Chin. Ann. of Math.
14B: 3(1993),387-396.

EXTENSION OF MULTIPLIERS AND INJECTIVE HILBERT MODULES

LIN HUAXIN*

Abstract

The author studies the problem whether a multiplier of a hereditary C^*-subalgebra B of a C^*-algebra A can be extended to a multiplier of A. One related problem is the Hahn-Banach extension theorem for Hilbert modules over C^*-algebras. It is shown that every self-dual Hilbert module over W^*-algebra or an injective C^*-algebra is injective.

Keywords Multiplier, Injective Hilbert modules, C^*-algebra
1991 MR Subject Classification 46L05

§0. Introduction

Let A be a C^*-algebra, B its hereditary C^*-subalgebra and x a multiplier of B. A natural question is whether x has a norm preserving extension to a multiplier of A. It is a non-commutative analogue of the following: Let X be a locally compact Hausdorff space, 0 an open subset of x and f a bounded continuous function on 0, can f be extended (norm-preservingly) to a bounded continuous function on x ? There is another non-commutative version of this problem: If B is a hereditary C^*-subalgebra of A and x is a left multiplier of B, can x be extended (norm-preservingly) to a left multiplier of A?

In the first section we study these problems. It turns out that the first problem is related to the problem whether a bounded A-module map from a right ideal R to A can be extended (norm-preservingly) to a bounded A-module map from A to A? A further problem is whether the Hahn-Banach extension theorem holds for Hilbert A-module: if Y is a Hilbert A-module, X a Hilbert A-submodule of Y and φ a bounded A-module map from X to A, is there an A-module map $\tilde{\varphi}$ from Y to A such that

$$\|\tilde{\varphi}\| = \|\varphi\| \text{ and } \tilde{\varphi}|_x = \varphi?$$

As in homology theory, the notion of injectivity was introduced in category whose objects are Banach modules over a (unital) Banach algebra and whose morphisms are contractive module maps, and the existence and uniqueness of injective envelope of a Banach module was proved by M. Hamana[7]. M. Hamana[7, Proposition 2] and M. Takesaki[17] also showed that the only unital C^*-algebras that are injective as injective Banach modules over itself are commutative AW^*-algebras. In section 2, we shall study the corresponding problems for Hilbert modules over C^*-algebras. We show that self-dual Hilbert modules over W^*-algebras and injective C^*-algebras are injective. Therefore the Hahn-Banach extension theorem holds

Manuscript received September 3, 1989.
*Department of Mathematics, East China Normal University, Shanghai 200062, China .

for those Hilbert modules. We also show that the only injective Hilbert modules over a W^*-algebra are self-dual Hilbert modules. Existence and uniqueness of injective envelopes of Hilbert modules over a W^*-algebra is established at the end of the paper. It is interesting to know that Hilbert modules over non-AW^*-algebras may not have injective envelopes. Applications of these extension theorems can be found in [11].

We shall denote by A^{**} the enveloping W^*-algebra of C^*-algebra A, $M(A)$ the multiplier algebra, the set

$$\{x \text{ in } A^{**} : xa,\ ax \text{ in } A \text{ for } a \text{ in } A\},$$

$LM(A)$ the left multipliers, the set

$$\{x \text{ in } A^{**} : xa \text{ in } A \text{ for } a\text{in } A\}$$

and $QM(A)$ the set

$$\{x \text{ in } A^{**} : axb \text{ in } A \text{ for } a,\ b \text{ in } A\}.$$

§1. MAW^*-Algebras and Self-Injective C^*- Algebras

Let A be a C^*-algebra. We consider the following extension properties:

(1) For every hereditary C^*-subalgebra B of A and x in $M(B)$ there is a y in $M(A)$ such that $yb = xb, by = bx$ for all b in B and $\|x\| = \|y\|$.

(2) For every hereditary C^*-subalgebra B of A and x in $LM(B)$ there is a y in $LM(A)$ such that $yb = xb$ for all b in B and $\|x\| = \|y\|$.

Definition 1.1. *Let R be a closed right ideal of A. A linear map if $\varphi : R \to A$ is called a module map $\varphi(ra) = \varphi(r)a$ for all a in A.*

We have the third extesion property:

(3) For every closed right ideal R of A and a bounded module map $\varphi : R \to A$ there is a module map $\tilde{\varphi} : A \to A$ such that $\tilde{\varphi}(r) = \varphi(r)$ for all r in R and $\|\tilde{\varphi}\| = \|\varphi\|$.

Definition 1.2. *A C^*-algebra A is called an MAW^*-algebra if $M(A)$ is an AW^*-algebra.*

Example 1.1. An abelian C^*-algebra $A = C_0(X)$ is an MAW^*-algebra if and only if X is an extremally disconnected space.

Theorem 1.1 *A C^*-algebra is MAW^*-algebra A if and only if it has the extension property* (1).

Proof. We first assume that A is an AW^*-algebra. Suppose that $x \in M(B)_{s.a}$ and $\|x\| \leq 1$. Define $r_n = (1/2)(2/3)^n$. Then $\|x\| \leq 3r_1$. Inductively, given x_n in $M(B)_{s.a.}$ with $\|x_n\| \leq 3r_n$, define

$$p_n = E_{(-\infty,-r_n)}(x_n),\quad p'_n p' = E_{(-\infty,-r_n]}(x_n),$$
$$q'_n = E_{[r_n,\infty)}(x_n),\quad q_n = E_{(r_n,\infty)}(x_n)$$

(the spectral projections corresponding to the sets $(-\infty,-r_n)$, $(-\infty,-r_n]$, $[r_n,\infty)$ and (r_n,∞) respectively). Then p_n and q_n are open projections of A; p'_n and q'_n are closed projections of A. Let B_n, C_n, B_n and C_n be the hereditary C^*-subalgebras of A corresponding to $p_n, q_n, 1-q'_n$ and $1-p'_n$ respectively. Suppose that R_n is the right ideal of right annihilators of C_n. Then $R_n = e_nA$, where e_n is a projection in A (see [9, Theorem 2.3]). Clearly,

$$p_{n+1} \geq p'_n \geq e_n \geq p_n.$$

Similarly there is a projection f_n in A such that $q_{n+1} \geq q'_n \geq f_n \geq q_n$. Accordingly, define

$$g_n = r_n e_n - r_n f_n, \quad \text{then} \quad \|g_n\| \leq r_n.$$

We now define $x_{n+1} = x_n - g_n$. Suppose that $B = pA^{**}p \cap A$, where p is an open projection; then

$$g_n \leq e_n + f_n \leq p_{n+1} + q_{n+1} \leq p.$$

Hence x_{n+1} is in $M(B)_{s.a.}$ and $\|x_{n+1}\| \leq 2r_n = 3r_{n+1}$. This completes the induction step.

Now put

$$y = \sum_{1 \leq n < \infty} g_n.$$

Then $\|y\| \leq 1$ and $y \in A$. For every b in B.

$$(\sum_{1 \leq i \leq n} g_i)b = \sum_{1 \leq i \leq n} (x_i - x_{i+1})b = (x_1 - x_{n+1})b.$$

Since $\|x_{n+1}\| \to 0$, $yb = xb$ for every b in B.

If A is an MAW^*-algebra, then $M(A)$ is an AW^*-algebra. If B is a hereditary C^*-subalgebra of A then B is a hereditary C^*-subalgebra of $M(A)$. So the conclusion follows from what we have just proved.

Now we suppose that A has the extension property (1). Assume that B is a hereditary C^*-subalgebras of $M(A)$, $B = pM(A)^{**}p \cap M(A)$. Set

$$B^{\perp} = \{x \in M(A) : xB = Bx = 0\}.$$

Then

$$B^{\perp} = qM(A)^{**}q \cap M(A)$$

for some open projection q of $M(A)$.

Clearly $p \perp q$. Let π be the universal representation of A. Then π acting on $M(A)$ is faithful. Thus there are open projections (of A) p_1 and q_1 such that $\pi(p_1) = \pi(p)$ and $\pi(q_1) = \pi(q)$. Moreover $p_1 \perp q_1$. Set $B_i = p_1 A^{**} p_1 \cap A$ and $C_1 = q_1 A^{**} q_1 \cap A$. Then $B_1 + C_1$ is a hereditary C^*-subalgebra of A and $p_1, q_1 \in M(B_1 + C_1)$. Therefore there is a u in $M(A)$ such that $ud = p_1 d$ and $du = dp_1$ for all d in $B_1 + C_1$, and $u = 1$. It follows that

$$up_1 = p_1 u = p_1, \quad uq_1 = q_1 u = 0.$$

Thus $u \in (B^{\perp})^{\perp}$ and u is a unit for B. It follows from the proof of [15, Proposition 1] that $M(A)$ is an AW^*-algebra.

Definition 1.3. *A C^*-algebra A with the extension property* (3) *is called a self-injective C^*-algebra.*

Proposition 1.1. *An abelian C^*-algebra A is self-injective if and only if $A = C(X)$ for some extremally disconnected space.*

Proof. (See [5,16.6])

Proposition 1.2. *Every W^*-algebra is self-injective.*

Proof. See Theorem 2.2.

Definition 1.4. *Let R be a closed ideal of A and p be the corresponding open projection of A in A^{**}. Set*

$$LM(R, A) = \{x \ \text{in} \ A^{**} : xr \in A \ \text{for all} \ r \in R \ \text{and} \ xp = x\}.$$

For $x \in LM(R, A)$, define $\varphi(r) = xr$ for r in R. Then φ is a bounded module map from R into A and $\|\varphi\| = \|x\|$. As [14, 3.12.3], we can show that for every bounded module map φ from R to A there is a unique x in $LM(R, A)$ such that $\varphi(r) = xr$ for all r in R and $\|\varphi\| = \|x\|$.

Proposition 1.3. *Every closed ideal of a self-injective C^*-algebra is self-injective and every closed ideal of an MAW^*-algebra is an MAW^*-algebra.*

Proof. Let I be a closed ideal of a self-injective C^*-algebra A and R is a closed right ideal of I. Suppose that x is in $LM(R, I) \subset LM(R, A)$. Then there is y in $LM(A, A) = LM(A)$ such that $\|y\| = \|x\|$ and $yr = xr$ for all r in R. Let z be the central open projection corresponding to I. Then , clearly, $LM(A)z \subset LM(I) = LM(I, I)$. So $yz \in LM(I)$ and $yzb = xb$ for all b in B.

The second statement follows from Theorem 1.1 and the above argument.

Theorem 1.2. *Every self-injective C^*-algebra has the extension property* (2). *Every unital C^*-algebra with extension property* (2) *has extension property* (1) *and consequentely is an AW^*-algebra.*

Proof. Suppose that A is a self-injective C^*-algebra and $B = pA^{**}p \cap A$, where p is an open projection. Set $R = pA^{**} \cap A$. Let x be in $LM(B) \subset LM(R, A)$. Then there is a y in $LM(A, A) = LM(A)$ such that $yr = xr$ for all r in R and $\|x\| = \|y\|$.

Therefore $yb = xb$ for all b in B. So A has the extension property (2).

Now suppose that A is a unital C^*-algebra with the extension property (2). Let B and C be two orthogonal hereditary C^*-subalgebras of A. Suppose that p is the open projection of A corresponding to B. Then p is in $LM(B + C)$. There is an e in $M(A) + A$ such that $ed = pd$ for all d in $B + C$. It follows from [15, Proposition 1.] that A is an AM^*-algebra.

Remark 1.1. Recall that a C^*-algebra A is called an injective C^*-algebra if given any self-adjoint linear subspace S, containing the unit, of a C^*-algebra B, any completely positive linear map of S into A extends to a completely positive linear map of B into A.

Theorem 1.3. *Every injective C^*-algebra is self-injective.*

Proof. Suppose that A is an injective C^*-algebra, R is a closed right ideal of A and φ is a bounded A-module map from R to A. It follows from [17, Proposition 2.8] that φ is completely bounded. By [17, 2.5.], both A and R are matricial normed right A-modules. Then from [17, Theorem 4.1] (It works for right A-modules.), we conclude that there is a bounded A-module map $\tilde{\varphi} : A \to A$ which extends φ with the same norm.

Remark 1.2. It follows from [8, Theorem 4.1] that every C^*-algebra has a unique injective envelope. Therefore every C^*-algebra can be essentially embedded into a self-injective C^*-algebra. However, since there are W^*-algebras that are not injective, a C^*-algebra may be essentially embedded into non-isomorphic self-injective C^*-algebras.

§2. Injective Hilbert Modules

It is known (see [7] and [17]) that the only unital C^*-algebras that are injective as Banach A-modules over themselves are commutative AW^*-algebras. As M. Hamana pointed out[7] that for a C^*-algebra, the category of Banach A-modules and contractive module maps is too large. It is then natural to consider Hilbert A-modules.

Definition 2.1[13]. *Let A be a C^*-algebra. A pre-Hilbert module over A is a right A-module H equipped with an A-valued "inner product", a function $\langle\cdot,\cdot\rangle : H\times H\to A$, with the following properties:*

(1) $\langle\cdot,\cdot\rangle$ is sesquilinear (we make the convention that the inner products are conjugate linear in the first variable).

(2) $\langle x,x\rangle\geq 0$; if $\langle x,x\rangle = 0$, then $x=0$.

(3) $\langle x,ya\rangle = \langle x,y\rangle a$ for all a in A.

(4) $\langle y,x\rangle = \langle x,y\rangle^*$ for all x and y in H.

For x in H, put $\|x\| = \|\langle x,x\rangle\|^{1/2}$. This is a norm on H. If H is complete, H is called a Hilbert module over A.

Remark 2.1. A C^*-algebra A is itself a Hilbert A-module with $\langle a,b\rangle = a^*b$. More generally, any (closed) right ideal of A is a Hilbert A-module.

Remark 2.2. If H is a Hilbert module over A, its A-dual, the bounded A-module maps from H to A, is denoted by $H^{\#}$. Each h in H gives rise to a map h in $H^{\#}$ defined by $h(y) = \langle h,y\rangle$ for y in H. We call H self-dual if $H = H^{\#}$, i.e., if every map in $H^{\#}$ arises by taking A-valued inner product with some fixed h in H. If we define scalar multiplication on $H^{\#}$ by

$$(\lambda\tau)(h) = \bar{\lambda}\tau(h)$$

for complex number λ,τ in $H^{\#}$ and h in H and add maps in $H^{\#}$ pointwise, then $H^{\#}$ becomes a linear space. $H^{\#}$ becomes a right A-module if we set

$$(\tau\cdot a)(h) = a^*\cdot\tau(h)$$

for τ in $H^{\#}$, a in A and h in H. When A is a W^*-algebra, $H^{\#}$ is a Hilbert A-module containing H as a closed (Hilbert) A-submodule (see [13] and [9]).

Remark 2.3. Let H be a Hilbert A-module and H_0 be a Hilbert A-submodule of H. We shall denote by $H_0^\perp$ the Hilbert A-module $\{x\in H : \langle x,h\rangle = 0 \text{ for all } h \text{ in } H_0\}$.

Definition 2.2. *Let A be a C^*-algebra. An A-module map i from Hilbert A-module H_1 to Hilbert A-module H_2 is called an embedding if $\langle i(x),i(y)\rangle = \langle x,y\rangle$ for all x and y in H_1. An embedding is called an H-isometry if it is a bijection.*

Lemma 2.1. *Suppose that H is a Hilbert module over a C^*-algebra A and H_0 is a closed submodule of H. If H_0 is self-dual, then $H = H_0 + H_0^\perp$.*

Proof. Define a module map $P: H\to H_0^{\#}(=H_0)$ by the following:

$$Ph(x) = \langle h,x\rangle \text{ forall } x \text{ in } H_0.$$

The $\|P\| = 1$ and $Ph = h$ for each h in H_0.

For every x in H_0 and h in H, $\langle (1-P)h,x\rangle = 0$; in particular $\langle (1-P)h,Ph\rangle = 0$. Therefore for every h in H,

$$h = (1-P)h + Ph, \text{and } \langle h,h\rangle = \langle Ph,Ph\rangle + \langle (1-p)h,(1-p)h\rangle.$$

Thus $H = H_0 + H_0^\perp$.

Corollary 2.1(see [6, Proposition 1]). *Suppose that H is any Hilbert module over a unital C^*-algebra and H_0 is a submodule of H. If H_0 is a direct summand of A^n for some integer n, then $H = H_0 + H_0^\perp$.*

Proof. It is clear that A^n is self-dual. Since H_0 is a direct summand of a self-dual Hilbert module, H_0 itself is self-dual.

Proposition 2.1. *Let H be a Hilbert module over a W^*-algebra A and H_0 a direct summand of H. Then there is an embedding i from $H_0^\#$ into $H^\#$ such that $i|_{H_0} = i_{H_0}$ and*

$$H^\# = i(H_0^\#) + i(H_0^\#)^\perp.$$

Proof. Let P be the A-module projection from H onto H_0. We can embed H_0 into $H^\#$ by defining $i(f)(x) = f(Px)$ for all f in H_0 and x in H. Define Q on $H^\#$ by $Qf(x) = f(Px)$ for all f in $H^\#$ and x in H. Thus $\langle Qf, x\rangle = \langle f, Px\rangle$ for every f in $H^\#$ and x in H. We have, for every x in H_0,

$$\langle (1-Q)f, x\rangle = \langle f, x\rangle - \langle Qf, x\rangle = 0.$$

For each $(1-Q)f$, define an element h in $(H_0^\#)^\# = H_0^\#$ by $h(x) = \langle (1-Q)f, x\rangle$ for every x in $i(H_0^\#)$. Since $\langle (1-Q)f, x\rangle = 0$ for all x in $H_0, h = 0$. Hence $\langle (1-Q)f, x\rangle = 0$ for all x in $i(H_0^\#)$.

Therefore $(1-Q)(H^\#)^\perp \cdot i(H_0^\#)$. Since $Q(H) = i(H_0^\#)$, we conclude that

$$H^\# = i(H_0^\#) + i(H_0^\#)^\perp.$$

Remark 2.4. Let H be a Hilbert module over a C^*-algebra $A, M_n(H)$ the space of $n \times n$ matrices over H and $M_n(A)$ the algebra of $n \times n$ matrices over A. Then $M_n(H)$ is an inner product module over $M_n(A)$. The inner product is defined by a formal matrix product:

$$\langle x, y\rangle_n = \left[\sum_k \langle x_{ku}, y_{kv}\rangle\right]_{u,v}, \quad for\ x, y,\ in\ M_n(H).$$

H is an L^∞-matricially normed A-module with respect to the family of norms $\|x\|_n = \|\langle x, x\rangle\|^{1/2}$, for x in $M_n(H)$ (see [17, 2.5.] and [12, 1.14]).

Let $M(H, A^{**})$ denote the set of bounded A-module maps of H into A^{**}. Then the A-valued inner product on H can be extended to an A^{**}-valued inner product on $M(H, A^{**})$ (see [13, 3.4]). By the above definition, $(M(H, A^{**}),\ \{\|\cdot\|_n\})$ is an L^∞-matricially normed A^{**}-module. Notice that $H^\#$ is a subspace of $M(H, A^{**})$.

Lemma 2.2. *Let H be a Hilbert module over a unital C^*-algebra A. Then $H^\#$ together with the family norms $\|\cdot\|_n$ is an L^∞-matricially normed (right) A-module*(see [12, Definition 3.11]).

Proof. As a subspace of $(M(H, A^{**}),\ \{\|\cdot\|_n\})$, $(H^\#, \{\|\cdot\|_n\})$ is an L^∞-matricially normed space. For each n, if $x \in M_n(H^\#)$ and $\alpha \in M_n(A)$, then

$$\langle x\alpha, x\alpha\rangle_n = \alpha^* \langle x, x\rangle_n \alpha$$

and

$$\begin{aligned} \|x\|_n =& \|\langle x\alpha, x\alpha\rangle_n\|^{1/2} = \|\alpha^*\langle x, x\rangle_n \alpha\|^{1/2} \\ \le& (\|\alpha^*\|\|\langle x, x\rangle_n\|\|\alpha\|)^{1/2} \\ \le& \|x\|_n\|\alpha\|. \end{aligned}$$

In particular, $\|x\alpha\|_n \le \|x\|_n\|\alpha\|$ for all $\alpha \in M_n(\mathbf{C})$. Therefore $(H, \{\|\cdot\|_n\})$ is an L^∞-matricially normed A-module.

Lemma 2.3. *Let H be a Hilbert module over an injective C^*-algebra A and $\tau \in H^\#$. Then there is a bounded A-module map $\tilde{\tau} : H^\# \to A$ which extends τ with the same norm.*

Proof. Let τ be in H^3. As in [17, Proposition 2.8], τ is completely bounded and $\|\tau\|_{cb} = \|\tau\|$. It follows from Lemma 2.2 and [12, Theorem 1.14] that $(H^\#, \{\|\cdot\|_n\})$ is a matricial normed A-module in the sence of [17, Definition 2.2]. By [17, Theorem 4.1] (The theorem works for right A-modules), there is a bounded A-module map $\tilde{\tau} : H^\# \to A$ which extends τ with the same norm.

Definition 2.3. *A Hilbert A-module H is said to be injective if it has the following property: for every Hilbert A-module Y, a closed (Hilbert) submodule X and Y and a bounded A-module map T from X to H, there is an A-module map $\tilde{T}$ from Y to H such that $\tilde{T}|_x = T$ and $\|\tilde{T}\| = \|T\|$.*

Theorem 2.1. *Every self-dual Hilbert module over a W^*-algebra or over an injective C^*-algebra A is injective.*

Proof. Let H be a self-dual Hilbert A-module. Suppose that Y is a Hilbert A-module, X is a Hilbert A-submodule of Y and T is a bounded A-module map from X to H. For any h in H define $T^*(h)$ in $X^\#$ by $T^\#(h)(x) = \langle h, T(x)\rangle$ for all x in X. So $\|T^*(h)(x)\| \le \|T(h)\|\|x\|$ for all x in X and h in H. Hence $\|T^*(h)\| \le \|T\|\|h\|$. If A is a W^*-algebra then, by [13, 3.2], the A-valued inner product $\langle\cdot,\cdot\rangle$ extends to $X^\# x X^\#$ in such a way as to make $X^\#$ into a self-dual Hilbert A-module. Thus $T^*(h)$ can be extended to an element in $(X^\#)^\# (= X^\#)$ with the same norm. If A is an injective C^*-algebra, it follows from Lemma 2.3 that $T^*(h)$ can be aslo extended to an element of $(X^\#)^\#$ with the same norm. We use the same notation $T^*(x)$ for the extension. For every x_0 in $X^\#$, define an element $\tilde{T}(x_0)$ in $H^\# = H$ by

$$\tilde{T}(x_0)(h) = T^*(h)(x_0) \quad \text{for all } h \text{ in } H.$$

Then

$$\|\tilde{T}(x_0)(h)\| = \|T^*(h)(x_0)\| \le \|T^*(h)\|\|x_0\| \le \|T\|\|h\|\|x_0\|.$$

It is then easy to varify that $\tilde{T}$ is an A-module map from $X^\#$ to H such that $\tilde{T}|_H = T$ and $\|\tilde{T}\| = \|T\|$.

Define $P : y \in X^\#$ by $(Py)(x) = \langle y, x\rangle$ for all x in X. Clearly P is an A-module map, $\|P\| = 1$ and $Px = x$ for all x in X. Then $\tilde{T}(P)$ is an A-module map from Y to A such that

$$\tilde{T}(P)|_x = T \quad \text{and} \quad \|\tilde{T}(P)\| = \|T\|.$$

The following corollary is the Hahn-Banach extension theorem for Hilbert modules over W^*-algebras and injective C^*-algebras.

Corollary 2.2. *Let H be a Hilbert module over a W^*-algebra A or an injective C^*-algebra. Suppose that H_0 is a closed submodule of H and φ is a bounded A-module map from H_0 to A. Then there is an A-module map $\tilde{\varphi}$ from H to A such that*

$$\|\tilde{\varphi}\| = \|\varphi\| \quad \text{and} \quad \tilde{\varphi}|_{h_0} = \varphi.$$

Proof. Since A is unital, A itself is a self-dual Hilbert A-module.

Definition 2.4. *A C^*-algebra A is called H-self-injective if A is an injective Hilbert module over itself.*

Remark 2.5. From [15] commutative AW^*-algebras are H-self-injective. By Corollary

2.2 W^*-algebras and injective C^*-algebras are H-self-injective. An H-self-injective C^*-algebra is a self-injective C^*-algebra, and hence, by Remark 2.1 an MAW^*-algebra.

Theorem 2.2. *Suppose that A is a W^*-algebra. A Hilbert A-module H is injective if and only if H is self-dual.*

Proof. Suppose that H is injective. Since H is a closed submodule of $H^\#$, there is an A-module map P from $H^\#$ into H such that $P_H = i_H$ and $\|P\| = 1$. Clearly, $P^2 = P$ and $(1-P)^2 = 1-P$. Fix an element h in $(1-P)(H^\#)$, define an element $h^\wedge$ in $(H^\#)^\# = H^\#$ by $h^\wedge(x) = \langle h, (1-P)x\rangle$ for every x in $H^\#$. Then $h^\wedge(x) = 0$ for all x in H, i.e., $h^\wedge = 0$. Since $(1-P)h = h$, $\langle h, h\rangle = \langle h, (1-P)h\rangle = h^\wedge(h) = 0$. Therefore $(1-P)(H^\#) = \{0\}$. So $H = H^\#$ and H is self-dule (see [13, 3.8]).

Definition 2.5. *Suppose that H is a Hilbert A-module. A Hilbert A-module H_1 is called an injective envelope of H if H_1 is an injective Hilbert A-module, H is a (Hilbert) A-submodule of H_1 and there is no proper injective (Hilbert) A-submodule of H_1 containing H.*

At this point one may expect that every Hilbert module over a unital C^*-algebra has an injective envelope. Unfortunately, this is not true. Since unital H-self-injective C^*-algebras are at least AW^*-algebras (Remark 2.5), we see from Theorem 2.3 that Hilbert modules over non-AW^*-algebras may not have injective envelopes.

Theorem 2.3. *Suppose that A is a unital C^*-algebra but not H-self-injective. Then any Hilbert A-module containing A has no injective envelope. In fact such A-modules cannot be embedded into injective A-modules.*

Proof. Suppose that H is a Hilbert A-module containing A and a closed A-submodule of an injective Hilbert A-module H_1. Then by Lemma 2.1, $H_1 = A + A^\perp$. Let X be a closed (Hilbert) A-submodule of a Hilbert A-module Y and φ be a bounded A-module map from X to A. So φ is a bounded A-module map from Hilbert A-module X to H_1. Let $\tilde{\varphi}$ be a norm preserving extension of φ to y and P be the A-module projection from H_1 to A. Then $P(\tilde{\varphi})$ is a norm preserving extension of φ from Y to A such that

$$P(\tilde{\varphi})|_x = \varphi \text{ and } \|P(\tilde{\varphi})\| = \|\varphi\|.$$

So A is H-self-injective, a contradiction.

Theorem 2.4. *Every Hilbert module over a W^*-algebra A has a unique (up to H-isometrics) injective envelope.*

Proof. We shall show that $H^\#$ is the injective envelope. If $H \subset Y \subset H^\#$ and Y is injective, then, by 3.17, Y is self-dual. Hence, by Lemma 2.1, Y is a direct summand of $H^\#$, i.e., $H^\# = Y + Y^\perp$. Since for every y in Y, $\langle y, x\rangle = 0$ for x in $H \subset Y$, $Y^\perp = \{0\}$. So $Y + H^\#$.

If x is another injective envelope of H, then by Theorem 2.2 X must be self-dual. Define an A-module map r from x to $H^\#$ by $r(x)(h) = \langle x, h\rangle$. Then $\|r\| = 1$. Since X is injective, there is a norm preserving A-module map i from $H^\#$ to X such that $i|_H = i_H$. We denote $i(r)$ by Φ. Then $\|\Phi\| = 1$. Let

$$X_0 = \{x \in X : \Phi(x) = x\}.$$

Then $X_0 \supset H$. Moreover X_0 is a closed A-module.

We claim that X_0 is injective. Suppose that $N \subset M$ are two Hilbert A-modules and φ is a bounded A-module map from N to X_0. Since X is injective, there is an A-module map $\tilde{\varphi}$ from M to X such that

$$\tilde{\varphi}|_N = \varphi \text{ and } \|\tilde{\varphi}\| = \|\varphi\|.$$

Then

$$\Phi(\tilde{\varphi})|_x = \varphi \text{ and } \|\Phi(\tilde{\varphi})\| = \|\varphi\|.$$

Therefore $X_0 = X$, i.e., Φ is the identity map and hence r is an isometry from X into $H^\#$. If h is in $r(X)^\#$, since X is self-dual, $\langle r^*(h), x\rangle = h(r(x))$ for all x in $r(X)$. On the other hand, by Corollary 2.2 there is an $h^\wedge$ in $(H^\#)^\# = H^\#$ such that $h(r(x)) = \langle h^\wedge, r(x)\rangle$ for all x in X. Now

$$\langle r(r^*(h)), x\rangle = \langle r^*(h), x\rangle = h(r(x)) = h(x) = \langle h^\wedge, x\rangle$$

for all x in H. So $r(r^*f) = h^\wedge$. Thus $\langle h^\wedge, y\rangle = \langle r(r^*(h)), y\rangle$ for all y in $H^\#$. In particular,

$$\langle r(r^*(h)), r(x)\rangle = \langle h^\wedge, r(x)\rangle$$

for all x in X. Therefore $r(X)$ is self-dual. As in the first part of the proof, $r(X) = H^\#$. Therefore $h^\wedge = h$. Hence for each x in $H^\#$, by [11, 3.4],

$$\langle r^*(h), r^*(x)\rangle = \langle r(r^*(h)), x\rangle = \langle h, x\rangle.$$

So r is an H-isometry.

Remark 2.6. AW^*-algebras, injective C^*-algebras, monotone complete C^*-algebras, unital self-injective C^*-algebras, unital H-self-injective C^*-algebras are all, in some senses, generalizations of commutative C^*-algebras $C(X)$ with X being stonean spaces. It is desirable to clarify the relationship between them. Here are some questions:

1) Is every self-injective C^*-algebra H-self-injective?

2) Is every unital self-injective C^*-algebra monotone complete?

Acknowledgment. The author would like to thank Professor C.A.Akemann for some useful discussions.

REFERENCES

[1] Akemann, C. A., The general Stone-Weierstrass problem, *J. Functional Anal.*, **4** (1969), 277-294.
[2] Akemann, C. A., A Gelfand representation theory for C^*-algebra, *Pac. J. Math.*, **39** (1971), 1-11.
[3] Akemann, C. A., Pedersen, G. K. & Tomiyama, J., Multipiers of C^*-algebras, *J. Functional Anal.*, **13** (1973), 277-301.
[4] Dixmier, J., C^*-algebras, North-Holland Publishing Company, Amsterdam New York Oxford, 1977.
[5] Dure, M. J. & Fillmore, P. A., Triviality theorem for Hilbert modules, Topics in modern operator theory (C. Apostal et al., eds), Berkhauser, Basel, 1981, 71-80.
[6] Gillman, L. & Jerison, M., Ring of continuous functions, Springer-Verlag, New York Heidelberg Berlin (1976).
[7] Hamana, M., Injective envelopes of Banach modules, *Tohoku Math. Journal*, **10** (1978), 439-453.
[8] Hamana, M., Injective envelopes of C^*-algebras, *J. Math. Soc. Japan*, **31**:1 (1979), 181-197.
[9] Kaplansky, I., Projections in Banach algebras, *Ann. of Math.*, **53** (1951), 235-249.
[10] Lin, H., Hilbert C^*-modules and their bounded module maps (preprint).
[11] Lin, H., Bounded module maps and pure completely positive maps (preprint in proparation).
[12] Ruan, Z. J., On matrically normed spaces associated with operator algebras, Ph. D. dissertation, University of California Los Angeles, 1987.
[13] Paschke, W. L., Inner product modules over B^*-algebras, *Trans. Amer. Math. Soc.*, **182** (1973), 443-459.

[14] Pedersen, G. K., C^*-algebras and their automorphism group, Academic Press, London New York San Francisco, 1979.

[15] Pedersen, G. K., SAW^*-algebras and corona C^*-algebras, contributions to non-commutative topology, *J. Operator Theory,* **15** (1986), 25-32.

[16] Takesaki, M., Theory of opertor algebras I, Spinger-Verlag, Heidelberg, 1979.

[17] Takesaki, M., On the Hahn-Banach type theorem and the Jordan decomposition of module linear mapping over some operator algebras, *Kodai Math. Sem. Rep.,* **12** (1960), 1-10.

[18] Wittstock, G., Extension of completely bounded C^*-module homomorphisms, operator algebras and group representations VII, (Monographs and Studies in Mathematics 18), Pitman, 1984.