Chin. Ann. of Math.
14B: 3(1993), 367-386.

DISTORTION THEOREM FOR BIHOLOMORPHIC MAPPINGS IN TRANSITIVE DOMAINS (III)

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Abstract

The distortion theorem for biholomorphic convex mappings in bounded symmetric domains are considered. Especially the distortion theorem for biholomorphic convex mappings in classical domain of type IV and two exceptional domains are given.

Keywords Distortion theorem, Biholomorphic mapping, Transitive domain
1991 MR Subject Classification 32H05

§1. Introduction

Let $M \subset C^n$ be a transitive domain, bounded or unbounded, m be a point in M. Let G be a Lie group consisting of some holomorphic automorphisms of M and acting transitively on M, K be an isotropy group of G which leaves m fixed. Then M is a realization of G/K.

Let $g \in G$, ψ_g denote the holomorphic automorphism corresponding to g, and $J_{\psi_g}(z)$ be the Jacobian of ψ_g at point $z \in M$. If M is unbounded, we must assume $|\det J_{\psi_k}(m)| = 1$ for all k in K. We denote by ψ_z a holomorphic automorphism of M which maps $z \in M$ to $m \in M$. Set $K_M(z,\bar{z}) = c\det J_{\psi_z}(z)\overline{\det J_{\psi_z}(z)}$ with c being constant and $K_M(z,\bar{z})$ is the Bergman kernel function for certain constant when M is bounded. Denote $K_M(m,\overline{m})^{-1} \frac{\partial}{\partial z_p} K_M(z,\bar{z})\Big|_{z=m}$ by C_p.

Suppose f is a biholomorphic mapping of M into C^n. Then we can use $K_M(z,\bar{z})$, C_p and the coefficients of the expansion of f to express the $\det J_f(z)$. This is a result of Gong and Zheng[1].

Let $M \subset C^n$ be a bounded symmetric domain. Then G is a semisimple, connected, noncompact Lie group with finite center, and K is a maximal compact subgroup of G. Let $\mathcal{G}$ be the Lie algebra of G, $\mathcal{K}$ is the maximal compact subalgebra of $\mathcal{G}$ which corresponds to K. Then $\mathcal{G}$ has the Cartan decomposition $\mathcal{G} = \mathcal{K} + \mathcal{P}$. Suppose $\mathfrak{A}$ is the maximal Abelian subspace in $\mathcal{P}$, and A is the analytic subgroup in G corresponding to $\mathfrak{A}$ in $\mathcal{G}$. Then G has Iwasawa decomposition $G = KAN$.

We can choose a basis of $\mathfrak{A}$, $X_1, \cdots, X_q$, where $q = \dim\mathfrak{A} = \operatorname{rank} G/K$, and for any $X \in \mathfrak{A}$, there exists a unique decomposition $X = x_1X_1 + \cdots x_qX_q$. If ξ is the mapping which realizes G/K onto M. For every $z \in M$, there exist $X \in \mathfrak{A}$ and $k \in K$, such that

$$z = \xi(ka \cdot O) = (\xi \exp Ad(k)X \cdot O)$$

Manuscript received September 28, 1989.
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where $O = eK$ is the identity coset in G/K. For $g \in G$, we have $\psi_g(z) = \xi(g^{-1}ka \cdot O)$. Let $\phi_g = \psi_{g^{-1}}$.

Moreover if ξ satisfies the following conditions:

1) $\Lambda = \xi(\exp X \cdot O) = (\tanh x_1, \cdots, \tanh x_q, 0, \cdots, 0)$;

2) $z = \xi(\exp Ad(k)X \cdot O) = \Lambda\widetilde{k}$;

3) $k \to \widetilde{k}'$ is the unitary representation of K;

4) ξ is the holomorphic diffeomorphism of G/K onto M;

then we say M is the canonical realization of G/K, or M is the canonical form of bounded symmetric domain, or M is the Harish-Chandra realization of G/K.

Obviously, if bounded symmetric domains M and N are holomorphically equivalent to each other, then M and N are two different realizations of the same G/K. Any bounded symmetric domain is holomorphically equivalent to a canonical form of a bounded symmetric domain.

The holomorphic mapping f which maps M into C^n is normalized if $f(m) = 0$ and $J_f(m) = I$, where I is the identity matrix, i.e.,

$$f(z) = z - m + \sum_{i,j} d_{i,j}(z_i - m_i)(z_j - m_j) + \cdots \tag{1.1}$$

where $d_{i,j} = (d_{ij}^{(1)}, \cdots, d_{ij}^{(n)})$, $m = (m_1, \cdots, m_n)$, $z = (z_1, \cdots, z_n)$.

A family S of normalized holomorphic mappings of M into C^n is called an A-invariant fmaily if the following condition is satisfied: the composition of any $f(z) \in S$ with any holomorphic automorphism of M, after normalization, remains a holomorphic mapping in S.

In [1], we proved

Theorem 1.1. *Suppose $M \subset C^n$ is a bounded symmetric domain which contains the origin, and it is the canonical realization of Hermite symmetric space G/K. Suppose f is a normalized biholomorphic mapping which maps M into C^n, $f \in S$ and S is a normalized A-invariant family. Let*

$$z = \xi(\exp Ad(k)X \cdot O) \in M, \quad X = \sum_{j=1}^{q} x_j X_j, \quad m = 0.$$

Then

$$\left|\log \frac{\det J_f(z)}{\sqrt{K_M(z,\bar{z})/K_M(0,0)}}\right| \le C(S) \sum_{p=1}^{q} \log \frac{1 + |\tanh x_p|}{1 - |\tanh x_p|} \tag{1.2}$$

where

$$C(S) = \sup\left\{\left|\sum_{i,j} k_{l,i} d_{ij}^{(j)}\right|, f \in S, l = 1, \cdots, q; \widetilde{k} = (k_{li}) \in U_n\right\}. \tag{1.3}$$

Inequality (1.2) implies the following distortion theorem

$$\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}\left(\prod_{p=1}^{q}\frac{1-|\tanh x_p|}{1+|\tanh x_p|}\right)^{C(S)}$$

$$\le|\det J_f(z)|\le\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}\left(\prod_{p=1}^{q}\frac{1+|\tanh x_p|}{1-|\tanh x_p|}\right)^{C(S)}. \qquad (1.4)$$

In [2], we gave the estimates of the upper and lower bounds of $C(S)$ for the family S of normalized biholomorphic convex mappings on the classical domains of types I, II and III. In this paper, we will discuss the bounds of $C(S)$ for family S of normalized biholomorphic convex mappings on bounded symmetric domains. We will especially give the estimates of $C(S)$ for which the domains are classical domain of type IV and two exceptional domains. We will make a conjecture about the precise value of $C(S)$ for the bounded symmetric domains which include the conjectures we made for the classical domains of types I, II and III at [1]. If the conjecture is true, then we can only use the Bergman kernel function of M to express the estimates of $|\det J_f(z)|$.

§2. Holomorphically Equivalent Bounded Symmetric Domains

Lemma 2.1. *Assumption as Theorem* 1.1, *then the* $C(S)$ *defined by* (1.3) *is the smallest positive number which makes* (1.2) *and* (1.4) *hold.*

Proof. In [1] section 4, we already proved that: when M is the canonical realization of G/K, then the equality

$$\log\det J_f(z)=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+2\int_0^1\sum_{p=1}^{q}\frac{\tanh x_p}{1-\rho^2\tanh^2 x_p}\sum_{i,j}k_{pi}d_{ij}^{(j)}(\rho)d\rho \qquad (2.1)$$

holds, where $d_{ij}^{(j)}$ was defined at Theorem 1 of [1], $\widetilde{k}=(k_{ij})\in U_n$. Since $d_{ij}^{(j)}(0)=d_{ij}^{(j)}$, we may take $|z|$ sufficiently small so that

$$\sum_{i,j}k_{pi}d_{ij}^{(j)}(\rho)=\sum_{i,j}k_{pi}d_{ij}^{(j)}+o(1).$$

Let $d=(\sum_j d_{1j}^{(j)},\cdots,\sum_j d_{nj}^{(j)})$, $k_p=(k_{p1},\cdots,k_{pn})$. We can express $\bar d$ as $(y_1,\cdots,y_q,0,\cdots,0)\widetilde{k}$ where $y_1,\cdots,y_q$ are real numbers and $\widetilde{k}\in U_n$.

By the definition (1.3) of $C(S)$, for any preassigned small number $\varepsilon>0$, there exists a holomorphic mapping $f(z)$ and p_0, $1\le p_0\le q$, such that $\left|\sum_{i,j}k_{p_0 i}d_{ij}^{(j)}\right|=C(S)-\varepsilon$, where $d_{ij}^{(j)}$ are the coefficients of the expansion of $f(z)$ at (1.1).

It is possible to choose a z such that $z=\xi(\exp Ad(k)X\cdot O)$ where the unitary representation of k is $\widetilde{k}'$. Then

$$\sum_{i,j}k_{pi}d_{ij}^{(j)}=k_pd'=k_p\overline{\widetilde{k}}'(y_1,\cdots,y_q,0,\cdots,0)'=y_p,\quad 1\le p\le q,$$

and (2.1) becomes

$$\log\det J_f(z)=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+2\sum_1^q\int_0^1\frac{y_p\tanh x_p}{1-\rho^2\tanh^2x_p}d\rho$$
$$+2\sum_1^q\int_0^1\frac{\tanh x_p}{1-\rho^2\tanh^2x_p}o(1)d\rho.$$

It is also possible to choose a z such that $y_{p_0}\tanh x_{p_0}\ge 0$ and $x_j=0$ when $j\ne p_0$. Then the previous equality becomes

$$\begin{aligned}\log\det J_f(z)&=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+(|y_{p_0}|+o(1))\log\frac{1+|\tanh x_{p_0}|}{1-|\tanh x_{p_0}|}\\&=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+(|y_{p_0}|+o(1))\sum_{p=1}^q\log\frac{1+|\tanh x_{p_0}|}{1-|\tanh x_{p_0}|}\\&=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+(C(S)-\varepsilon+o(1))\sum_{p=1}^q\log\frac{1+|\tanh x_{p_0}|}{1-|\tanh x_{p_0}|}.\end{aligned}$$

Since ε is any preassigned small number, the right hand side of (1.4) cannot be improved.

Similarly, it is also possible to choose a z such that $y_{p_0}\tanh x_{p_0}\le 0$ and $x_j=0$ when $j\ne p_0$. Hence for any preassigned small number ε, we can choose a z such that

$$\log\det J_f(z)=\log\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}+(C(S)-\varepsilon+o(1))\sum_{p=1}^q\log\frac{1-|\tanh x_{p_0}|}{1+|\tanh x_{p_0}|}.$$

It means the left hand side of (1.4) cannot be improved. Combining these two results, we prove that $C(S)$ is the smallest positive number such that (1.2) holds.

Lemma 2.2. *If $f(z)$ is a normalized holomorphic mapping* (1.1) *which is defined on M, then $\frac{|\det J_f(z)|}{\sqrt{K_M(z,\bar z)/K_M(m,\overline{m})}}$ is a biholomorphic invariant.*

Proof. Suppose H is the biholomorphic mapping which maps M onto N. Since f is a normalized holomorphic mapping on M, the mapping $h=J_H(m)\circ f\circ H^{-1}$ is a normalized holomorphic mapping on N. Then

$$\det J_h(w)=\det J_H(m)\det J_f(z)\det J_{H^{-1}}(w). \tag{2.2}$$

Since $N=H(M)$, we have

$$K_N(w,\overline{w})=K_M(z,\bar z)|\det J_{H^{-1}}(w)|^2, \tag{2.3}$$
$$K_N(\widetilde{m},\overline{\widetilde{m}})=K_M(m,\overline{m})|\det J_{H^{-1}}(m)|^2=K_M(m,\overline{m})|\det J_H(m)|^{-2} \tag{2.4}$$

where $H(m)=\widetilde{m}$. By (2.3), (2.4), we get

$$\sqrt{\frac{K_N(w,\overline{w})}{K_N(\widetilde{m},\overline{\widetilde{m}})}}=\sqrt{\frac{K_M(z,\bar z)}{K_M(m,\overline{m})}}|\det J_{H^{-1}}(w)||\det J_H(m)|. \tag{2.5}$$

By (2.2), (2.5), we obtain

$$\sqrt{\frac{K_N(w,\overline{w})}{K_N(\widetilde{m},\overline{\widetilde{m}})}}\sqrt{\frac{K_M(m,\overline{m})}{K_M(z,\bar z)}}|\det J_f(z)|=|\det J_h(w)|,$$

i.e.,

$$\frac{|\det J_f(z)|}{\sqrt{K_M(z,\bar{z})/K_M(m,\overline{m})}} = \frac{|\det J_h(w)|}{\sqrt{K_N(w,\overline{w})/K_N(\widetilde{m},\overline{\widetilde{m}})}}.$$

From Lemmas 1 and 2, we immediately get

Lemma 2.3. *If M, N are holomorphically equivalent to each other, $C(S_M)$ and $C(S_N)$ are the smallest numbers such that (1.4) holds on M and N respectively, then $C(S_M) = C(S_N)$. In particular, when M and N are holomorphically equivalent to each other and M is the canonical realization of G/K, then $C(S_M) = C(S_N)$.*

Based upon these lemmas, the estimates of $C(S)$ will be given. As a simple application, we have

Theorem 2.1. *If M is a bounded symmetric domain which is the canonical realization of G/K, and S is the family of normalized biholomorphic convex mappings on M, then $C(S) \le 2n-1$.*

Proof. Since M is the canonical realization, any point $z = (z_1, \cdots, z_n) \in M$ can be expressed as $(\lambda_1, \cdots, \lambda_q, 0, \cdots, 0)\widetilde{k}$ where $|\lambda_1| < 1, \cdots, |\lambda_q| < 1, \widetilde{k} \in U_n$. Since M is bounded, there exist $r_{q+1}, \cdots, r_n$, such that $|r_{q+1}z_{q+1}| < 1, \cdots, |r_n z_n| < 1$ hold for all $z \in M$. The linear transformation $w = zA$ transforms M to N where $A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & 0 & \\ & & & r_{q+1} & & \\ & 0 & & & \ddots & \\ & & & & & r_n \end{pmatrix}$.

If S_M and S_N are families of normalized biholomorphic convex mappings on M and N respectively, then $C(S_M) = C(S_N)$ by Lemma 2.3. But $w = (w_1, \cdots, w_n) \in N$ has the property: $\{w_j e_j \in N\}$ forms a unit disc, $j = 1, \cdots, n$, where e_j denotes the n-dimensional vector for which each entry is zero except that the j-th entry is one. N is convex since M is convex. Using the method which we used at the Lemma 3 of [2], we get $|d_{jk}^{(m)}| \le 2$, where $d_{jk}^{(m)}$ are the coefficients of the expansion (1.1) of the mapping in S_N. By the Lemma 2 of [2], we get $C(S_N) \le 2n-1$. We prove Theorem 2.1 by $C(S_N) = C(S_M)$.

§3. Jacobian of Holomorphic Automorphism Mappings

On purpose to get the estimate of the lower bound of $C(S)$, usually we try to find a mapping in S, and find a value such that (1.4) holds. Of course we expect this mapping is the extremal mapping. Now we take the mapping as F_b which we will define below.

We start from the Lie group of holomorphic automorphisms. We denote the non-zero roots (including multiplicity) of adjoint representations of $\mathfrak{A}$ in Lie algebra $\mathcal{G}$ by

$$\{\pm\alpha_j, j = q+1, \cdots, 2n\}$$

where α_j are positive roots. Then the basis $X_1, \cdots, X_q$ of $\mathfrak{A}$ and $Y_{\alpha_j} + Y_{-\alpha_j}$, $j = q+1, \cdots, 2n$ form a basis of $\mathcal{P}$, where $Y_{\pm\alpha_j}$ are the eigenvectors corresponding to the non-zero root $\pm\alpha_j$, and $Y_{\alpha_j} - Y_{-\alpha_j}$, $j = q+1, \cdots, 2n$ form a part of the basis of $\mathcal{K}$. The mapping

$$\text{Exp}: \ x \to k(x)a(X) \cdot O$$

is a diffeomorphism of an open set in $2n$-dimensional Euclidean space to a condensed open set in G/K, where $x = (x_1, \cdots, x_{2n})$, O is the identity coset in G/K,

$$k(x) = \exp \sum_{j=q+1}^{2n} x_j(Y_{\alpha_j} - Y_{-\alpha_j}), \quad a(X) = \exp \sum_{j=1}^{q} x_j X_j.$$

Let $g \in G$, we evaluate $J_{\phi_g}(z)$. For any $z = (z_1, \cdots, z_n) \in M$, we can express z as $(\lambda_1, \cdots, \lambda_q, 0, \cdots, 0)\widetilde{k}$, where $\lambda_j = \tanh x_j$, $j = 1, \cdots, q$, $\widetilde{k} \in U_n$. Let $z_j = u_j + iu_{n+j}$. Then $u = (u_1, \cdots, u_{2n})$ gives the real coordinates of the point z, and we have real orthogonal matrix $P(k)$ such that

$$(u_1, \cdots, u_{2n}) = (\lambda_1, \cdots, \lambda_q, 0, \cdots, 0)P(k).$$

For any $k \in K$, the action of ϕ_k on M is just as a linear transformation action on M, and the absolute value of the determinant of the Jacobian is equal to one. So we only need to consider the Jacobian $J_{\phi_{a_1}}$ of ϕ_{a_1}, $a_1 \in A$. Let F_{a_1} be the normalization of ϕ_{a_1}. Letting a_1 approach to the boundary b, we obtain a normalized biholomorphic mapping F_b. M is convex since M is the canonical realization of G/K. ϕ_{a_1} and F_{a_1} are convex mappings, which implies F_b is a convex mapping.

Let $k(y)a(Y) \cdot O = a_1 k(x)a(X) \cdot O$ where $y = (y_1, \cdots, y_{2n})$, $k(y) = \exp \sum\limits_{j=q+1}^{2n} y_j(Y_{\alpha_j} - Y_{-\alpha_j})$, $a_1(Y) = \exp Y = \exp \sum\limits_{j=1}^{q} y_j X_j$.

We take the local coordinates $t = \{t_1, \cdots, t_{2n}\}$ at a neighborhood of $k(x)a(X) \cdot O$. The points at the neighborhood of $k(x)a(X) \cdot O$ can be expressed as

$$\{k(x)k(t)a(X+T) \cdot O, \quad T = \sum_{j=1}^{q} t_j X_j\}.$$

ϕ_a acting on each point of this neighborhood, we get a neighborhood of $k(y)a(Y) \cdot O$, its local coordinates are $\{s = (s_1, \cdots, s_{2n})\}$, the points in the neighborhood can be expressed as

$$\{k(y)k(s)a(Y+S) \cdot O, \quad S = \sum_{j=1}^{q} s_j X_j\}.$$

We have the following diagram:

$$\begin{array}{ccc} t & \longrightarrow & k(x)k(t)a(x+T) \cdot O \\ \downarrow & & \downarrow{\scriptstyle a_1} \\ s & \longrightarrow & k(y)k(s)a(Y+S) \cdot O \end{array}$$

$$\begin{array}{cc} \xrightarrow{\xi} & z = (\lambda_1(t), \cdots, \lambda_q(t), 0, \cdot, 0)\widetilde{k}(t)\widetilde{k}(x) \\ & \downarrow{\scriptstyle \phi_{a_1}} \\ \xrightarrow{\xi} & w = (\eta_1(s), \cdots, \eta_q(s), 0, \cdots, 0)\widetilde{k}(s)\widetilde{k}(y) \end{array}$$

$$\longrightarrow u = (u_1, \cdots, u_{2n}) = (\lambda_1(t), \cdots, \lambda_q(t), 0, \cdots, 0)P(k(t))P(k(x))$$

$$\downarrow$$

$$\longrightarrow v = (v_1, \cdots, v_{2n}) = (\eta_1(s), \cdots, \eta_q(s), 0, \cdots, 0)P(k(s))P(k(y)).$$

At this diagram, $k(y)k(s)a(Y+S)\cdot O = ak(x)k(t)a(X+T)\cdot O$, $\lambda_j(t) = \tanh(x_j+t_j)$, $\eta_j(t) = \tanh(y_j+s_j)$, $\lambda_j = \tanh x_j$ and $\eta_j = \tanh y_j$, $j = 1, \cdots, q$.

Our goal is to evaluate $\left|\frac{\partial(w_1,\cdots,w_n)}{\partial(z_1,\cdots,z_n)}\right|$. Obviously,

$$\left|\frac{\partial(v_1,\cdots,v_{2n})}{\partial(u_1,\cdots,u_{2n})}\right| = \left|\frac{\partial(w_1,\cdots,w_n)}{\partial(z_1,\cdots,z_n)}\right|^2$$

and

$$\frac{\partial(v_1,\cdots,v_{2n})}{\partial(u_1,\cdots,u_{2n})} = \frac{\partial(v_1,\cdots,v_{2n})}{\partial(s_1,\cdots,s_{2n})}\cdot\frac{\partial(s_1,\cdots,s_{2n})}{\partial(t_1,\cdots,t_{2n})}\cdot\left(\frac{\partial(u_1,\cdots,u_{2n})}{\partial(t_1,\cdots,t_{2n})}\right)^{-1}.$$

We try to evaluate these three Jacobians. At first we evaluate the last one.

Since $U = (u_1, \cdots, u_{2n}) = (\lambda_1(t), \cdots, \lambda_q(t), 0, \cdots, 0)P(k(t))P(k(x))$, the value of $\frac{\partial u}{\partial t_r}$ at $t=0$ is

$$\left.\frac{\partial u}{\partial t_r}\right|_{t=0} = \left.\frac{\partial(u_1,\cdots,u_{2n})}{\partial t_r}\right|_{t=0} = (0,\cdots,0,1-\lambda_r^2,0,\cdots,0)P(k(x)) = (1-\lambda_r^2)e_rP(k(x)).$$

When $1 \le r \le q$ and e_r is a $2n$ vector, each entry is zero except that the r-th entry is 1. The value of $\frac{\partial u}{\partial t_l}$ at $t=0$ is

$$\left.\frac{\partial u}{\partial t_l}\right|_{t=0} = \left.\frac{\partial(u_1,\cdots,u_{2n})}{\partial t_l}\right|_{t=0} = (\lambda_1,\cdots,\lambda_q,0,\cdots,0)\frac{\partial}{\partial t_l}P(k(t))P(k(x)).$$

But

$$\begin{aligned}\left.\frac{\partial}{\partial t_l}P(k(t))\right|_{t=0} &= P\left(\frac{\partial}{\partial t_l}k(t)\right)\Bigg|_{t=0}\\ &= P\left(\frac{\partial}{\partial t_l}\exp\sum_{j=q+1}^{2n}t_j(Y_{\alpha_j}-Y_{-\alpha_j})\right)\Bigg|_{t=0} = P(Y_{\alpha_l}-Y_{-\alpha_l}).\end{aligned}$$

We get

$$\left.\frac{\partial u}{\partial t_l}\right|_{t=0} = (\lambda_1,\cdots,\lambda_q,0,\cdots,0)P(Y_{\alpha_l}-Y_{-\alpha_l})P(k(x))$$

when $q+1 \le l \le 2n$. Combining these results, we have

$$\begin{pmatrix} \frac{\partial u_1}{\partial t_1}, & \cdots, & \frac{\partial u_{2n}}{\partial t_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial t_q}, & \cdots, & \frac{\partial u_{2n}}{\partial t_q} \\ \frac{\partial u_1}{\partial t_{q+1}}, & \cdots, & \frac{\partial u_{2n}}{\partial t_{q+1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial t_{2n}}, & \cdots, & \frac{\partial u_{2n}}{\partial t_{2n}} \end{pmatrix} = \begin{pmatrix} 1-\lambda_1^2, & 0, & \cdots, & 0, & 0, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & 1-\lambda_q^2, & 0, & \cdots, & 0 \\ & & & * & & & \end{pmatrix} P(k(x)).$$

On the first matrix on the right hand side, the 1-th column vector is

$$(\lambda_1,\cdots,\lambda_q,0,\cdots,0)P(Y_{\alpha_l}-Y_{-\alpha_l})$$

when $q+1 \le l \le 2n$. We have

$$\left.\frac{\partial(u_1,\cdots,u_{2n})}{\partial(t_1,\cdots,t_{2n})}\right|_{t=0} = \prod_{j=1}^{q}(1-\lambda_j^2)A(\lambda_1,\cdots,\lambda_q). \tag{3.1}$$

Similarly,

$$\left.\frac{\partial(v_1,\cdots,v_{2n})}{\partial(s_1,\cdots,s_{2n})}\right|_{t=0} = \prod_{j=1}^{q}(1-\eta_j^2)A(\eta_1,\cdots,\eta_q), \tag{3.2}$$

where $A(\lambda_1,\cdots,\lambda_q)$ is a function of $\lambda_1,\cdots,\lambda_q$.

Now we try to evaluate $\left.\frac{\partial(s_1,\cdots,s_{2n})}{\partial(t_1,\cdots,t_{2n})}\right|_{t=0}$. Since $k(y)k(s)a(Y+S)\cdot O = a_1k(x)k(t)a(X+T)\cdot O$, there exist $k_1 \in K$, such that $k(y)k(s)a(Y+S)k_1 = a_1k(x)k(t)a(X+T)$. Taking the adjoint representation on both sides, since the representation of a is diagonal and the real representation of k is orthogonal, we have

$$\begin{aligned}&Ada_1Adk(x)Adk(t)Ad(a(X+T)^2)(Adk(t))'(Adk(x))'(Ada_1)'\\ =&Adk(y)Adk(s)Ad(a(Y+S)^2)(Adk(s))'(Adk(y))'.\end{aligned} \tag{3.3}$$

We take the derivative with respect to t_l on both sides of (3.3), and evaluate the value at $t=0$. Because $Ad\exp X = \mathrm{Exp}\, ad\, X$, where Exp means the exponential of matrix, we have

$$\begin{aligned}\left.\frac{\partial}{\partial t_l}Adk(s)\right|_{t=0} &= \left.\frac{\partial}{\partial t_l}\mathrm{Exp}\sum_{j=q+1}^{2n}s_j ad(Y_{\alpha_j}-Y_{-\alpha_j})\right|_{t=0}\\ &= \sum_{j=q+1}^{2n}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0} ad(Y_{\alpha_j}-Y_{-\alpha_j}),\\ \left.\frac{\partial}{\partial t_l}Ad(a(Y+S)^2)\right|_{t=0} &= Ad(a(Y)^2)\left.\frac{\partial}{\partial t_l}Ad(a(2s))\right|_{t=0}\\ &= Ad(a(Y)^2)\left.\frac{\partial}{\partial t_l}\mathrm{Exp}\sum_{j=1}^{q}2s_j adX_j\right|_{t=0}\\ &= Ad(a(Y)^2)ad\left(2\sum_{j=1}^{q}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0}X_j\right)\\ &= 2Ad(a(Y)^2)\sum_{j=1}^{q}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0} adX_j.\end{aligned}$$

But $(Adk(s))' = (Adk(s))^{-1} = Adk(-s)$. We take the derivative with respect to t_l on the right hand side of (3.3), the value at $t=0$ is

$$Adk(y)\sum_{j=q+1}^{2n}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0} ad(Y_{\alpha_j}-Y_{-\alpha_j})Ad(a(Y)^2)(Adk(y))' - Adk(y)Ad(a(Y)^2)\sum_{j=q+1}^{2n}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0}\cdot$$

$$\cdot ad(Y_{\alpha_j}-Y_{-\alpha_j})(Adk(y))' + Adk(y)Ad(a(Y)^2)2\sum_{j=0}^{q}\left.\frac{\partial s_j}{\partial t_l}\right|_{t=0} adX_j(Adk(y))'.$$

We consider the left hand side of (3.3). Since

$$\begin{aligned}
\frac{\partial}{\partial t_l} Ad(a(X+T)^2)\Big|_{t=0} &= Ad(a(x)^2)\,\frac{\partial}{\partial t_l} Ada(2t)\Big|_{t=0} \\
&= Ad(a(x)^2)\,\frac{\partial}{\partial t_l}\mathrm{Exp}\sum_{j=1}^{q} 2t_j\cdot adX_j\Big|_{t=0} \\
&= Ad(a(x)^2)2adX_l \\
&= 2Ad(a(x)^2)adX_l \quad \text{if } 1\le l\le q; \\
\frac{\partial}{\partial t_l} Adk(t)\Big|_{t=0} &= \frac{\partial}{\partial t_l}\mathrm{Exp}\sum_{j=q+1}^{2n} t_j ad(Y_{\alpha_j}-Y_{-\alpha_j})\Big|_{t=0} \\
&= ad(Y_{\alpha_l}-Y_{-\alpha_l}) \quad \text{if } q+1\le l\le 2n;
\end{aligned}$$

the value of the derivative with respect to t_l on the left hand side of (3.3) at $t=0$ is

$$Ada_1\cdot Adk(x)\cdot 2Ad(a(x)^2)adX_l\cdot (Adk(x))'(Ada_1)' \quad \text{if } 1\le l\le q;$$

$$Ada_1\cdot Adk(x)\cdot ad(Y_{\alpha_l}-Y_{-\alpha_l})\cdot Ad(a(x)^2)(Adk(x))'(Ada_1)' - Ada_1\cdot Adk(x)\cdot$$

$$\cdot Ad(a(x)^2)\cdot ad(Y_{\alpha_l}-Y_{-\alpha_l})(Adk(x))'(Ada_1)' \quad \text{if } q+1\le l\le 2n.$$

Combining the previous equalities, we have

$$\begin{aligned}
&Adk(y)\Bigg[\sum_{j=q+1}^{2n}\frac{\partial s_j}{\partial t_l}\Big|_{t=0} ad(Y_{\alpha_j}-Y_{-\alpha_j})\cdot Ad(a(Y)^2) \\
&-Ad(a(Y)^2)\sum_{j=q+1}^{2n}\frac{\partial s_j}{\partial t_l}\Big|_{t=0} ad(Y_{\alpha_l}-Y_{-\alpha_j}) + Ad(a(Y)^2)2\sum_{j=1}^{q}\frac{\partial s_j}{\partial t_l}\Big|_{t=0} adX_j\Bigg](Adk(y))' \\
&=\begin{cases} Ada_1\cdot Adk(x)\cdot 2Ad(a(X)^2)adX_l\cdot (Adk(x))'(Ada_1)', & \text{if } 1\le l\le q; \\ Ada_1\cdot Adk(x)[ad(Y_{\alpha_l}-Y_{-\alpha_l})\cdot Ad(a(x)^2) - Ad(a(x)^2)ad(Y_{\alpha_l}-Y_{-\alpha_l})]\cdot \\ \cdot (Adk(x))'(Ada_1)', \quad \text{if } q+1\le l\le 2n. \end{cases}
\end{aligned} \tag{3.4}$$

Letting $t=0$ at (3.3), we get

$$Ada_1 Adk(x)Ad(a(x)^2)(Adk(x))'(Ada_1)' = Adk(y)Ad(a(Y)^2)(Adk(y))'.$$

Taking the inverse on both sides, we have

$$(Ada_1)^{-1}(Adk(x))Ad(a(x)^{-2})(Adk(x))'(Ada_1)^{-1} = Adk(y)Ad(a(Y)^{-2})(Adk(y))'.$$

Multiplying the right hand side of the previous equality to the left hand side of (3.4), and the left hand side of the previous equality to the right hand side of (3.4), we get

$$Adk(y)Ad(a(Y)^{-2})\sum_{j=q+1}^{2n}\frac{\partial s_j}{\partial t_l}\bigg|_{t=0} ad(Y_{\alpha_j}-Y_{-\alpha_j})Ad(a(Y)^2)(Adk(y))'-Adk(y)$$

$$\cdot\sum_{j=q+1}^{2n}\frac{\partial s_j}{\partial t_l}\bigg|_{t=0} ad(Y_{\alpha_l}-Y_{-\alpha_j})\cdot(Adk(y))'+Adk(y)\cdot 2\sum_{j=1}^{q}\frac{\partial s_j}{\partial t_l}\bigg|_{t=0} adX_j(Ad(k(y))'$$

$$=\begin{cases}(Ada_1)^{-1}Adk(x)\cdot 2adX_l(Adk(x))'(Ada_1)', & \text{if } 1\le l\le q;\\ (Ada_1)^{-1}Adk(x)Ad(a(x)^{-2})ad(Y_{\alpha_l}-Y_{-\alpha_l})Ad(a(x)^2)(Adk(x))'(Ada_1)' & \\ -(Ada_1)^{-1}Adk(x)Ad(a(x))^2ad(Y_{\alpha_l}-Y_{-\alpha_l})(Adk(x))'(Ada_1)', & \text{if } q+1\le l\le 2n.\end{cases}\tag{3.5}$$

For any $X\in\mathfrak{A}$, we have

$$Ada^{-1}(X)adY_{\alpha_l}Ada(x)=ad(e^{-adX}Y_{\alpha_l})=ad(e^{-\alpha_l(x)}Y_{\alpha_l})$$

because $[X,Y_{\alpha_j}]=\alpha_j(X)Y_{\alpha_j}$ holds for all $X\in\mathfrak{A}$. Similarly

$$Ada^{-1}(X)adY_{-\alpha_l}Ada(x)=ad(e^{\alpha_l(x)}Y_{-\alpha_l}).$$

Therefore

$$Ad(a(Y)^{-2})ad(Y_{\alpha_j}-Y_{-\alpha_j})Ad(a(Y)^2)=\frac{e^{-2\alpha_l(Y)}-e^{2\alpha_l(Y)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l})$$
$$+\frac{e^{-2\alpha_l(Y)}+e^{2\alpha_l(Y)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l});$$

$$Ad(a(X)^{-2})ad(Y_{\alpha_j}-Y_{-\alpha_j})Ad(a(X)^2)=\frac{e^{-2\alpha_l(X)}-e^{2\alpha_l(X)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l})$$
$$+\frac{e^{-2\alpha_l(X)}+e^{2\alpha_l(X)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l}).$$

Substituting this formula into (3.5), we have

$$Adk(y)\Bigg[\sum_{j=q+1}^{2n}\frac{\partial s_j}{\partial t_l}\bigg|_{t=0}\bigg\{\frac{e^{-2\alpha_l(Y)}-e^{2\alpha_l(Y)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l})$$
$$+\frac{(e^{-\alpha_l(Y)}-e^{\alpha_l(Y)})^2}{2}ad(Y_{\alpha_l}-Y_{-\alpha_l})\bigg\}+2\sum_{j=1}^{q}\frac{\partial s_j}{\partial t_l}\bigg|_{t=0}adX_j\Bigg](Adk(y)')$$

$$=\begin{cases}2(Ada_1)^{-1}Adk(x)\cdot adX_l(Adk(x))'(Ada_1)', & \text{if } 1\le l\le q;\\ (Ada_1)'^{-1}Adk(x)\Big[\frac{e^{-2\alpha_l(X)}-e^{2\alpha_l(X)}}{2}ad(Y_{\alpha_l}+Y_{-\alpha_l})+\frac{(e^{-\alpha_l(X)}-e^{\alpha_l(X)})^2}{2}\cdot & \\ \cdot ad(Y_{\alpha_l}-Y_{-\alpha_l})\Big](Adk(x))'(Ada_1)', & \text{if } q+1\le l\le 2n.\end{cases}\tag{3.6}$$

$\{X_1,\cdots,X_q,Y_{\alpha_{q+1}}+Y_{-\alpha_{q+1}},\cdots,Y_{\alpha_{2n}}+Y_{-\alpha_{2n}}\}$ is a basis of $\mathcal{P}$, and $\{Y_{\alpha_{q+1}}-Y_{-\alpha_{q+1}},\cdots,Y_{\alpha_{2n}}-Y_{-\alpha_{2n}}\}$ is a part of a basis of $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{K}$ are invariant subspaces under the adjoint representation of K in $\mathcal{G}$. When we choose the basis of $\mathcal{G}$ as above, the adjoint representation matrix of $k\in K$ on the column vector can be expressed as $\begin{pmatrix}\widetilde{k}_1 & 0\\ 0 & \widetilde{k}_2\end{pmatrix}$, where $\widetilde{k}_1$ is acting on $\mathcal{P}$, and $\widetilde{k}_2$ is acting on K.

Let

$$B(Y) = \operatorname{diag}\left[e^{-\alpha_{q+1}(Y)} + e^{\alpha_{q+1}(Y)}, \cdots, e^{-\alpha_{2n}(Y)} + e^{\alpha_{2n}(Y)}\right],$$

$$C(Y) = \operatorname{diag}\left[e^{-\alpha_{q+1}(Y)} - e^{\alpha_{q+1}(Y)}, \cdots, e^{-\alpha_{2n}(Y)} - e^{\alpha_{2n}(Y)}\right].$$

Taking the basis $\{X_1, \cdots, X_q, Y_{\alpha_{q+1}} + Y_{-\alpha_{q+1}}, \cdots, Y_{\alpha_{2n}} + Y_{-\alpha_{2n}}, Y_{\alpha_{q+1}} - Y_{-\alpha_{q+1}}, \cdots, Y_{\alpha_{2n}} - Y_{-\alpha_{2n}}, \cdots\}$ as coordinates, then the left hand side can be expressed as

$$\left(\frac{\partial s_1}{\partial t_l}, \cdots, \frac{\partial s_{2n}}{\partial t_l}, \frac{\partial s_{q+1}}{\partial t_l}, \cdots, \frac{\partial s_{2n}}{\partial t_l}, 0, \cdots, 0\right)_{t=0} \cdot \begin{pmatrix} 2I_q & 0 & 0 & 0 \\ 0 & \frac{1}{2}B(Y)C(Y) & 0 & 0 \\ 0 & 0 & \frac{1}{2}C(Y)^2 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{k}_1(y) & 0 \\ 0 & \widetilde{k}_2(y) \end{pmatrix}.$$

Since $a_1 \in A$, we have $a_1 = \exp R$, $R = \sum_{j=1}^{q} r_j X_j \in \mathfrak{A}$. If $\widetilde{a}_1$ is the adjoint representation matrix of a_1^{-1}, we have

$$\begin{aligned} Ad(a_1^{-1})ad(Y_{\alpha_j} - Y_{-\alpha_j})Ada_1 =& \frac{1}{2}(e^{-\alpha_j(R)} - e^{\alpha_j(R)})ad(Y_{\alpha_j} + Y_{-\alpha_j}) \\ &+ \frac{1}{2}(e^{-\alpha_j(R)} + e^{\alpha_j(R)})ad(Y_{\alpha_j} - Y_{-\alpha_j}), \\ Ad(a_1^{-1})ad(Y_{\alpha_j} + Y_{-\alpha_j})Ada_1 =& \frac{1}{2}(e^{-\alpha_j(R)} + e^{\alpha_j(R)})ad(Y_{\alpha_j} + Y_{-\alpha_j}) \\ &+ \frac{1}{2}(e^{-\alpha_j(R)} - e^{\alpha_j(R)})ad(Y_{\alpha_j} - Y_{-\alpha_j}), \\ Ad(a_1^{-1})adX_jAda_1 =& adX_j. \end{aligned}$$

Relative to the basis of $\mathcal{G}$, we get

$$\widetilde{a}_1 = \begin{pmatrix} I_q, & 0, & 0, & 0 \\ 0, & \frac{1}{2}B(R), & \frac{1}{2}C(R), & 0 \\ 0, & \frac{1}{2}C(R), & \frac{1}{2}B(R), & 0 \\ 0, & 0, & 0, & I \end{pmatrix}.$$

Then the right hand side of (3.6) can be expressed as

$$\begin{cases} 2e_l \begin{pmatrix} \widetilde{k}_1(x) & 0 \\ 0 & \widetilde{k}_2(x) \end{pmatrix} \widetilde{a}_1, \quad \text{if } 1 \le l \le q; \\ \left(\frac{1}{2}\left(e^{-2\alpha_l(x)} - e^{2\alpha_l(x)}\right) e_l + \frac{1}{2}\left(e^{-\alpha_l(x)} - e^{\alpha_l(x)}\right)^2 e_{2n+l-q}\right) \begin{pmatrix} \widetilde{k}_1(x) & 0 \\ 0 & \widetilde{k}_2(x) \end{pmatrix} \widetilde{a}_1, \\ \quad \text{if } q+1 \le l \le 2n. \end{cases}$$

Combining all these results, we have

$$\begin{pmatrix} \frac{\partial s_1}{\partial t_1}, & \cdots, & \frac{\partial s_{2n}}{\partial t_1}, & \frac{\partial s_{q+1}}{\partial t_1}, & \cdots, & \frac{\partial s_{2n}}{\partial t_1}, & 0, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial s_1}{\partial t_{2n}}, & \cdots, & \frac{\partial s_{2n}}{\partial t_{2n}}, & \frac{\partial s_{q+1}}{\partial t_{2n}}, & \cdots, & \frac{\partial s_{2n}}{\partial t_{2n}}, & 0, & \cdots, & 0 \end{pmatrix}_{t=0}$$

$$\cdot \begin{pmatrix} 2I_q & 0 & 0 & 0 \\ 0 & \frac{1}{2}B(Y)C(Y) & 0 & 0 \\ 0 & 0 & \frac{1}{2}C(Y)^2 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{k}_1(y) & 0 \\ 0 & \widetilde{k}_2(y) \end{pmatrix}$$

$$= \begin{pmatrix} 2I_q & 0 & 0 & 0 \\ 0 & \frac{1}{2}B(X)C(X) & \frac{1}{2}C(X)^2 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{k}_1(X) & 0 \\ 0 & \widetilde{k}_2(X) \end{pmatrix} \begin{pmatrix} I_q & 0 & 0 & 0 \\ 0 & \frac{1}{2}B(R) & \frac{1}{2}C(R) & 0 \\ 0 & \frac{1}{2}C(R) & \frac{1}{2}B(R) & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

From the previous equality we get

$$\begin{pmatrix} \frac{\partial s_1}{\partial t_1}, & \cdots, & \frac{\partial s_{2n}}{\partial t_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial s_1}{\partial t_{2n}}, & \cdots, & \frac{\partial s_{2n}}{\partial t_{2n}} \end{pmatrix}_{t=0} \begin{pmatrix} 2I_q & 0 \\ 0 & \frac{1}{2}B(Y)C(Y) \end{pmatrix} \widetilde{k}_1(y)$$

$$= \begin{pmatrix} 2I_q & 0 \\ 0 & \frac{1}{2}B(X)C(X) \end{pmatrix} \widetilde{k}_1(X) \begin{pmatrix} I_q & 0 \\ 0 & \frac{1}{2}B(R) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}C(X)^2E(x)\frac{1}{2}C(R) \end{pmatrix}, \tag{3.7}$$

where $E(x)$ is the $2n-q$ square matrix which consists of the first $2n-q$th rows and columns of $\widetilde{k}_2(x)$.

Taking determinants on both sides of the previous equality, we have

$$\left.\frac{\partial(s_1,\cdots,s_{2n})}{\partial(t_1,\cdots,t_{2n})}\right|_{t=0} 2^q \det(\frac{1}{2}B(Y)C(Y))$$

$$=2^q \det(\frac{1}{2}B(X)C(X)) \det(\frac{1}{2}B(R)) \det\left(\widetilde{k}_1(x) + \begin{pmatrix} 0 & 0 \\ 0 & D(x)E(x)D(R) \end{pmatrix}\right), \tag{3.8}$$

where $D(X) = C(X)B(X)^{-1}$. After simplication, (3.8) becomes

$$\left.\frac{\partial(s_1,\cdots,s_{2n})}{\partial(t_1,\cdots,t_{2n})}\right|_{t=0} = \frac{\det(\frac{1}{2}B(X)\frac{1}{2}C(X))}{\det(\frac{1}{2}B(Y)\frac{1}{2}C(Y))} \det(\frac{1}{2}B(R)) \det(A(x,R)), \tag{3.9}$$

where

$$A(x,R) = \widetilde{k}_1(x) + \begin{pmatrix} 0 & 0 \\ 0 & D(x)E(x)D(R) \end{pmatrix}.$$

From (3.1), (3.2) and (3.9), we get

$$\left.\frac{\partial(v_1,\cdots,v_{2n})}{\partial(u_1,\cdots,u_{2n})}\right|_{t=0} = \left.\frac{\partial(v_1,\cdots,v_{2n})}{\partial(s_1,\cdots,s_{2n})}\right|_{t=0} \left.\frac{\partial(s_1,\cdots,s_{2n})}{\partial(t_1,\cdots,t_{2n})}\right|_{t=0} \left.\left(\frac{\partial(u_1,\cdots,u_{2n})}{\partial(t_1,\cdots,t_{2n})}\right)^{-1}\right|_{t=0}$$

$$= \prod_{j=1}^{q} \frac{1-\eta_j^2}{1-\lambda_j^2} \cdot A(\eta_1,\cdots,\eta_q)A^{-1}(\lambda_1,\cdots,\lambda_q) \frac{\det(\frac{1}{2}B(X)\frac{1}{2}C(X))}{\det(\frac{1}{2}B(Y)\frac{1}{2}C(Y))}$$

$$\cdot \det(\frac{1}{2}B(R)) \det(A(x,R)). \tag{3.10}$$

$\det \frac{\partial w}{\partial z} \neq 0$, since ϕ_{a_1} is a biholomorphic mapping and $\left.\frac{\partial(v_1,\cdots,v_{2n})}{\partial(u_1,\cdots,u_{2n})}\right|_{t=0} \neq 0$. (3.10) is a continuous function on M. Let $X \to 0$, the limiting values on both sides exist. The limiting value of right hand side is nonzero because the left hand side is nonzero. Therefore

$$\lim_{x\to 0}\det(\frac{1}{2}C(X))A^{-1}(\lambda_1,\cdots,\lambda_q)$$

is equal to a constant, and denote it by c_0. Taking $X=sD$, $D=\sum_{j=1}^{q}d_jX_j$,

$$\lim_{s\to 0}\frac{1}{s}\frac{1}{2}(e^{-\alpha_l(sD)}-e^{\alpha_l(sD)})=-\alpha_l(D)$$

hold for any $D\in\mathfrak{A}$. But $\lim_{s\to 0}\frac{1}{s}\tanh s\alpha_j=\alpha_j$. We have

$$\lim_{s\to 0}\det(\frac{1}{2}C(sD))A^{-1}(\tanh sd_1,\cdots,\tanh sd_q)=(-1)^{2n-q}\prod_{j=q+1}^{2n}\alpha_j(D)A^{-1}(d_1,\cdots,d_q)$$

for any $D\in\mathfrak{A}$.

We get the value of $A(d_1,\cdots,d_q)=(-1)^{2n-q}c_0\prod_{j=q+1}^{2n}\alpha_j(D)$. Set

$$\widetilde{\alpha}_j(x)=\sum_{r=1}^{q}\tanh x_r\alpha_j(X_r).$$

We rewrite (3.10) as

$$\left.\frac{\partial(v_1,\cdots,v_{2n})}{\partial(u_1,\cdots,u_{2n})}\right|_{t=0}=\prod_{j=1}^{q}\frac{1-\eta_j^2}{1-\lambda_j^2}\prod_{j=q+1}^{2n}\frac{\widetilde{\alpha}_j(Y)}{\widetilde{\alpha}_j(X)}\cdot\frac{\det(\frac{1}{2}B(X)\cdot\frac{1}{2}C(X))}{\det(\frac{1}{2}B(Y)\cdot\frac{1}{2}C(Y))}$$
$$\cdot\det(\frac{1}{2}B(R))\det(A(x,R)).$$

Finally we have

Theorem 3.1. *Suppose $M\subset C^n$ is a bounded symmetric domain which contains the origin, and it is the canonical realization of the Hermite symmetric space G/K. If $z\in M$, then $z=\xi(\exp Ad(k)X\cdot O)$ is the realization of G/K onto M where $k\in K$, $X=\sum_{j=1}^{q}x_jX_j\in\mathfrak{A}$, $O=eK$ is the identity coset in G/K. If $a_1=\exp R\in A$, $R=\sum_{j=1}^{q}r_jX_j\in\mathfrak{A}$ and ϕ_{a_1} denotes the holomorphic automorphism corresponding to a_1, which maps $z=(z_1,\cdots,z_n)$ to $w=(w_1,\cdots,w_n)$, where*

$$z=\xi(\exp Ad(k(\widetilde{x}))X\cdot O),\quad k(\widetilde{x})\in K,\quad X=\sum_{j=1}^{q}x_jX_j\in\mathfrak{A},$$

$$w=\xi(\exp Ad(k(\widetilde{y}))Y\cdot O),\quad k(\widetilde{y})\in K,\quad Y=\sum_{j=1}^{q}y_jY_j\in\mathfrak{A},$$

then

$$\left|\frac{\partial(w_1,\cdots,w_n)}{\partial(z_1,\cdots,z_n)}\right|^2=\prod_{j=1}^{q}\frac{1-\eta_j^2}{1-\lambda_j^2}\prod_{j=q+1}^{2n}\frac{\widetilde{\alpha}_j(Y)}{\widetilde{\alpha}_j(X)}\cdot\frac{\det(\frac{1}{2}B(X)\frac{1}{2}C(X))}{\det(\frac{1}{2}B(Y)\frac{1}{2}C(Y))}$$
$$\cdot\det(\frac{1}{2}B(R))\det(\frac{1}{2}A(\widetilde{x},R)),\tag{3.11}$$

where $\lambda_j=\tanh x_j$, $\eta_j=\tanh y_j$, $j=1,\cdots,q$; $\{\alpha_j,j=q+1,\cdots,2n\}$ are nonzero positive roots of adjoint representation of $\mathfrak{A}$ in $\mathcal{G}$, i.e., $[X,Y_{\alpha_j}]=\alpha_j(X)Y_{\alpha_j}$ holds for any $X\in\mathfrak{A}$, *where* Y_{α_j} *is the corresponding eigenvector of nonzero positive root* α_j,

$$B(X) = diag\left[e^{-\alpha_{q+1}(X)} + e^{\alpha_{q+1}(X)}, \cdots, e^{-\alpha_{2n}(X)} + e^{\alpha_{2n}(X)}\right],$$

$$C(X) = diag\left[e^{-\alpha_{q+1}(X)} - e^{\alpha_{q+1}(X)}, \cdots, e^{-\alpha_{2n}(X)} - e^{\alpha_{2n}(X)}\right]$$

and $A(\widetilde{x}, R) = \widetilde{k}_1(\widetilde{x}) + \begin{pmatrix} 0 & 0 \\ 0 & D(x)E(\widetilde{x})D(R) \end{pmatrix}$, $D(X) = C(X)B(X)^{-1}$, $E(x)$ *is a* $2n-q$ *matrix which consists of the first* $2n-q$*th rows and columns of the matrix* $\widetilde{k}_2(\widetilde{x})$, $\widetilde{k}(\widetilde{x}) = \begin{pmatrix} \widetilde{k}_1(\widetilde{x}) & 0 \\ 0 & \widetilde{k}_2(\widetilde{x}) \end{pmatrix}$ *is the adjoint representation matrix of* $\widetilde{k}(x) \in \mathcal{K}$ *in* $\mathcal{G}$; $\widetilde{x} = (x_{q+1}, \cdots, x_{2n})$, $\widetilde{y} = (y_{q+1}, \cdots, y_{2n})$.

§4. Bergman Kernel Function and the Jacobian of F_b

If $z = \xi(ka \cdot O)$, where $k \in K$, $a \in A$, $O = eK$ is the identity coset in G/K, then

$$\frac{K_M(z,\bar{z})}{K_M(0,0)} = |\det J_{\psi_{ka}}(\xi(ka \cdot O))|^2 = |\det J_{\psi_a}(\xi(a \cdot O))|^2.$$

Taking $\widetilde{x} = 0$, $R = -X$, $Y = R + X = 0$ in (3.11), we have

$$\frac{K_M(z,\bar{z})}{K_M(0,0)} = \prod_{j=1}^{q} \frac{1}{1-\lambda_j^2} \prod_{j=q+1}^{2n} \frac{1}{\widetilde{\alpha}_j(X)} (-1)^{2n-q} \det(\frac{1}{2}B(X)\frac{1}{2}C(X)) \cdot \det(\frac{1}{2}B(X)) \det(A(0,-X)).$$

Obviously,

$$\det(A(0,-X)) = \det\left(I - \begin{pmatrix} 0 & 0 \\ 0 & D(X)^2 \end{pmatrix}\right) = \det(I - D(X)^2) = \det(\frac{1}{2}B(X))^{-2}.$$

For any point $z = \xi(ka(X) \cdot O) \in M$, we can choose k, X such that $\alpha_j(X) \geq 0$, $j = q+1, \cdots, 2n$. Then we have

$$\frac{K_M(z,\bar{z})}{K_M(0,0)} = \det(\frac{1}{2}C(X)) \prod_{j=1}^{q} \frac{1}{1-\lambda_j^2} \prod_{j=q+1}^{2n} \frac{(-1)^{2n-q}}{\widetilde{\alpha}_j(X)} = \prod_{j=1}^{q} \frac{1}{1-\lambda_j^2} \prod_{j=q+1}^{2n} \frac{e^{\alpha_j(X)} - e^{-\alpha_j(X)}}{2\widetilde{\alpha}_j(X)}. \tag{4.1}$$

But

$$e^{\alpha_j(X)} = \exp\left(\sum_{k=1}^{q} x_k \alpha_j(X_k)\right) = \prod_{j=1}^{q} \left(\frac{1+\lambda_k}{1-\lambda_k}\right)^{\frac{1}{2}\alpha_j(X_k)}$$

since

$$e^{2x_j} = \frac{1+\lambda_j}{1-\lambda_j}, \quad \alpha_j(X) = \prod_{k=1}^{q} x_k \alpha_j(X_k).$$

Hence $\prod_{j=q+1}^{2n} e^{\alpha_j(X)} = \prod_{k=1}^{q} \left(\frac{1+\lambda_k}{1-\lambda_k}\right)^{\rho(X_k)}$ where $2\rho = \sum_{j=q+1}^{2n} \alpha_j$. Thus

$$\frac{K_M(z,\bar{z})}{K_M(0,0)} = \prod_{k=1}^{q} \frac{(1+\lambda_k)^{\rho(X_k)-1}}{(1-\lambda_k)^{\rho(X_k)+1}} \prod_{j=q+1}^{2n} \frac{1-e^{-2\alpha_j(X)}}{\widetilde{\alpha}_j(X)} = \prod_{k=1}^{q} (1-\lambda_k^2)^{-b_k-1}. \tag{4.2}$$

Theorem 4.1. *The Bergman kernel function of M can be expressed as (4.1) or (4.2), where $b_k = \max\{\rho(\sigma X_k), \sigma \in W\}$, W is the Weyl group of G/K.*

Let $X \to 0$ at (3.11). Then

$$\left|\frac{\partial(w_1,\cdots,w_n)}{\partial(z_1,\cdots,z_n)}\right|^2_{z=0} = \prod_{j=1}^{q}(1-\theta_j^2)\prod_{j=q+1}^{2n}\widetilde{\alpha}_j(R)/\det(\frac{1}{2}C(R)),$$

where $R = \sum_{j=1}^{q} r_j X_j \in \mathfrak{A}$, $\theta_j = \tanh r_j$, $j = 1,\cdots,q$. If F_a is the normalization of ϕ_a, then

$$\left|\det J_{F_{a_1}}(z)\right|^2 = \prod_{j=1}^{q}\frac{(1-\eta_j^2)\widetilde{\alpha}_j(Y)}{(1-\lambda_j^2)\widetilde{\alpha}_j(X)(1-\theta_j^2)\widetilde{\alpha}_j(R)} \cdot \frac{\det(\frac{1}{2}B(X)\frac{1}{2}C(X))\frac{1}{2}B(R)\frac{1}{2}C(R))}{\det(\frac{1}{2}B(Y)\frac{1}{2}C(Y))}\det(A(\widetilde{x},R)).$$

By Theorem 3.1,

$$\left|\det J_{F_{a_1}}(z)\right|^2 = \frac{K_M(z,\bar{z})}{K_M(0,0)}\cdot\frac{K_M(\theta,\bar{\theta})}{K_M(\eta,\bar{\eta})}\cdot\frac{\det(\frac{1}{2}B(X)\frac{1}{2}B(R))}{\det(\frac{1}{2}B(Y))}\det(A(\widetilde{x},R)),$$

where

$$\theta = \xi(ka(R)\cdot O), \quad \eta = \xi(ka(Y)\cdot O).$$

Let $a_1 = a_1(R)$ approach to infinity along the geodesic $\exp tR_0$. $\theta = \xi(ka(R)\cdot O) = \xi(ka_1\cdot O)$ tends to a point b on the characteristic boundary of M if and only if $\prod_{j=q+1}^{n}|\alpha_j(R_0)| \neq 0$. In particular, we can take R_0 such that $\alpha_j(R_0) < 0$, and then letting $R = tR_0$, $t \to \infty$ we get a point b at the characteristic manifold of M. The mappings $F_a \to F_b$, and

$$|J_{F_b}(z)|^2 = \frac{K_M(z,\bar{z})}{K_M(0,0)}\lim_{\theta\to b}\frac{K_M(\theta,\bar{\theta})}{K_M(\eta,\bar{\eta})}\lim_{t\to\infty}\frac{\det(\frac{1}{2}B(x)\frac{1}{2}B(tR_0))}{\det(\frac{1}{2}B(Y))}\det(A(\widetilde{x},tR_0)).$$

In particular, we take $z = \xi(a(X)\cdot O)$. Then

$$A(\widetilde{x},R) = I + \begin{pmatrix}0 & 0\\ 0 & D(X)D(R)\end{pmatrix}.$$

Because $D(R) \to I$ when $R = tR_0$, $t \to +\infty$, we have

$$\lim_{t\to\infty}\det(A(\widetilde{x},tR_0)) = \det\left(I + \begin{pmatrix}0 & 0\\ 0 & D(X)\end{pmatrix}\right) = \det(I + D(X)) = \prod_{j=q+1}^{2n}\frac{2e^{-\alpha_j(X)}}{e^{\alpha_j(X)}+e^{-\alpha_j(X)}}.$$

Moreover,

$$\det(\frac{1}{2}B(X)\frac{1}{2}B(tR_0))\det{}^{-1}(\frac{1}{2}B(Y)) = \prod_{j=q+1}^{2n}\frac{(e^{-\alpha_j(X)}+e^{\alpha_j(X)})(e^{-\alpha_j(tR_0)}+e^{\alpha_j(tR_0)})}{2(e^{-\alpha_j(X+tR_0)}+e^{\alpha_j(X+tR_0)})}$$

since $Y = X + tR_0$. The right hand side of the previous equality is $\prod_{j=q+1}^{2n}\frac{e^{-\alpha_j(X)}+e^{\alpha_j(X)}}{2e^{-\alpha_j(X)}}$ when $t \to +\infty$. Finally, we have

$$|J_{F_b}(z)| = \sqrt{\frac{K_M(z,\bar{z})}{K_M(0,0)}}\lim_{\theta\to b}\sqrt{\frac{K_M(\theta,\bar{\theta})}{K_M(\eta,\bar{\eta})}} \tag{4.3}$$

when $z = \xi(a(X)\cdot O)$.

Theorem 4.2. *Assumptions as Theorem 3.1, then (4.3) holds where $\theta = \xi(ka(R)\cdot O)$.*

§5. Distortion Theorem for Biholomorphic Convex Mappings in Classical Domain of Type IV

We can use Theorem 4.2 to evaluate the lower bound of $C(S)$ in classical domain of type IV.

If $R_{\rm IV}$ is the classical domain of type IV which is defined as

$$1+|ww'|^2-2w\overline{w}'>0,\quad 1-|ww'|>0,$$

where $w=(w_1,\cdots,w_n)\in C^n$. The linear transformation $z=wP_0$ where $P_0=\begin{pmatrix}\begin{pmatrix}1&-i\\ i&1\end{pmatrix}&0\\ 0&\sqrt{2}A\end{pmatrix}$ transforms $R_{\rm IV}$ onto M, where

$$M=\{z=e^{i\theta}(\lambda_1,\lambda_2,0,\cdots,0)k\in C^n, k=P_0^{-1}\Gamma P_0,\quad \Gamma\in SO(n),|\lambda_1|<1,|\lambda_2|<1\},$$

and it is the canonical realization of $R_{\rm IV}$.

The Bergman kernel function $K_{\rm IV}(w,\overline{w})$ of $R_{\rm IV}$ is

$$\frac{1}{V(R_{\rm IV})}\cdot\frac{1}{(1+|ww'|^2-2\overline{w}w')^n}\quad (V(R_{\rm IV})\text{ is the volume of }R_{\rm IV}),$$

where $w=zP_0=e^{i\theta}(\lambda_1,\lambda_2,0,\cdots,0)P_0^{-1}\Gamma=e^{i\theta}\left(\frac{\lambda_1+i\lambda_2}{2},\frac{i}{2}(\lambda_1-i\lambda_2),0,\cdots,0\right)\Gamma$, i.e.,

$$K_{\rm IV}(w,\overline{w})=\frac{1}{V(R_{\rm IV})}\cdot\frac{1}{(1-\lambda_1^2)^n(1-\lambda_2^2)^n}.\tag{5.1}$$

Because the difference between $K_{\rm IV}(w,\overline{w})$ and $K_{\rm IV}(z,\overline{z})$ is only a constant, we have

$$\lim_{\theta\to b}\sqrt{\frac{K_M(\theta,\bar\theta)}{K_M(\eta,\bar\eta)}}=\lim_{\theta\to b}\sqrt{\frac{(1-\eta_1^2)^n(1-\eta_2^2)^n)}{(1-\theta_1^2)^n(1-\theta_2^2)^n}}.$$

Since $\eta_j=\frac{\lambda_j+\theta_j}{1+\lambda_j\theta_j}$, we get

$$\frac{1-\eta_j^2}{1-\theta_j^2}=\frac{(1-\eta_j)(1+\eta_j)}{1-\theta_j^2}=\frac{(1-\lambda_j)(1-\theta_j)(1+\lambda_j)(1+\theta_j)}{(1-\theta_j^2)(1+\lambda_j\theta_j)^2}=\frac{1-\lambda_j^2}{(1+\lambda_j\theta_j)^2}.$$

The right hand side approaches $(1+\lambda_j)(1-\lambda_j)^{-1}$ when $\theta_j\to 1$. We obtain

$$\lim_{\theta\to b}\sqrt{\frac{K_M(\theta,\bar\theta)}{K_M(\eta,\bar\eta)}}=\left(\frac{(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)}\right)^{\frac{n}{2}}.\tag{5.2}$$

By Theorem 1.1, (5.1), Theorem 4.2 and (5.2), we have

Theorem 5.1. *Let S be the family of normalized holomorphic convex mappings in $R_{\rm IV}$ which map $R_{\rm IV}$ into C^n. If $f\in S$, then*

$$\frac{\prod_{j=1}^{2}(1-\lambda_j)^{C_4(S)-\frac{n}{2}}}{\prod_{j=1}^{2}(1+\lambda_j)^{C_4(S)+\frac{n}{2}}}\le|\det J_f(z)|\le\frac{\prod_{j=1}^{2}(1+\lambda_j)^{C_4(S)-\frac{n}{2}}}{\prod_{j=1}^{2}(1-\lambda_j)^{C_4(S)+\frac{n}{2}}},\tag{5.3}$$

where $z=e^{i\theta}\left(\frac{\lambda_1+\lambda_2}{2},\frac{i}{2}(\lambda_1-\lambda_2),0,\cdots,0\right)\Gamma,\Gamma\in SO(n)$, $1\ge\lambda_1\ge\lambda_2\ge 0$, and $C_4(S)$ satisfies

$$\frac{1}{2}n\le C_4(S)\le n.\tag{5.4}$$

All the conclusions in the theorem are proved except the right hand side inequality of (5.4). We can prove it as follows.

Let $f \in S$, $f(z) = z + \sum_{i,j} d_{ij} z_i z_j + \cdots$. Then $F(z) = \frac{1}{2}(f(z) + f(-z))$ belongs to the image of R_{IV} under $f(z)$. Let $\phi(z) = f^{-1}(F(z))$. Then ϕ is a holomorphic mapping which maps M into M. Obviously,

$$\phi(z) = \sum_{i,j} d_{i,j} z_i z_j + \cdots$$

and $\phi(z_i e_i + z_j e_j) = 2d_{ij} z_i z_j + \cdots$, when $i \neq j$. Take $z = z_i e_i + z_j e_j$ such that $z = e^{it}(\lambda, 0, \cdots, 0)\Gamma$, $\Gamma \in SO(n)$. It is always possible because we just take $\lambda_1 = \lambda_2 = \lambda$ in the expression of z. There exists θ such that $z = \lambda e^{it} \cos\theta e_i + \lambda e^{it} \sin\theta e_j$. Then $\phi(z_i e_i + z_j e_j) = 2d_{ij}\lambda^2 \cos\theta \sin\theta e^{2it} + \cdots$. Multiplying it by e^{-2it}, and integrating from 0 to π, we get $\frac{1}{2\pi}\int_0^{2\pi} \phi(z_i e_i + z_j e_j) e^{-2it} dt = d_{ij}\lambda^2 \sin 2\theta \in R_{IV}$. Taking $\theta = \frac{\pi}{4}$, we have $|\lambda^2 d_{ij}^{(k)}| \leq 1$. Letting $\lambda \to 1$, we get $|d_{ij}^{(k)}| \leq 1$ when $i \neq j$. Similarly, we can prove $|d_{ii}^{(k)}| \leq 1$. We get the right hand side inequality of (5.4) by the definition of $C_4(S)$.

Conjecture 5.1. *$C_4(S) = \frac{n}{2}$, the convex mapping $F_b(z)$ which we construct at Section 4 is an extremal mapping.*

If the conjecture is true then (5.3) becomes

$$\prod_{j=1}^{2} (1 + \lambda_j)^{-n} \leq |\det J_f(z)| \leq \prod_{j=1}^{2} (1 - \lambda_j)^{-n}.$$

§6. Distortion Theorem for Holomorphic Convex Mappings in Exceptional Classical Domains

Now we consider two exceptional classical domains. Let $R_{\mathrm{V}} \subset C^{16}$ be the canonical realized exceptional classical domain in C^{16} and $R_{\mathrm{VI}} \subset C^{27}$ be the canonical realized exceptional classical domain in C^{27}; let $K_{\mathrm{V}}(z, \bar{z})$ and $K_{\mathrm{VI}}(z, \bar{z})$ be the Bergman kernel functions of R_{V} and R_{VI} respectively.

By Theorems 1.1, 2.1 and 4.2, we have

Theorem 6.1. *If S is the family of normalized biholomorphic convex mappings on R_{V} which map R_{V} into C^{16}, and $f \in S$, then*

$$\sqrt{\frac{K_V(z,\bar{z})}{K_V(0,0)}} \left(\prod_{j=1}^{2} \frac{1 - |\tanh x_j|}{1 + |\tanh x_j|} \right)^{C_5(S)}$$

$$\leq |\det J_f(z)| \leq \sqrt{\frac{K_V(z,\bar{z})}{K_V(0,0)}} \left(\prod_{j=1}^{2} \frac{1 + |\tanh x_j|}{1 - |\tanh x_j|} \right)^{C_5(S)} \tag{6.1}$$

and

1) $C_5(S) \leq 31$,

2) $\lim\limits_{\theta \to b} \sqrt{\frac{K_V(\theta,\bar{\theta})}{K_V(\eta,\bar{\eta})}} \leq \left(\prod_{j=1}^{2} \frac{1+|\tanh x_j|}{1-|\tanh x_j|} \right)^{C_5(S)}$,

3) $\lim\limits_{\theta \to b} \sqrt{\frac{K_V(\theta,\bar{\theta})}{K_V(\eta,\bar{\eta})}} \geq \left(\prod_{j=1}^{2} \frac{1-|\tanh x_j|}{1+|\tanh x_j|} \right)^{C_5(S)}$,

where

$$\theta = \xi(k_1 a(R)\cdot O) = (\theta_1, \theta_2, 0, \cdots, 0)\widetilde{k}_1 \in R_{\rm V}, R = \sum_{j=1}^{2} r_j X_j, \theta_j = \tanh r_j,$$

$$\eta = \xi(k_2 a(Y)\cdot O) = (\eta_1, \eta_2, 0, \cdots, 0)\widetilde{k}_2 \in R_{\rm V}, Y = \sum_{j=1}^{2} y_j X_j, \eta_j = \tanh y_j,$$

$$z = \xi(k_3 a(X)\cdot O) = (\lambda_1, \lambda_2, 0, \cdots, 0)\widetilde{k}_3 \in R_{\rm V}, X = \sum_{j=1}^{2} x_j X_j, \lambda_j = \tanh x_j,$$

and

$$\eta_j = \frac{\lambda_j + \theta_j}{1 + \lambda_j \theta_j}, \quad j = 1, 2;$$

b is a point in characteristic boundary of $R_{\rm V}$.

Conjecture 6.1. *The mapping F_b which we construct at Section 4 is an extremal mapping of* (6.1).

Similarly, we have

Theorem 6.2. *If S is the family of normalized biholomorphic convex mappings in $R_{\rm VI}$ which map $R_{\rm VI}$ into C^{27}, and $f \in S$, then*

$$\sqrt{\frac{K_{\rm VI}(z,\bar{z})}{K_{\rm VI}(0,0)}}\left(\prod_{j=1}^{3}\frac{1-|\tanh x_j|}{1+|\tanh x_j|}\right)^{C_6(S)} \le |\det J_f(z)| \le \sqrt{\frac{K_{\rm VI}(z,\bar{z})}{K_{\rm VI}(0,0)}}\left(\prod_{j=1}^{3}\frac{1+|\tanh x_j|}{1-|\tanh x_j|}\right)^{C_6(S)} \tag{6.2}$$

and

1) $C_6(S) \le 53$,

2) $\lim\limits_{\theta\to b}\sqrt{\frac{K_{\rm VI}(\theta,\bar{\theta})}{K_{\rm VI}(\eta,\bar{\eta})}} \le \left(\prod_{j=1}^{3}\frac{1+|\tanh x_j|}{1-|\tanh x_j|}\right)^{C_6(S)}$,

3) $\lim\limits_{\theta\to b}\sqrt{\frac{K_{\rm VI}(\theta,\bar{\theta})}{K_{\rm VI}(\eta,\bar{\eta})}} \ge \left(\prod_{j=1}^{3}\frac{1-|\tanh x_j|}{1+|\tanh x_j|}\right)^{C_6(S)}$,

where

$$\theta = \xi(k_1 a(R)\cdot O) = (\theta_1, \theta_2, \theta_3, 0, \cdots, 0)\widetilde{k}_1 \in R_{\rm VI}, R = \sum_{j=1}^{3} r_j X_j, \theta_j = \tanh r_j,$$

$$\eta = \xi(k_2 a(Y)\cdot O) = (\eta_1, \eta_2, \eta_3, 0, \cdots, 0)\widetilde{k}_2 \in R_{\rm VI}, Y = \sum_{j=1}^{3} y_j X_j, \eta_j = \tanh y_j,$$

$$z = \xi(k_3 a(X)\cdot O) = (\lambda_1, \lambda_2, \lambda_3, 0, \cdots, 0)\widetilde{k}_3 \in R_{\rm VI}, X = \sum_{j=1}^{3} x_j X_j, \lambda_j = \tanh x_j,$$

and

$$\eta_j = \frac{\lambda_j + \theta_j}{1 + \lambda_j \theta_j}, \quad j = 1, 2, 3;$$

b is a point in characteristic boundary of R_{VI}.

Conjecture 6.2. *The mapping* F_b *which we construct at Section* 4 *is an extremal mapping of* (6.2).

From the process of the proof of Theorem 1.1, we can get a more precise form of the theorem.

Theorem 6.3. *Assumptions as Theorem* 1.1, *then*

$$\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}\prod_{j=1}^{q}\left(\frac{1-|\tanh x_p|}{1+|\tanh x_p|}\right)^{C_p(S)}$$
$$\le|\det J_f(z)|\le\sqrt{\frac{K_M(z,\bar z)}{K_M(0,0)}}\prod_{j=1}^{q}\left(\frac{1+|\tanh x_p|}{1-|\tanh x_p|}\right)^{C_p(S)} \tag{6.3}$$

holds where $C_p(S)=\sup\{|\sum\limits_j d_{pj}^{(j)}|, f\in S\}$.

Let $\lambda_j=\tanh x_j$, $j=1,\cdots,q$. For $\lambda_je_j+\lambda_le_l\in M, 1\le j,l\le q$, if there exists $k\in K$, such that $(\lambda_je_j+\lambda_le_l)k=\lambda_le_j+\lambda_je_l$, then $C_j(S)=C_l(S)$, and the estimation formula (1.4) cannot improve.

If S is the family of normalized biholomorphic mappings, then

$$\prod_{j=1}^{q}\left(\frac{1-|\tanh x_p|}{1+|\tanh x_p|}\right)^{C_p(S)}\le\lim_{\theta\to b}\sqrt{\frac{K_M(\theta,\bar\theta)}{K_M(\eta,\bar\eta)}}\le\prod_{j=1}^{q}\left(\frac{1+|\tanh x_j|}{1-|\tanh x_j|}\right)^{C_p(S)},$$

where

$$\theta=\xi(ka(R)\cdot O)=(\theta_1,\cdots,\theta_q,0,\cdots,0)\widetilde{k}, R=\sum_{j=1}^{q}r_jX_j,\theta_j=\tanh r_j,$$

$$\eta=\xi(ka(Y)\cdot O)=(\eta_1,\cdots,\eta_q,0,\cdots,0)\widetilde{k}, Y=\sum_{j=1}^{q}y_jX_j,\eta_j=\tanh y_j,$$

and

$$\eta_j=\frac{\lambda_j+\theta_j}{1+\lambda_j\theta_j},\quad j=1,\cdots,q;$$

b is a point in the characteristic boundary of M.

Finally, we make a conjecture for the distortion theorem in bounded symmetric domains as follows.

Conjecture. *Let* $M\subset C^n$ *be a bounded symmetric domain which contains the origin. It is the canonical realization of Hermite symmetric space* G/K. S *is the family of normalized biholomorphic convex mappings which map* M *into* C^n. $K_M(z,\bar z)$ *is the Bergman kernel function of* M. *For any* $z\in M$, z *can be expressed as*

$$z=\xi(ka(X)\cdot O)=\xi(\exp Ad(k)X\cdot O)=(\lambda_1,\cdots,\lambda_q,0,\cdots,0)\widetilde{k},$$

where $X = \sum_{j=1}^{q} x_j X_j$, $\lambda_j = \tanh x_j$. *If* $f \in S$, *then*

$$\sqrt{\frac{K_M(z,\bar{z})}{K_M(0,0)}} \lim_{\theta_1,\cdots,\theta_q \to -1} \sqrt{\frac{K_M(\theta,\bar{\theta})}{K_M(\eta^{(1)},\bar{\eta}^{(1)})}}$$

$$\leq |\det J_f(z)| \leq \sqrt{\frac{K_M(z,\bar{z})}{K_M(0,0)}} \lim_{\theta_1,\cdots,\theta_q \to -1} \sqrt{\frac{K_M(\theta,\bar{\theta})}{K_M(\eta^{(2)},\bar{\eta}^{(2)})}}, \tag{6.4}$$

where

$$\theta = \xi(ka(R)\cdot O) = (\theta_1,\cdots,\theta_q,0,\cdots,0)\widetilde{k} \in M, R = \sum_{j=1}^{q} r_j X_j, \theta_j = \tanh r_j,$$

$$\eta^{(l)} = \xi(k_l a(Y_l)\cdot O) = (\eta_1^{(l)},\cdots,\eta_q^{(l)},0,\cdots,0)\widetilde{k_l} \in M,$$

$$Y_l = \sum_{j=1}^{q} y_j^{(l)} X_j, \eta_j^{(l)} = \tanh y_j^{(l)}, \quad l = 1,2,$$

$$\eta_j^{(1)} = \frac{-|\lambda_j| + \theta_j}{1 - |\lambda_j|\theta_j}, \quad \eta_j^{(2)} = \frac{|\lambda_j| + \theta_j}{1 + |\lambda_j|\theta_j}, \quad j = 1,\cdots,q.$$

The estimation (6.4) is precise, the mapping $F_b(z)$ which we construct at section 4 is an extremal mapping. $F_b(z)$ makes the equality of (6.4) hold.

If M is the unit disc, the conjecture is true.

If M is the ball, the conjecture coincides with the conjecture at [3].

If M is a classical domain, the conjecture coincides with the conjectures at [2] and at section 5 of this paper.

If the conjecture is true, then the distortion of normalized biholomorphic convex mappings can be expressed by Bergman kernel function only.

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