tt^{-1})+\log(1+\log b_T t^{-1/2}))\}^{-1/2}$ satisfies

(a′) $\lambda(t+a,a)$ is a non-increasing function of a,

(b′) for any $\varepsilon>0$, there exists $\theta_0=\theta_0(\varepsilon)>1$ such that for any $1<\theta\le\theta_0$ and k

$$\sup_{t\ge 0}\lambda(t+\theta^k,\theta^k)/\lambda(t+\theta^{k+1},\theta^{k+1})\le\theta^{1/2}(1+\varepsilon),$$

$$\sup_{n\ge 1}\lambda(\theta^{(n-1)k},\theta^k)/\lambda(\theta^{nk},\theta^k)\le 1+\varepsilon,$$

then, if

(iv′) $\lim_{T\to\infty}\frac{\log Ta_T^{-1}+\log(1+\log b_T a_T^{-1/2})}{\log\log\log T}=\infty$

is satisfied, we have

$$\overline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{R\in L^*_{t+a_T}(a_T)}\lambda(t+a_T,a_T)|W(R)|=1 \quad \text{a.s.,} \tag{2.1}$$

$$\overline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{R\in \widetilde{L}_{t+a_T}(a_T)}\lambda(t+a_T,a_T)|W(R)|=1 \quad \text{a.s.} \tag{2.2}$$

Furthermore, we have the following theorem.

Theorem 2.1. *Suppose that* $0<a_T\le T$, $\beta(\cdot,\cdot)$, $\lambda(\cdot,\cdot)$ *are as above. If*

(v) $\lim_{T\to\infty}\frac{\log Ta_T^{-1}+\log(1+\log b_T a_T^{-1/2})}{\log\log T}=r$, $0\le r\le\infty$,

then

$$\overline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{R\in L^*_{t+a_T}(a_T)}\beta(t+a_T,a_T)|W(R)|=\alpha_r \quad \textit{a.s.,} \tag{2.3}$$

$$\overline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{R\in \widetilde{L}_{t+a_T}(a_T)}\beta(t+a_T,a_T)|W(R)|=\alpha_r \quad \textit{a.s.} \tag{2.4}$$

Proof. (1) At first, we show that

$$\overline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{R\in \widetilde{L}_{t+a_T}(a_T)}\beta(t+a_T,a_T)W(R)\le\alpha_r \quad \text{a.s.} \tag{2.5}$$

We know that (2.5) holds true for $r=\infty$ by Theorem 3 in [1]. If $0\le r<\infty$, let $a_{T_k}=\theta^k$. By the same argument as in [1], Theorem 2, we have

the left hand side of (2.5)

$$\le\overline{\lim_{k\to\infty}}\sup_{0\le t\le T_k-a_{T_k}}\sup_{R\in\widetilde{L}_{t+a_{T_k}}(a_{T_k})}\lambda(t+a_{T_k},a_{T_k})W(R)\frac{\beta(t+a_{T_k},a_{T_k})}{\lambda(t+a_{T_k},a_{T_k})}$$

$$\le 1\times\overline{\lim_{k\to\infty}}\sup_{0\le t\le T_k-a_{T_k}}\left(\frac{\log\frac{t+a_{T_k}}{a_{T_k}}+\log\left(1+\log\frac{b_{t+a_{T_k}}}{\sqrt{a_{T_k}}}\right)}{\log\frac{t+a_{T_k}}{a_{T_k}}+\log\left(1+\log\frac{b_{t+a_{T_k}}}{\sqrt{a_{T_k}}}\right)+\log\log(t+a_{T_k})}\right)^{1/2}$$

$$=\alpha_r,$$

which proves that (2.5) holds true.

(2) In order to prove Theorem 2.1, we need only to show that

$$\varliminf_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{R\in L^*_{t+a_T}(a_T)} \beta(t+a_T,a_T)|W(R)| \ge \alpha_r \quad \text{a.s.} \tag{2.6}$$

It is clear that (2.6) is true for $r=0$. If $0<r\le\infty$, by (v) we have

$$\frac{\lambda(T,a_T)}{\log\log\log T} = \frac{\lambda(T,a_T)}{\log\log T}\cdot\frac{\log\log T}{\log\log\log T} \to \infty.$$

It follows from (2.1) that for any positive number $\{T_n\}$, $\lim_{n\to\infty} T_n=\infty$, we have

$$\varliminf_{T_n\to\infty} \sup_{0\le t\le T_n-a_{T_n}} \sup_{R\in L^*_{t+a_{T_n}}(a_{T_n})} \lambda(t+a_{T_n},a_{T_n})|W(R)| = 1 \quad \text{a.s.}$$

Therefore for any positive number $\{T_n\}$, $T_n\uparrow\infty$,

$$\varliminf_{T_n\to\infty} \sup_{0\le t\le T_n-a_{T_n}} \sup_{R\in L^*_{t+a_{T_n}}(a_{T_n})} \beta(t+a_{T_n},a_{T_n})|W(R)|$$

$$\ge 1\times \varliminf_{n\to\infty} \sup_{0\le t\le T_n-a_{T_n}} \frac{\beta(t+a_{T_n},a_{T_n})}{\lambda(t+a_{T_n},a_{T_n})} = \varliminf_{n\to\infty}\frac{\beta(T_n,a_{T_n})}{\lambda(T_n,a_{T_n})} = \alpha_r.$$

By the arbitrariness of $\{T_n\}$ we can prove (2.6).

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