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A CLASS OF HOMOGENEOUS LEFT INVARIANT OPERATORS ON THE NILPOTENT LIE GROUP G^{d+2}**

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Abstract

This paper is devoted to a class of homogeneous left invariant operators L_λ on the nilpotent Lie group G^{d+2} of the form

$$L_\lambda = -\sum_{j=1}^{d} X_j^2 - i\sum_{m=1}^{2} \lambda_m T_m, \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2,$$

where $\{X_1, \cdots, X_d, T_1, T_2\}$ is a base of left invariant vector fields on G^{d+2}. With aid of harmonic analysis on nilpotent Lie groups and the method of increment operators, for all admissible L_λ, subelliptic estimate and an explicit inverse are given and the hypoellipticity and the global solvability are obtained. Also, the structure of the set of admissible points λ is described exhaustively.

Keywords Lie group, Homogeneous left invariant operators, Hypoellipticity, Global solvability

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§1. Introduction

Since the 70's, harmonic analysis on nilpotent Lie groups has become an active area, whcih affords a new powerful tool for the analysis of LPDOs. In recent fifteen years, a lot of important results have been obtained[1−5,11−14]. In particular, investigation for left invariant operators on the Heisenberg group H_n or more general nilpotent Lie groups is the most widespread.

This paper is devoted to a family of second order operators of the form

$$L_\lambda = -\sum_{j=1}^{d} X_j^2 - i\sum_{m=1}^{2} \lambda_m T_m, \tag{1.1}$$

where $\{X_1, \cdots, X_d, T_1, T_2\}$ is a base of left invariant vector fields associated to the Lie group G^{d+2}, and λ_1, λ_2 are complex numbers. The operator L_λ may be regarded as a nontrivial extension of the operators discussed in [1], which plays an essential role for studying the parametrix and hypoellipticity of second order operators with the form

$$L = -\sum_{j,k=1}^{d} g_{jk}(z,r) Z_j Z_k - iV + C(z,r)$$

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on a smooth manifold U with dimension $d+2$. In general, a quadratic change of coordinates can not convert simultaneously the matrices $A_m = [a_{jk}^{(m)}](m = 1, 2)$ into ones in normal form (see §2), so that one cannot directly apply the method in [1] to the operators L_λ of (1.1). To overcome this difficulity, we employ the method of increment operators (proposed in [9,10]) so that the operator L_λ of (1.1) is changed into the one discussed in [1]. We establish a subelliptic estimate to show the invertibility for the operator L_λ and construct explicitly the inverse of the operator L_λ. Moreover, the set of admissible points $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{C}^2$, characterizing the invertibility of the operator L_λ, is described exhaustively. Further, we obtain the hypoellipticity and solvability for the operator L_λ. For the hypoellipticity of left invariant operators on general nilpotent Lie groups, it is well know that B.Helffer and J.Nourrigat in [5] and L.P.Rothschild in [11] obtained general results characterized by the unitary representations of the operators. But, for the operator given in this paper, it seems difficult to verify their conditions. The conditions of hypoellipticity by us, characterized by the parameter λ, are easily verified. As applications, the hypoellipticity and solvability for heat operators and Schrödinger operators on the Heisenberg group H_n and the generalized Kolmogorov operators on R^{d+2} and the operator $\Delta_{x,l}^d - \lambda\nabla_t$ are discussed.

§2. The Group G^{d+2} and the Operator L_λ

The group G^{d+2} is the Lie group whose underlying manifold is R^{d+2} with coördinates $(x_1, \cdots, x_d, t_1, t_2) = (x, t)$ and whose group structure is given by

$$(x,t)\cdot(y,s) = (x+y, t+s+\frac{1}{2}yAx), \tag{2.1}$$

where $yAx = (yA_1x, yA_2x)$ and the matrices $A_m = [a_{jk}^{(m)}](m = 1, 2)$ are skew-symmetric. It is easy to verify that the group G^{d+2} is a two step nilpotent Lie group. In particular, when $A_2 = [a_{jk}^{(2)}] = 0$, $d = 2n + m(n, m \in I_+)$, and

$$\begin{cases} a_{j,n+j}^{(l)} = -a_{n+j,j}^{(l)}, & j = 1, 2, \cdots, n, \\ a_{jk}^{(l)} = 0, & \text{otherwise}, \end{cases}$$

we have

$$G^{2n+m+2} \cong H_n \times R^{m+l} \text{ (studied in [16])},$$

where H_n is the Heisenberg group of degree n.

A base of the left invariant vector fields on G^{d+2} is $\{X_1, \cdots, X_d, T_1, T_2\}$:

$$\begin{cases} X_j = \dfrac{\partial}{\partial x_j} + \dfrac{1}{2}\displaystyle\sum_{k=1}^{d}\sum_{m=1}^{2} a_{jk}^{(m)} x_k \dfrac{\partial}{\partial t_m}, & j = 1, 2, \cdots, d, \\ T_m = \dfrac{\partial}{\partial t_m}, & m = 1, 2. \end{cases} \tag{2.2}$$

Their commutation relations are as follows:

$$\begin{aligned} &[X_j, X_k] = \sum_{m=1}^{2} a_{jk} T_m, \quad j, k = 1, 2, \cdots, d, \\ &[X_j, T_m] = [T_m, T_n] = 0, \quad m, n = 1, 2. \end{aligned} \tag{2.3}$$

Definition 2.1. *For $\rho > 0$, we define the dilations on G^{d+2} to be $\rho(x,t) = (\rho x, \rho^2 t) = (\rho x_1, \cdots, \rho x_d, \rho^2 t_1, \rho^2 t_2), (x,t) \in G^{d+2}$, and the dilations on $L^2(G^{d+2})$ by*

$$\delta_{\rho f}(x,t) = f(\rho x, \rho^2 t), \quad f(x,t) \in L^2(G^{d+2}). \tag{2.4}$$

Definition 2.2. *We say that a function f in $L^2(G^{d+2})$ is G-homogeneous of degree m if*

$$\delta_{\rho f}(x,t) = \rho^m f(x,t), \quad \rho > 0. \tag{2.5}$$

Definition 2.3. *An operator Q on $L^2(G^{d+2})$ is G-homogeneous of order m if it satisfies*

$$\delta_\rho^{-1} Q \delta_\rho = \rho^m Q, \quad \rho > 0. \tag{2.6}$$

We introduce the G-norm $\|\cdot\|$ of $(x,t) \in G^{d+2}$:

$$\|(x,t)\| = (|x|^4 + |t|^2)^{1/4},$$

and denote the Euclidean norm of (x,t) by $|(x,t)|$. Then it is easy to verify the following relation:

$$C^{-1}(1 + |(x,t)|)^{1/2} \le 1 + \|(x,t)\| \le C(1 + |(x,t)|), \quad C > 1. \tag{2.7}$$

For any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, we define the operator

$$L_\lambda = -\sum_{j=1}^{d} X_j^2 - i \sum_{m=1}^{2} \lambda_m T_m, \tag{2.8}$$

where X_j, T_m are given by (2.2). The operator L_λ is left invariant and G-homogeneous of order 2 on G^{d+2}.

§3. The Increment Operator and a Subelliptic Estimate for the Operator L_λ

Put

$$\|u\|_{G_0} = \left(\|u\|^2 + \sum_{j=1}^{d} \|X_j^2 u\|^2 + \sum_{m=1}^{2} \|T_m u\|^2 + \sum_{j=1}^{d} \|X_j X_k u\|^2 \right)^{1/2}, \quad u \in C_0^\infty(G^{d+2}),$$

where $\|\cdot\|$ denotes the L^2-norm. Let G_0 denote the completion of the space $C_0^\infty(G^{d+2})$ in the $\|\cdot\|_{G_0}$-norm.

By the same argument as (2.12)-(2.22) in [1], we can obtain the following lemmas.

Lemma 3.1. *$\|\cdot\|_{G_0}$ is independent of the choice of $X_1, \cdots, X_d$, also of coordinates $x_1, \cdots, x_d$.*

Lemma 3.2. $\|u\|_{G_0} \sim \|u\| + \sum_{j=1}^{d} \|X_j^2 u\| + \sum_{m=1}^{2} \|T_m u\|$, $u \in C_0^\infty(G^{d+2})$.

To discuss the invertibility of the operator in L^2, we need to set up a subelliptic estimate.

For given $u(x,t) \in S(G^{d+2})$, we have

$$L_\lambda u(x,t) = \left[-\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{d} \sum_{m=1}^{2} a_{jk}^{(m)} x_k \frac{\partial}{\partial t_m} \right)^2 - i \sum_{m=1}^{2} \lambda_m \frac{\partial}{\partial t_m} \right] u(x,t).$$

Taking the Fourier transformation in the two sides of the above equality in the variable $t=(t_1,t_2)$ and denoting the dual variable by $\tau=(\tau_1,\tau_2)$, we get

$$(F_tL_\lambda(u)(x,\tau)=\left[-\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}+\frac{i}{2}\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k\right)^2+\sum_{m=1}^{2}\lambda_m\tau_m\right](F_tu)(x,\tau). \quad (3.1)$$

Let

$$L_\lambda^\tau=-\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}+\frac{i}{2}\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k\right)^2+\sum_{m=1}^{2}\lambda_m\tau_m, \quad (3.2)$$

then (3.1) shows that

$$(F_tL_\lambda u)(x,\tau)=L_\lambda^\tau(F_tu)(x,\tau). \quad (3.3)$$

Moreover, by introducing an auxiliary variable $x_0\in R^1$, we obtain

$$\begin{aligned}&e^{ix_0}L_\lambda^\tau v(x)\\&=\left[-\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k\frac{\partial}{\partial x_0}\right)^2\right.\\&\quad\left.-i\left(\sum_{m=1}^{2}\lambda_m\tau_m\right)\frac{\partial}{\partial x_0}\right](e^{ix_0}v(x)),\quad v(x)\in S(R^d). \quad (3.4)\end{aligned}$$

Put

$$P_\lambda^\tau=-\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k\frac{\partial}{\partial x_0}\right)^2-i\left(\sum_{m=1}^{2}\lambda_m\tau_m\right)\frac{\partial}{\partial x_0}. \quad (3.5)$$

Then, if we regard τ_1, τ_2 as two parameters, the operator P_λ^τ is just one discussed in [1].

Definition 3.1. *We call the operators P_λ^τ a family of the increment operators (with the variable x_0) associated to the operator L_λ.*

(3.4) shows that

$$P_\lambda^\tau(e^{ix_0}v(x))=e^{ix_0}L_\lambda^\tau v(x),\quad v(x)\in S(R^d). \quad (3.6)$$

Combining (3.6) with (3.3) yields

$$(P_\lambda^\tau(e^{ix_0}(F_tu)))(x,\tau)=e^{ix_0}((F_tL_\lambda u)(x,\tau)),\quad u(x,t)\in S(G^{d+2}). \quad (3.7)$$

From the operator P_λ^τ, we can derive the nilpotent Lie group G_τ^{d+1} whose underlying manifold is R^{d+1} with coordinates $(x_0,x)=(x_0,x_1,\cdots,x_d)$ and whose group law is given by

$$(x_0,x)\cdot(y_0,y)=\left(x_0+y_0+\frac{1}{2}\sum_{j,k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k y_j,x+y\right). \quad (3.8)$$

A base of the left invariant vector fields on the group G_τ^{d+1} is $\{X_0^\tau,X_1^\tau,\cdots,X_d^\tau\}$:

$$\begin{cases}X_0^\tau=\dfrac{\partial}{\partial x_0},\\ X_j^\tau=\dfrac{\partial}{\partial x_j}+\dfrac{1}{2}\displaystyle\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau_m x_k\frac{\partial}{\partial x_0},\ j=1,2,\cdots,d.\end{cases} \quad (3.9)$$

Let

$$\|w\|_{G_\tau} = \left(\|w\|^2 + \sum_{j=0}^{d} \|X_j^\tau w\|^2 + \sum_{j,k=1}^{d} \|X_j^\tau X_k^\tau w\|^2 \right)^{1/2}, \quad w \in C_0^\infty(G_\tau^{d+1}),$$

and denote the completion of $C_0(G_\tau^{d+1})$ in the $\|\cdot\|_{G_\tau}$ norm by G_τ.

Let

$$\pm i f_1(\tau), \pm i f_2(\tau), \cdots, \pm i f_{n_\tau}(\tau), \quad f_j > 0$$

be the nonzero eigenvalues of the skew-symmetric matrix $A(\tau) = \sum_{m=1}^{2} \tau_m A_m$, repeated according to multility. Then, $f_j(\tau)$ are positive homogeneous of degree 1 in τ and continuous functions of t in view of Theorem 6.1 of Chapter 2 in [7]. Moreover, the plane R^2 is divided into $2m$ conic domains by the straight lines $c_1, c_2, \cdots, c_m$ passing the origin so that n_τ is a constant in every conic domain.

For given $\tau = (\tau_1, \tau_2) \in R^2 \backslash \{0\}$, put

$$F_\alpha(\tau) = \sum_{j=1}^{n_\tau} (2\alpha_j + 1) f_j(t), \quad \alpha \in I_+^{n_\tau},$$

$$\lambda \cdot \tau = \lambda_1 \tau_1 + \lambda_2 \tau_2, \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2,$$

and let Λ^τ be the subset of R as follows:

$$\Lambda^\tau = \begin{cases} R, & \text{if } n_\tau = 0, \\ \{\nu \in R : |\nu| \geq F_0(\tau)\}, & \text{if } 0 < 2n_\tau < d, \\ \{\nu \in R : |\nu| = F_\alpha(\tau), \alpha \in I_+^{n_\tau}\}, & \text{if } 2n_\tau = d. \end{cases}$$

Definition 3.2. *Let $\lambda \in \mathbb{C}^2$. We say that the point λ is an admissible point of the operator L_λ if $\lambda \cdot \tau \notin \Lambda^\tau$ for each $\tau \in R^2 \backslash \{0\}$.*

Theorem 3.1. *The operator L_λ given by (2.8) satisfies the subelliptic estimate*

$$\|u\|_{G_0} \leq C(\|L_\lambda u\| + \|u\|), \quad u(x,t) \in S(G^{d+2}), \tag{3.11}$$

if and only if λ is admissible.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.3. *For given $\tau \in R^2 \backslash \{0\}$, the increment operator P_λ^τ given by (3.5) satisfies the estimate*

$$\|w\|_{G_\tau} \leq C(\tau)(\|P_\lambda^\tau w\| + \|w\|), \quad w \in S(G_\tau^{d+1}), \tag{3.12}$$

if and only if $\lambda \cdot \tau \notin \Lambda^\tau$.

Proof. We take the Fourier transform of $w(x_0, x)$ in the variables $(x_0, x_{n_\tau} + 1, \cdots, x_d)$ and denote the dual variables by $(\zeta_0, \zeta_1, \cdots, \zeta_{n_\tau}, \eta_1, \cdots, \eta_{d-2n_\tau})$. By Plancherel's theorem and a translation in the variables $x_1, \cdots, x_{n_\tau}$, (3.12) is equivalent to the following inequality for each $(\zeta_0, \eta) \in (R\backslash\{0\}) \times R^{d-2n_\tau}$:

$$\|v\|^2 + \sum_{j=1}^{n_\tau} [\|D_j^2 v\|^2 + \|(\zeta x_j f_j(\tau))^2 v\|^2] + (\zeta^2 + \sum_{j=1}^{d-2n_\tau} \eta_j^4)\|v\|^2$$

$$\leq C(\tau)(\|(P^\tau)_{\zeta,\eta} v\|^2 + \|v\|^2), \quad v \in S(R^n), \tag{3.13}$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad j = 1, 2, \cdots, n,$$

$$(P_\lambda^\tau)_{\xi,\eta} = \sum_{j=1}^{n_\tau}[D_j^2 + (\zeta x_j f_j(\tau))^2] + \sum_{m=1}^{2} \lambda_m \tau_m \zeta + |\eta|^2.$$

Using the unitary dilation of $L^2(R^n)$:

$$\rho^U v(x) = \rho^{n/2} v(\rho x) = \rho^{n/2} v(\rho x_1, \cdots, \rho x_n), \quad \rho > 0,$$

we know that (3.13) is equivalent to

$$\|v\|^2 + \sum_{j=1}^{n_\tau}(\|D_j v\|^2 + \|(x_j f_j(\tau))^2 v\|^2) + \sum_{j=1}^{d-2n_\tau} \eta_j^4 \|v\|^2 \le C(\tau)\|(P_\lambda^\tau)_{\pm 1,\tau} v\|^2, \quad v(x) \in S(R^{n_\tau}). \tag{3.14}$$

If $n_\tau \neq 0$, according to the proof of Theorem 2.10 in [1], (3.14) holds for all $\eta \in R^{d-2n_\tau}$ if and only if $\lambda \cdot \tau \notin \Lambda^\tau$. If $n_\tau = 0$, (3.13) actually is

$$1 + \zeta_0^2 + \sum_{j=1}^{d} \eta_j^4 \le C(\tau)(|(\lambda \cdot \tau)\zeta_0 + |\eta|^2|^2 + 1), \tag{3.15}$$

$(\zeta_0, \eta) \in (R\backslash\{0\}) \times R^d$. We easily show that (3.15) holds for some constant $C(\tau)$ and all $(\zeta_0, \eta) \in (R\backslash\{0\}) \times R^d$ if and only if $\lambda \cdot \tau \notin R$. This completes the proof of the lemma.

Lemma 3.4. *Let λ be admissible. Then if $|\tau| = 1$, there is a constant C independent of τ such that*

$$\|w\|_{G_\tau} \le C(\|P_\lambda^\tau w\| + \|w\|), \quad w \in S(G_\tau^{d+1}). \tag{3.16}$$

Proof. According to Lemma 3.3 and its proof, we have (3.12), which is equivalent to (3.14). By the proof of Theorem 2.10 in [1], (3.14) is equivalent to

$$\|Q^\tau v\| + (1 + |\eta|^2)\|v\| \le C(\tau)\|(Q^\tau + |\eta|^2 \pm \lambda \cdot \tau)v\|, \quad v \in S(R^n), \tag{3.17}$$

$\eta \in R^{d-2n}$, where

$$Q^\tau = \sum_{j=1}^{n_\tau} Q_j^\tau = \sum_{j=1}^{n_\tau}[D_j^2 + (x_j f_j(\tau))^2].$$

Thus, it suffices to show that for the case $|\tau| = 1$ the constant $C(\tau)$ in (3.17) may be chosen such that it is independent of τ.

Let $\{\phi_\alpha(x)\}$ be the sequence of the Hermite's functions on R^n. Then each $v(x) \in S(R^n)$ has a (unique) decomposition:

$$v(x) = \sum_\alpha v_\alpha \phi_\alpha(x), \quad v_\alpha \text{ complex numbers.}$$

Define the unitary operators H^τ on $L^2(R^n)$ as follows:

$$(H^\tau v)(x) = \left[\prod_{j=1}^{n_\tau} f_j(\tau)\right]^{1/2} v(\sqrt{f_1(\tau)}x_1, \cdots, \sqrt{f_n(\tau)}x_n),$$

$\tau \in R^2\backslash\{0\}$; we then have

$$(H^\tau)^{-1}Q^\tau H^\tau \phi_\alpha(x) = F_\alpha(\tau)\phi_\alpha(\tau)$$

and hence

$$(H^\tau)^{-1}Q^\tau H^\tau v(x) = \sum_\alpha F_\alpha(\tau)v_\alpha\phi_\alpha(x).$$

Consequently, replacing $v(x)$ by $H^\tau v(x)$ in (3.17), we obtain

$$\sum_\alpha |v_\alpha|^2 + \sum_\alpha F_\alpha^2(\tau)|v_\alpha|^2 \le C(\tau)\sum_\alpha |F_\alpha(\tau) \pm \lambda\cdot\tau|^2|v_\alpha|^2. \tag{3.18}$$

Further, since λ is admissible and f_j are continuous, it follows that

$$\inf_{\substack{\alpha\in I_+^{n_\tau} \\ |\tau|=1}} |F_\alpha(\tau) \pm \lambda\cdot\tau| = d_\lambda > 0$$

and hence

$$\frac{1+F_\alpha^2(\tau)}{|F_\alpha(\tau)\pm\lambda\cdot\tau|^2} \le \frac{1}{d_\lambda^2} + \left(1+\frac{2|\lambda|}{d_\lambda}\right)^2 = C_\lambda < \infty.$$

Thus, $C(\tau) = C_\lambda$ is required in (3.17). If $n_\tau = 0$, (3.12) is actually equivalent to (3.15), and since

$$\inf_{|\tau|=1} |\mathrm{Im}(\lambda\cdot\tau)| = e_\lambda > 0,$$

we have

$$\begin{aligned}\frac{1+\xi^2+\sum_{j=1}^d \eta_j^4}{|(\lambda\cdot\tau)\xi + |\eta|^2|^2+1} &\le \frac{1+\xi^2+\sum_{j=1}^d \eta_j^4}{e_\lambda^2\xi^2+|\eta|^4+1} \\ &\le 1+\frac{1}{e_\lambda^2} = C_\lambda < \infty,\end{aligned}$$

which implies that (3.15) holds for $C(\tau) = C_\lambda$ independent of τ, and so does (3.12) which yields (3.16).

Proof of Theorem 3.1. We first prove the necessity. Suppose that (3.11) holds. Then by Lemma 3.3, it suffices to show that for each $\tau \in R^2\backslash\{0\}$ there is a constant $C(\tau)$ such that (3.12) holds.

By virtue of Lemma 3.2, (3.11) is equivalent to

$$\|u\|^2 + \sum_{j=1}^d \|X_j^2u\|^2 + \sum_{m=1}^2 \|T_mu\|^2 \le C(\|L_\lambda u\|^2 + \|u\|^2), \quad u \in S(G^{d+2}). \tag{3.19}$$

Taking the Fourier transformation of $u(x,t)$ in the variable t and denoting the dual variable by $\tau' = (\tau_1', \tau_2')$, from Plancherel's theorem we know that (3.19) is equivalent to the following inequality for every $\tau' \in R^2\backslash\{0\}$:

$$(1+|\tau'|^2)\|v\|^2 + \sum_{j=1}^d \|(\overline{X}_j^{\tau'})^2v\|^2 \le C(\|L_\lambda^{\tau'}v\|^2 + \|v\|^2), \quad v \in S(R^d), \tag{3.20}$$

where

$$\overline{X}_j^{\tau'}=\frac{\partial}{\partial x_j}+\frac{i}{2}\sum_{k=1}^{d}\sum_{m=1}^{2}a_{jk}^{(m)}\tau'_m x_k,\quad j=1,2,\cdots,d,$$

$$L_\lambda^{\tau'}=-\sum_{j=1}^{d}(\overline{X}_j^{\tau'})^2+\lambda\cdot\tau'.$$

By the argument as above, (3.12) is equivalent to

$$(1+\tau_0^2)\|v\|^2+\sum_{j=1}^{d}\|(\overline{X}_j^{\tau_0\tau})^2v\|^2\le C(\tau)(\|L_\lambda^{\tau_0\tau}v\|^2+\|v\|^2),\quad v\in S(R^d),\tag{3.21}$$

$\tau_0\in R\backslash\{0\}$. For given $\tau_0\in R\backslash\{0\}$ and $\tau\in R^2\backslash\{0\}$, taking $\tau'=\tau_0\tau$ in (3.20) we get

$$(1+\tau_0^2|\tau|^2)\|v\|^2+\sum_{j=1}^{d}\|\overline{X}_j^{\tau_0\tau}v\|^2\le C(\|L_\lambda^{\tau_0\tau}v\|^2+\|v\|^2),\quad v\in S(R^d).$$

Hence, when $|\tau|\ge 1$, (3.21) holds for $C(\tau)=C$; while for $|\tau|<1$, we have

$$\begin{aligned}|\tau|^2[(1+\tau_0^2)\|v\|^2+\sum_{j=1}^{d}\|(\overline{X}_j^{\tau_0\tau})^2v\|^2]&\le(1+\tau_0^2|\tau|^2)\|v\|^2+\sum_{j=1}^{d}\|(\overline{X}_j^{\tau_0\tau})^2v\|^2\\&\le C(\|L_\lambda^{\tau_0\tau}v\|^2+\|v\|^2),\quad v\in S(R^d),\end{aligned}$$

which implies that (3.21) holds for $C(\tau)=\frac{C}{|\tau|^2}$. To sum up, for each $\tau\in R^2\backslash\{0\}$ there is a constant

$$C(\tau)=\begin{cases}C, & \text{if } |\tau|\ge 1,\\ \dfrac{C}{|\tau|^2}, & \text{if } 0<|\tau|<1,\end{cases}$$

such that (3.21) holds. So λ is admissible.

Conversely, suppose that λ is admissible. Then there is a constant C independent of τ such that (3.21) holds for all $\tau_0\in R\backslash\{0\}$, i.e.,

$$(1+\tau_0^2)\|v\|^2+\sum_{j=1}^{d}\|(\overline{X}_j^{\tau_0\tau})^2v\|^2\le C(\|L^{\tau_0\tau}v\|^2+\|v\|^2),\quad v\in S(R^d),\tag{3.21'}$$

if $|\tau|=1$. For given $\tau'\in R^2\backslash\{0\}$, (3.20) follows by taking $|\tau_0|=|\tau'|$ and $\tau=\tau'/\tau_0$ in (3.21').

§4. The Inverse and Hypoellipticity of the Operator L_λ

Let the space S^∞ be defined by Definition 3.7 of [1]. Then Proposition 3.9 of [1] shows the pseudo-differential operator Q with the symbol $q(x,\xi)\in S^\infty$ maps $S(V)$ into $C^\infty(V)$.

By Proposition 3.21 in [1], we note that the symbol $\sigma(Q_\lambda^\tau)(x_0,x,\xi_0,\xi)$ of the left inverse Q_λ^τ (if it exists) for the operator P_λ^τ is independent of x_0. So, we may denote it by $q_\lambda^\tau(x,\xi_0,\xi)$.

Theorem 4.1. *Suppose that for each $\tau\in R^2\backslash\{0\}$ the increment operator P_λ^τ given by (3.5) has a two-sided inverse Q_λ^τ with the symbol $q_\lambda^\tau(x,\xi_0,\xi)$. Then the operator L_λ given by (2.8) has a two-sided inverse M_λ with the symbol*

$$\sigma(M_\lambda)(x,t,\xi,\tau)=q_\lambda^\tau(x,1,\xi).$$

The proof of Theorem 4.1 needs the aid of the following lemma.

Lemma 4.1. *For given $\tau \in R^2\backslash\{0\}$, suppose that the operator P_λ^τ has a two-sided inverse Q_λ^τ with the symbol $q_\lambda^\tau(x, \xi_0, \xi)$. Then the operator L_λ^τ given by (3.2) has a two-sided inverse M_λ^τ with the symbol*

$$\sigma(M_\lambda^\tau)(x, \xi) = q_\lambda^\tau(x, 1, \xi).$$

Proof. Since $Q_\lambda^\tau \cdot P_\lambda^\tau = I_d$ which implies that

$$Q_\lambda^\tau \cdot P_\lambda^\tau(e^{ix_0}v(x)) = e^{ix_0}v(x), \quad v(x) \in S(R^d),$$

we have

$$\begin{aligned} e^{ix_0}v(x) &= \int e^{i(x_0\xi_0+x\xi)}q_\lambda^\tau(x, \xi_0, \xi)P_\lambda^\tau(\widetilde{e^{ix_0}v(x)})(\xi_0, \xi)d\!\!{}^-\xi_0 d\!\!{}^-\xi \\ &= \int e^{i(x_0\xi_0+x\xi)}q_\lambda^\tau(x, \xi_0, \xi)\widetilde{e^{ix_0}L_\lambda^\tau v(x)}(\xi_0, \xi)d\!\!{}^-\xi_0 d\!\!{}^-\xi \\ &= \int e^{i(x_0\xi_0+x\xi)}q_\lambda^\tau(x, \xi_0, \xi)\sigma(\xi_0 - 1)\widetilde{L_\lambda^\tau v}(\xi)d\!\!{}^-\xi_0 d\!\!{}^-\xi \\ &= e^{ix_0}\int e^{ix\xi}q_\lambda^\tau(x, 1, \xi)\widetilde{L_\lambda^\tau v}(\xi)d\!\!{}^-\xi, \quad v(x) \in S(R^d), \end{aligned}$$

and hence

$$\int e^{ix\xi}q_\lambda^\tau(x, 1, \xi)\widetilde{L_\lambda^\tau v}(\xi)d\!\!{}^-\xi = v(x), \quad v(x) \in S(R^d).$$

We note that the above calculations are formal in the usual sense since $e^{ix_0}v(x) \notin S(G_\tau^{d+1})$, but it holds in the distribution sense or the above equality follows from calculating the left side of the equality

$$Q_\lambda^\tau \cdot P_\lambda^\tau(e^{ix_0-\varepsilon x^2/2}v(x)) = e^{ix_0-\varepsilon x^2/2}v(x) \quad (\varepsilon > 0)$$

and letting $\varepsilon \to 0$. So we may think that the above calculation is reasonable. We denote the pseudodifferential operator with the symbol $q_\lambda^\tau(x, 1, \xi)$ by M_λ^τ, where τ is a parameter. Then above equality shows that

$$M_\lambda^\tau \cdot L_\lambda^\tau = I_d.$$

Thus

$$M_\lambda^\tau = (L_\lambda^\tau)^{-1}$$

and

$$\sigma(M_\lambda^\tau)(x, \xi) = q_\lambda^\tau(x, 1, \xi).$$

Proof of Theorem 4.1. According to Lemma 4.1, the operator L_λ^τ has a two-sided inverse M_λ^τ with the symbol $q_\lambda^\tau(x, 1, \xi)$. Let

$$\begin{aligned} v(x, \tau) &= (F_t u)(x, \tau) \\ &= \int e^{-it\tau}u(x, t)dt, \quad u(x, t) \in S(G^{d+2}). \end{aligned}$$

Then

$$L_\lambda^\tau \cdot M_\lambda^\tau v(x, \tau) = v(x, \tau).$$

Taking the inverse Fourier transformation for the above equality in the variable τ, we thus get from (3.3)

$$(L_\lambda \cdot (F_\tau^{-1}M_\lambda^\tau v))(x, t) = u(x, t). \tag{4.1}$$

Since

$$(F_\tau^{-1}(M_\lambda^\tau v))(x,t) = \int e^{it\tau}\int e^{ix\xi} q_\lambda^\tau(x,1,\xi)(F_x v)(\xi,\tau) d\xi d\tau$$
$$= \int e^{i(x\xi+t\tau)} q_\lambda^\tau(x,1,\xi)\hat{u}(\xi,\tau) d\xi d\tau,$$

by writing the pseudodifferential operator with the symbol $q_\lambda^\tau(x,1,\xi)$ as M_λ, where τ is the dual variable, we obtain

$$(F_\tau^{-1}(M_\lambda^\tau v))(x,t) = (M_\lambda u)(x,t)$$

or

$$(F_t(M_\lambda u))(x,\tau) = (M_\lambda^\tau(F_t u))(x,\tau), \quad u(x,t) \in S(G^{d+2}). \tag{4.2}$$

Combining (4.2) with (4.1) then yields

$$L_\lambda \cdot M_\lambda = I_d.$$

Similarly, with the aid of $M_\lambda^\tau \cdot L_\lambda^\tau = I_d$ and (4.2), we also obtain

$$M_\lambda \cdot L_\lambda = I_d.$$

Thus

$$M_\lambda = (L_\lambda)^{-1}$$

and

$$\sigma(M_\lambda)(x,t,\xi,\tau) = q_\lambda^\tau(x,1,\xi),$$

which implies the theorem.

Next, we shall give an explict expression of the symbol for the inverse operator.

Let $N_{A(\tau)}$ denote the square root of the matrix $(A^*(\tau)A(\tau))$, where $A^*(\tau)$ denotes the formal transfer of $A(\tau)$. Then the matrix $N_{A(\tau)}$ is symmetric and has nonzero eigenvalues $f_j(\tau)$, $j = 1,2,\cdots,2n_\tau$, where $f_j(\tau)$'s are the nonzero eigenvalues of the matrix $A(\tau) = \sum_{m=1}^{2}\tau_m A_m$ and $f_{n_\tau+j}(\tau) = f_j(\tau)$, $j = 1,2,\cdots,n_\tau$. We appoint that $f_j(\tau) = 0$ for $2n_\tau < j \le d$ and write

$$\mathrm{detch}(N_{A(\tau)}s) = \prod_{j=1}^{d}\mathrm{ch}(f_j(\tau)s),$$

$$< \frac{\mathrm{th}(N_{A(\tau)}s)}{N_{A(\tau)}}\xi,\xi >= \sum_{j=1}^{d}\frac{\mathrm{th}(f_j(\tau)s)}{f_j(\tau)}\xi_j^2.$$

Remark 4.1. For any matrix N, $\frac{\mathrm{th}(Ns)}{N}$ is defined by

$$\frac{\mathrm{th}(Ns)}{N} = s - N^2\cdot\frac{s^3}{3} - \cdots.$$

Lemma 4.2(cf. Theorem 6.16 in [1]). *Let $\lambda \in \mathbb{C}^2$. Then if $\lambda\cdot\tau \notin \Lambda^\tau$ for given $\tau \in R^2\backslash\{0\}$, the operator P_λ^τ has a two-sided inverse Q_λ^τ with the symbol*

$$\sigma(Q_\lambda^\tau)(x,\xi_0,\xi) = q_\lambda^\tau(\xi_0,\sigma^\tau(x,\xi_0,\xi)),$$

where $\sigma_j^\tau(x,\xi_0,\xi) = \sigma(i^{-1}X_j^\tau)$ with X_j^τ given by (3.9), and the function $q_\lambda^\tau(\xi_0,\xi)$ is defined *as follows:*

$$q_\lambda^\tau(\xi_0,\xi) = \int_0^{+\infty} e^{-(\lambda\cdot\tau)\xi_0 s}\cdot G^\tau(\xi_0,\xi,s)ds \tag{4.3}$$

if $|(\lambda\cdot\tau)^R| < F_0(\tau)$ *and* $n_\tau \neq 0$, *where*

$$G^\tau(\xi_0,\xi,s) = [\text{detch}\,(N_{A(\tau)}|\xi_0|s)]^{-1/2}\cdot\exp\left[-<\frac{th(N_{A(\tau)}|\xi_0|s)}{N_{A(\tau)}|\xi_0|}\xi,\xi>\right].$$

If $n_\tau = 0$,

$$q^\tau(\xi_0,\xi) = \left[(\lambda\cdot\tau)\xi_0 - \sum_{j=1}^d \xi_j^2\right]^{-1}.$$

For those λ not satisfying $|(\lambda\cdot\tau)^R| < F_0(\tau)$, we can obtain $q_\lambda^\tau(\xi_0,\xi)$ from (4.3) by an analytic extension, changing the contour of the integral (4.3) if $0 < 2n_\tau < d$ or repeating integration by part in (4.3) if $2n_\tau = d$.

Combining Lemma 4.2 with Theorem 4.1, we can obtain

Theorem 4.2. *Let* $\lambda\in\mathbb{C}^2$. *Then if* $\lambda\in\mathcal{A}$, *the operator* L_λ *has a two-sided (pseudodifferential) inverse* M_λ *with the symbol*

$$\sigma(M_\lambda)(x,t,\xi,\tau) = m_\lambda(\sigma(x,t,\xi,\tau),\tau),$$

where $\sigma = (\sigma_1,\cdots,\sigma_d)$ *and* $\sigma_j(x,t,\xi,\tau) = \sigma(i^{-1}X_j)$ *with* X_j *given by* (2.2). *The function* $m_\lambda(\xi,\tau)$ *is defined as follows:*

$$m_\lambda(\xi,\tau) = \int_0^{+\infty} e^{-(\lambda\cdot\tau)s}\cdot G(\xi,\tau,s)ds \tag{4.4}$$

if $|(\lambda\cdot\tau)^R| < F_0(\tau)$, *where*

$$G(\xi,\tau,s) = [\text{detch}(N_{A(\tau)}s)]^{-1/2}\cdot\exp\left[-<\frac{th(N_{A(\tau)}s)}{N_{A(\tau)}}\xi,\xi>\right].$$

If $n_\tau = 0$, *then* $m_\lambda(\xi,\tau) = (\lambda\cdot\tau + \sum_{j=1}^d \xi_j^2)^{-1}$.

The desirable $m_\lambda(\xi,\tau)$ can be obtained from (4.4) by the analytic extension mentioned in Lemma 4.2.

Proof. The case $n_\tau = 0$ follows from letting $f_j(\tau)\to 0$ in (4.4).

Define the operator M_λ by

$$M_\lambda u(x,t) = \int e^{i(x\xi+t\tau)}\sigma(M_\lambda)(x,t,\xi,\tau)\hat{u}(\xi,\tau)d\xi d\tau. \tag{4.5}$$

Then, since $\sigma(M_\lambda)(x,t,\xi,\tau) = m_\lambda(\sigma(x,t,\xi,\tau),\tau)$ and $m_\lambda(\xi,\tau)$ is G-homogeneous of degree -2 in (ξ,τ), we know that, for any compact set $K\subset R^{d+2}$, $\sigma(M_\lambda)(x,t,\xi,\tau)$ satisfies the following estimate $|\sigma(M_\lambda)(x,t,\xi,\tau)| \le C_0(K)(1+|(\xi,\tau)|)^{-2}$ and hence (4.5) is well defined. Furthermore, for any $\alpha = (\alpha_x,\alpha_t)\in I_+^{d+2}$ we have

$$\partial_{x,t}^\alpha(M_\lambda)(x,t,\xi,\tau) = \partial_{x,t}^\alpha m_\lambda((x,t,\xi,\tau),\tau)$$

$$= \sum_{\substack{\alpha^{(k)}=(\alpha_1^{(k)},\cdots,\alpha_\alpha^{(k)})\\ |\alpha^{(k)}|=\alpha_{x,k}\\ k=1,2,\cdots,\alpha}} \partial_\sigma^{\alpha^{(1)}+\cdots+\alpha^{(\alpha)}} m_\lambda(\sigma(x,t,\xi,\tau),\tau)\cdot\prod_{j,k=1}^d\left(\frac{1}{2}\sum_{m=1}^2 a_{jk}^{(m)}\tau_m\right)j^{(k)}.$$

Note that $\partial_\sigma^{\alpha^{(1)}+\cdots+\alpha^{(\alpha)}} m_\lambda(\xi,\tau)$ is G-homogeneous of degree $-2-|\alpha_x|$ in (ξ,τ), we get for any $\alpha=(\alpha_x,\alpha_t)\in I_+^{d+2}$ and any compact set $K\subset R^{d+2}$

$$|\partial_{x,t}^\alpha \sigma(M_\lambda)(x,t,\xi,\tau)| \le C_\alpha(K)(1+|(\xi,\tau)|)^{-2-|\alpha_x|-|\alpha_x|}$$
$$= C_\alpha(K)(1+|(\xi,\tau)|)^{-2},$$

which implies that the integral

$$\int e^{i(x\xi+t\tau)} D_{x,t}(\sigma(M_\lambda)(x,t,\xi,\tau))\hat{u}(\xi,\tau) d\xi d\tau$$

converges. Consequently, for any $\alpha\in I_+^{d+2}$ we get

$$\partial_{x,t}^\alpha (M_\lambda u)(x,t) = \int \partial_{x,t}^\alpha [e^{i(x\xi+t\tau)}\sigma(M_\lambda)(x,t,\xi,\tau)]\cdot \hat{u}(\xi,\tau) d\xi d\tau.$$

This shows that M_λ maps $S(G^{d+2})$ to $C^\infty(G^{d+2})$. Also, Theorem 4.1 and Lemma 4.2 imply that $M_\lambda=(L_\lambda)^{-1}$ if λ is admissible. We thus have proved the theorem.

Corollary 4.1. *Let $\lambda\in\mathbb{C}^2$ and suppose that λ is admissible. Then operator L_λ is globally solvable, that is, for any $f(x,t)\in C^\infty(G^{d+2})$ there is $u(x,t)\in C^\infty(G^{d+2})$ such that*

$$L_\lambda u(x,t) = f(x,t).$$

Corollary 4.2. *The operator L_λ is hypoelliptic when λ is admissible.*

Remark. The main results of this paper have been extended to the case of $G^{n_1+n_2}$ by the same idea given here (see [17]).

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