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ISOMETRIC OPERATORS ON Π_K SPACES*

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Abstract

The authors obtain all generalized triangle models of U-dilation of an isometric operator on Π_K and prove that an isometric operator on Π_K has Wold decomposition and the unilateral parts of generalized Wold decomposition for an isometric operator on Π_K are uniquely determined up to unitary equivalence. Then a necessary and sufficient condition is got under which an isometric operator on Π_K has a regular Wold decomposition.

Keywords Pontryagin space, Isometric operator, Indefinite inner product, Unitary dilation, Wold decomposition.
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In this paper, we give definitions of generalized triangle model, generalized Wold decomposition, Wold decomposition, and regular Wold decomposition of an isometric operator on Pontryagin space Π_K. In first section, we obtain all forms of U-dilations of an isometric operator on Π_K under any generalized standard decomposition. In second section, we obtain two results that any isometric operator on Π_K has Wold decomposition and the unilateral parts of generalized Wold decompositions for an isometric operator on Π_K are unitarily equivalent to another. In last section, we get a necessary and sufficient condition under which an isometric operator on Π_K has regular Wold decomposition and give a class of isometric operators on Π_ℓ which do not have regular Wold decompositions. Our necessary and sufficient condition is simpler than B. W. McEnnis' in [3].

§1. U-Dilation of Isometric Operator on Π_K

In [1], Yan Shaozong obtained all forms of U-dilations of contractions on Π_K under a regular decomposition of Π_K. In [2], we showed that any contraction on Π_K is of the triangle model under a standard decomposition of Π_K. Naturally, we desire to find all forms of U-dilations of contractions on Π_K relative to a standard decomposition of Π_K. In this section, we settle this problem in the case of isometric operators on Π_K.

Definition 1.1. *If V is a linear operator on Π_K such that*

$$(Vx, Vy) = (x, y), \text{ for any } x, y \in \Pi_K,$$

then V is called an isometric operator or isometry.

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Remark 1.1. In the case of Π_K, any isometric operator must be bounded.

Definition 1.2. *If* $\Pi_K = \Pi_\ell \oplus \{Z \dot{+} Z^*\} \oplus \Pi_m$ *such that* Π_ℓ, Π_m *are complete subspaces of* Π_K *and* $\{Z, Z^*\}$ *is a Hilbert dual pair, then* $\Pi_K = \Pi_\ell \oplus \{Z + Z^*\} \oplus \Pi_m$ *is called a generalized standard decomposition.*

Definition 1.3. *If* V *is an isometric operator on* Π_K *and there is a generalized standard decomposition* $\Pi_K = \Pi_\ell \oplus \{Z + Z^*\} \oplus \Pi_m$ *such that*

$$V = \begin{array}{c} \begin{bmatrix} S & A & B & C \\ & U_\ell & 0 & E \\ & & V_m & F \\ & & & S^{-1*} \end{bmatrix} \begin{matrix} Z \\ \Pi_\ell \\ \Pi_m \\ Z^* \end{matrix} \end{array}, \tag{1.1}$$

where $S : Z \to Z$; $U_\ell : \Pi_\ell \to \Pi_\ell$; $V_m : \Pi_m \to \Pi_m$; $E : Z^* \to \Pi_\ell$; $F : Z^* \to \Pi_m$; *and* $T : Z^* \to Z$; S *is nonsingular on* Z; U_ℓ *is unitary on* Π_ℓ; V_m *is an isometric operator on* Π_m. $A = -SE^\dagger U_\ell, B = -SF^\dagger V_n, C = -S(E^\dagger E + F^\dagger F)/2 + ST$, *and* $T = -T^*$ *provided that* Z *is identical with* Z^*, *then* $V = \{S, U_\ell, V_m, E, F, C\}$ *is called the generalized triangle model of* V, *where* $\dagger$ *and* $*$ *denote the adjoint operations in indefinite and definite inner products respectively.*

It generalizes [4], Chapter 3, §2.

Definition 1.4. *Suppose that* T *is a contraction on* Π_K *and* H *is a Hilbert space. If there exists a unitary* U *on* $\Pi_K \oplus H$, *which is a unitary with respect to the indefinite inner product, such that*

$$T = PU|\Pi_K,$$

where P *is the projection from* $\Pi_K \oplus H$ *onto* Π_K, *then* U *is called the U-dilation of* T.

Under the generalized triangle model (1.1), the U-dilations of V must be of the following form:

$$U = \begin{bmatrix} S & A & B & C & W \\ 0 & U_\ell & 0 & E & X \\ 0 & 0 & V_m & F & Y \\ 0 & 0 & 0 & S^{-1*} & Q \\ M & L & J & K & U_0 \end{bmatrix} \begin{matrix} Z \\ \Pi_\ell \\ \Pi_m \\ Z^* \\ H \end{matrix}$$

By a direct calculation, $U^\dagger$ is of the following form

$$U^\dagger = \begin{bmatrix} S^{-1} & E^\dagger & F^\dagger & C^* & K^* \\ 0 & U & V_m^\dagger & B^\dagger & J^\dagger \\ 0 & 0 & 0 & S^* & M^* \\ Q^* & X^\dagger & Y^\dagger & W^* & U_0^* \end{bmatrix}.$$

$UU^\dagger = I$ is equivalent to the following equations:

$$\text{(I)} \quad \begin{cases} SS^{-1} + WQ^* = I, \\ XQ^* = 0, \\ YQ^* = 0, \\ QQ^* = 0, \\ U_0Q^* = 0. \end{cases}$$

$$\text{(II)}\quad \begin{cases} SE^\dagger + AU_\ell^\dagger + WX^\dagger = 0, \\ U_\ell U_\ell^\dagger + XX^\dagger = I, \\ YX^\dagger = 0, \\ QX^\dagger = 0, \\ ME^\dagger + LU_\ell^\dagger + U_0X^\dagger = 0. \end{cases}$$

$$\text{(III)}\quad \begin{cases} SF^\dagger + BV_m^\dagger + WY^\dagger = 0, \\ XY^\dagger = 0, \\ V_mV_m^\dagger + YY^\dagger = I, \\ QY^\dagger = 0, \\ MF^\dagger + JV_m^\dagger + U_0Y^\dagger = 0, \end{cases}$$

$$\text{(IV)}\quad \begin{cases} SC^* + AA^\dagger + BB^\dagger + CS^* + WW^* = 0, \\ U_\ell A^\dagger + ES^* + XW^* = 0, \\ V_mB^\dagger + FS^* + YW^* = 0, \\ S^{*-1}S^* + QW^* = I, \\ MC^* + LA^\dagger + JB^* + KS^* + U_0W^* = 0. \end{cases}$$

$$\text{(V)}\quad \begin{cases} SK^* + AL^\dagger + BJ^\dagger + CM^* + WU_0^* = 0, \\ U_\ell L^\dagger + EM^* + XU_0^* = 0, \\ V_mJ^\dagger + FM^* + YU_0^* = 0, \\ S^{*-1}M^* + QU_0^* = 0, \\ MK^* + LL^\dagger + JJ^\dagger + KM^* + U_0U_0^* = I. \end{cases}$$

$U^\dagger U = I$ is equivalent to the following equations:

$$\text{(I}'\text{)}\quad \begin{cases} S^{-1}S + K^*M = I, \\ L^\dagger M = 0 \\ M^*M = 0, \\ J^\dagger M = 0, \\ Q^*S + U_0^*M = 0. \end{cases}$$

$$\text{(II}'\text{)}\quad \begin{cases} S^{-1}A + E^\dagger U_\ell + K^*L = 0, \\ U_\ell^\dagger U_\ell + L^\dagger L = 0, \\ J^\dagger L = 0, \\ M^*L = 0, \\ Q^*A + X^\dagger U_\ell + U_0^*L = 0. \end{cases}$$

$$\text{(III}'\text{)}\quad \begin{cases} S^{-1}B + F^\dagger V_m K^*J = 0, \\ L^\dagger J = 0, \\ V_m^\dagger V_m + J^\dagger J = I, \\ M^*J = 0, \\ Q^*B + Y^\dagger V_m + U_0^*J = 0. \end{cases}$$

$$\text{(IV}'\text{)}\quad \begin{cases} S^{-1}C + E^\dagger E + F^\dagger F + C^*S^{*-1} + K^*K = 0, \\ U_\ell^\dagger E + A^\dagger S^{*-1} + L^\dagger K = 0, \\ V_m^\dagger F + B^\dagger S^{*-1} + J^\dagger K = 0, \\ S^*S^{*-1} + M^*K = I, \\ Q^*C + X^\dagger E + Y^\dagger F + W^*S^{*-1} + U_0^*K = 0. \end{cases}$$

$$
(\mathrm{V}')\quad \begin{cases} S^{-1}W + E^\dagger X + E^\dagger Y + C^* Q + K^* U_0 = 0, \\ U_\ell^\dagger X + A^\dagger Q + L^\dagger U_0 = 0, \\ V_m^\dagger Y + B^\dagger Q + J^\dagger U_0 = 0, \\ S^* Q + M^* U_0 = 0, \\ Q^* W + X^\dagger X + Y^\dagger Y + W^* Q + U_0^* U_0 = I. \end{cases}
$$

If $UU^\dagger = I$ and $U^\dagger U = I$ hold, then we have

$$Q = 0,\ X = 0,\ M = 0,\ L = 0,\ J = 0$$

from the above equations. Therefore, equations (I)-(V) and (I')-(V') are simplified as follows:

$$SF^\dagger + BV_m^\dagger + WY^\dagger = 0. \tag{1.2}$$

$$V_m V_m^\dagger + YY^\dagger = I. \tag{1.3}$$

$$U_0 Y^\dagger = 0. \tag{1.4}$$

$$SC^* + AA^\dagger + BB^\dagger + CS^* + WW^* = 0. \tag{1.5}$$

$$V_m B^\dagger + FS^* + YW^* = 0. \tag{1.6}$$

$$KS^* + U_0 W^* = 0. \tag{1.7}$$

$$YU_0^* = 0. \tag{1.8}$$

$$U_0 U_0^* = I. \tag{1.9}$$

$$Y^\dagger V_m = 0. \tag{1.10}$$

$$S^{-1}C + E^\dagger E + F^\dagger F + C^* S^{*-1} + K^* K = 0. \tag{1.11}$$

$$Y^\dagger F + W^* S^{*-1} + U_0^* K = 0. \tag{1.12}$$

$$Y^\dagger Y + U_0^* U_0 = I. \tag{1.13}$$

In fact, by Definition 1.3 the above equations can be simplified further. Since $C = -S(E^\dagger E + F^\dagger F)/2 + ST$, hence $K = 0$, equations (1.7) and (1.12) are reduced to

$$U_0 W^* = 0, \tag{1.7$'$}$$

$$Y^\dagger F + W^* S^{*-1} = 0. \tag{1.12$'$}$$

In (1.5), substitute $-SF^\dagger Y$ for W. Hence

$$
\begin{aligned}
& SC^* + AA^* + BB^\dagger + CS^* + SF^\dagger TT^\dagger FS^* \\
=& SC^* + SE^\dagger ES^* + SF^\dagger V_m V_m^\dagger FS^* + CS^* + SF^\dagger YY^\dagger FS^* \\
=& SC^* + SE^\dagger ES^* + SF^\dagger V_m V_m^\dagger FS^* + CS^* + SF^\dagger (I - V_m V_m^\dagger) FS^* \\
=& SC^* + SE^\dagger ES^* + SF^\dagger FS^* + CS^* \\
=& S(C^* S^{*-1} + E^\dagger E + F^\dagger F + S^{-1} C) S^* \\
=& 0,
\end{aligned}
$$

where (1.3) is used. Again substitute the expression of W in (1.2), and then we have

$$
\begin{aligned}
SF^\dagger + BV_m^\dagger + (-SF^\dagger Y)Y^\dagger &= SF^\dagger + BV_m^\dagger - SF^\dagger + SF^\dagger V_m V_m^\dagger \\
&= -SF^\dagger V_m V_m^\dagger + SF^\dagger V_m V_m^\dagger = 0
\end{aligned}
$$

Note that equation (1.6) is the adjoint of (1.2). So, (1.2) and (1.6) hold naturally. Consequentially, $U^\dagger U = I$ and $UU^\dagger = I$ are equivalent to

$$\begin{cases} V_m V_m^\dagger + YY^\dagger = I, \\ U_0 Y^\dagger = 0, \\ U_0 U_0^* = I, \\ Y^\dagger V_m = 0, \\ Y^\dagger Y + U_0^* U_0 = I. \end{cases} \tag{A}$$

$$W = -SF^\dagger Y. \tag{11''}$$

It is clear that the equation system (A) determines all forms of U-dilations of the isometry V_m on $(\Pi_m, (\cdot,\cdot))$. By (A), we can solve Y. Thus $W = -SF^\dagger Y$.

Theorem 1.1. *Suppose that V is an isometric operator on $\Pi_K, \Pi_K = \Pi_\ell \oplus \{Z \dot{+} Z^*\} \oplus \Pi_m$, and the corresponding generalized triangle model is*

$$V = \begin{bmatrix} S & A & B & C \\ & U_\ell & 0 & E \\ & & V_m & F \\ & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ \Pi_\ell \\ \Pi_m \\ Z^* \end{matrix}.$$

Then, the U-dilation of V exists, all forms of U-dilations of V are

$$U = \begin{bmatrix} S & A & B, & -SF^\dagger Y & C \\ & U_\ell & 0 & 0 & E \\ & & V_m & Y & F \\ & & 0 & U_0 & 0 \\ & & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ \Pi_\ell \\ \Pi_m, \\ H \\ Z^* \end{matrix} \tag{1.14}$$

where $V_0 = \begin{bmatrix} V_m & Y \\ & U_0 \end{bmatrix}$ is a unitary on indefinite inner product space $(\Pi_m \oplus H, (\cdot,\cdot) \oplus (\cdot,\cdot)_H)$. Moreover, in order that U is a minimal U-dilation of V, it is nesessary and sufficient that $\begin{bmatrix} V_m & Y \\ & U_0 \end{bmatrix}$ is a minimal U-dilation of V_m.

Remark 1.2. Since all forms of U-dilations of V are of (1.14), it follows that $U^n, n = 1, 2, \ldots$, are also the U-dilations of V^n.

Remark 1.3. U is called a minimal U-dilation of V, if $\Pi_K \oplus H = \bigvee_{-\infty}^{+\infty} U^n \Pi_K$.

Proof of Theorem 1.1. Since

$$(B, -SF^\dagger Y) = -S(F^\dagger, 0) \begin{bmatrix} V_m & Y \\ & U_0 \end{bmatrix},$$

U is a unitary on $\Pi_K \oplus H$. Naturally, (1.14) are all forms of U-dilations of V. It is sufficient for us to show that the necessary and sufficient condition for U to be a minimal U-dilation of V is that V_0 is a minimal U-dilation of V_m. Assume that U is a minimal U-dilation of V, i.e. $\Pi_K \oplus H = \bigvee_{-\infty}^{+\infty} U^n \Pi_K$. If $\bigvee_{-\infty}^{+\infty} V_0^n \Pi_m$ is a proper reduced subspace to V_0, then

$$V_0 = \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix} \begin{matrix} \bigvee_{-\infty}^{+\infty} V_0^n \Pi_m \\ (\bigvee_{-\infty}^{+\infty} V_0^n \Pi_m)^\perp \end{matrix}.$$

Since $\overset{+\infty}{\underset{-\infty}{\vee}} V_0^n \Pi_m \supset \Pi_m$ and V_1 and V_2 are unitary operators, we have

$$V_0 = \begin{bmatrix} \begin{bmatrix} V_{m'} & Y \\ & U_0' \end{bmatrix} & \\ & V_2 \end{bmatrix} \begin{matrix} \Pi_m \\ H_1 \\ (\overset{+\infty}{\underset{-\infty}{\vee}} V^0 \Pi_m)^\perp \end{matrix}.$$

$Y = (Y_1, 0)$, and $U_0 = \begin{bmatrix} U_0' & \\ & V_2 \end{bmatrix}$.

Letting

$$U' = \begin{bmatrix} S & A & (B, & -SF^*Y) & C \\ 0 & U_\ell & 0 & 0 & E \\ & & V_m & Y_1 & F \\ & & & U_0' & 0 \\ & & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ \Pi_\ell \\ \Pi_m \\ H_1 \\ Z^* \end{matrix},$$

we know that U' is also a unitary; furthermore

$$U = \begin{bmatrix} U' & \\ & V_2 \end{bmatrix} \begin{matrix} \Pi_K \oplus H_\ell \\ (\overset{+\infty}{\underset{-\infty}{\vee}} V_0^n \Pi_m)^\perp \end{matrix}.$$

This contradicts the fact that U is a minimal U-dilation. Conversely, when V is a minimal U-dilation of V_m, using the same method, we can show easily that U is also a minimal U-dilation of V.

§2. Wold Decompositions of Isometric Operators on Π_K Spaces

As we know, for any isometry on Hilbert space Wold decomposition exists, i.e.

$$H = \overset{\infty}{\underset{0}{\oplus}} V^n (VH)^\perp \oplus \overset{\infty}{\underset{n=0}{\cap}} V^n H,$$

V is a unitary on $\overset{\infty}{\underset{n=0}{\cap}} V^n H$ and V is a unilateral shift on $\overset{\infty}{\underset{0}{\oplus}} V^n (VH)^\perp$; moreover, subspaces $\overset{\infty}{\underset{n=0}{\cap}} V^n H$ and $\overset{\infty}{\underset{0}{\oplus}} V^n (VH)^\perp$ are reduced subspaces to V. Is the $\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K$ a regular subspace? In general, it is not true (see Example 3.1). $\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K$ is possibly a degenerate subspace. So we have to generalize Wold decomposition. We define generalized Wold decomposition, Wold decomposition and regular Wold decomposition and prove that any isometric operator on Π_K space has Wold decomposition and the unilateral parts of isometric operator on Π_K are uniquely determined up to unitary equivalence.

Definition 2.1. *Let V be an isometry on Π_K. A complete subspace L of Π_K will be called wandering for V if $V^p L \perp V^q L$ for every pair of integers $p, q \geq 0, p \neq q$; since V is an isometry it suffices to suppose that $V^n L \perp L$ for $n = 1, 2, \ldots$.*

One can then form $M_+(L) = \overset{\infty}{\underset{n=0}{\vee}} V^n L$, however $M_+(L)$ is possibly degenerate. We let $[M_+(L)]_0$ be a nondegenerate subspace of $M_+(L)$ ($[M_+(L)]_0$ is not unique).

Definition 2.2. *Let V be an isometric operator on Π_K and $\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K = Z \oplus \Pi_u$, where*

$$Z = \Big[\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K\Big] \cap \Big[\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K\Big]^\perp, \quad \Pi_u = \Big[\overset{\infty}{\underset{n=0}{\cap}} V^n \Pi_K\Big]_0.$$

If there is a generalized standard decomposition $\Pi_K = \Pi_u \oplus \{Z \dot{+} Z^*\} \oplus \Pi_s$ such that $V = \{S, U, V_s, C, D, B\}$ is a generalized triangle model, where U is a unitary on Π_u and V_s is a *unilateral operator on* Π_s*, then* $V = \{S, U, V_s, C, D, B\}$ *is called Wold decomposition of* V*. If there are complete subspaces* $\Pi_{K'}$ *and* P *such that* $\Pi_K = \Pi_{K'} \oplus P$ *and* $V = U \oplus V_s$*, where* U *is a unitary on* $\Pi_{K'}$ *and* V_s *is a unilateral operator on* P*, then* $V = U \oplus V_s$ *is called a regular Wold decomposition of* V*.*

Definition 2.3 *Let* V *be an isometry on* Π_K*. If there is a generalized standard decomposition* $\Pi_K = \Pi_{u'} \oplus \{Z' \dot{+} Z'^*\} \oplus \Pi_{s'}$ *such that* $V = \{S', U', V_{s'}, C', D', B'\}$ *is a generalized triangle model, where* U' *is a unitary on* Π_K *and* $V_{s'}$ *is a unilateral operator on* $\Pi_{s'}$*, then* $V = \{S', U', V_{s'}, C', D'B'\}$ *is called a generalized Wold decomposition of* V*.*

Next we discuss Wold decomposition and generalized Wold decomposition.

Theorem 2.1. *Any isometric operator on* Π_K *space has Wold decomposition, i.e. there exists a generalized standard decomposition* $\Pi_K = \Pi_u \oplus \{Z \dot{+} Z^*\} \oplus \Pi_s$ *such that*

$$V = \begin{bmatrix} S & F & G & B \\ & U & 0 & C \\ & & V_s & D \\ & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ \Pi_u \\ \Pi_s \\ Z^* \end{matrix},$$

where

$$\Pi_u \oplus Z = \bigcap_{n=0}^{\infty} V^n \Pi_K, \quad L = \Pi_K \ominus V\Pi_K, \quad M_+(L) = \bigvee_{n=0}^{\infty} V^n L,$$

$$Z = M_+(L) \cap M_+(L)^\perp, \quad \Pi_s = [M_+(L)]_0, \quad M_+(L) = Z \oplus \Pi_s,$$

and S *is unique.* U *and* V_s *are uniquely determined up to unitary equivalence by the choice of the subspace* Z^**.*

Proof. We first prove the existence of Wold decomposition.

At first we prove $(\bigcap_{n=0}^{\infty} V^n \Pi_K) = M_+(L)^\perp$. If $x \in \bigcap_{n=0}^{\infty} V^n \Pi_K$, then there exists $x_m \in \Pi_K$ such that $x = V^m x_m, m = 1, 2, \ldots$. Let $m > n$,

$$(V^m x_m, V^n \ell) = (V^{m-n} x_m, \ell) = 0,$$

where $\ell \in L$, i.e. $x \perp V^n L$, and $x \perp M_+(L)$. So $\bigcap_{n=0}^{\infty} V^n \Pi_K \subset M_+(L)^\perp$. Conversely, let $y \in M_+(L)^\perp$ (see B. W. McEnnis [3]).

$$M_+(L) = L \oplus VL \oplus \cdots \oplus V^n L \oplus V^n M_+(L),$$

and

$$\Pi_K \ominus V^n \Pi_K = L \oplus VL \oplus \cdots \oplus V^{n-1} L.$$

Hence $y \in [\Pi_K \ominus V^n \Pi_K]^\perp = V^n \Pi_K$, i.e. $y \in \bigcap_{n=0}^{\infty} V^n \Pi_K$. So $(\bigcap_{n=0}^{\infty} V^n \Pi_K)^\perp \subset M_+(L)$, and we proved $\bigcap_{n=0}^{\infty} V^n \Pi_K = M_+(L)^\perp$. Let

$$Z = M_+(L) \cap (M_+(L)^\perp = [\bigcap_{n=0}^{\infty} V^n \Pi_K] \cap [\bigcap_{n=0}^{\infty} V^n \Pi_K]^\perp.$$

Then $M_+(L) = Z \oplus [M_+(L)]_0$, and $\bigcap_{n=0}^{\infty} V^n \Pi_K = Z \oplus [\bigcap_{n=0}^{\infty} V^n \Pi_K]_0$, where $[M_+(L)]_0$ and $[\bigcap_{n=0}^{\infty} V^n \Pi_K]_0$ are nondegenerated parts of $M_+(L)$ and $\bigcap_{n=0}^{\infty} V^n \Pi_K$ respectively. They are not unique. Let $\Pi_u = [\bigcap_{n=0}^{\infty} V^n \Pi_K]_0$, $\Pi_s = [M_+(L)]_0$, and there exists a neutral subspace $Z^* \subset \Pi_K$ such that $\{Z, Z^*\}$ is a pair of Hibert dual and $\Pi_K = \Pi_u \oplus \{Z \dot{+} Z^*\} \oplus \Pi_s$. As

$$V^\dagger V | \bigcap_{n=0}^{\infty} V^n \Pi_K = VV^\dagger | \bigcap_{n=0}^{\infty} V^n \Pi_K = I | \bigcap_{n=0}^{\infty} V^n \Pi_K,$$

$V(\Pi_u \oplus Z)$ and $V^\dagger\ (\Pi_u \oplus Z) \subset \Pi_u \oplus Z$.

Similarly to triangle model theory of semiunitary on Π_K established by Yan Shaozong[4], we can find the following generalized triangle model:

$$V = \begin{bmatrix} S & F & G & B \\ & U & 0 & C \\ & & V_s & D \\ & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ \Pi_u \\ \Pi_s \\ Z^* \end{matrix},$$

where S, U, V_s, C, D, B are independent variants. S is an invertible operator on Z. U is a unitary on $(\Pi_u, (\cdot,\cdot))$. V_s is an isometric operator on $(\Pi_s, (\cdot,\cdot))$. C, D, B are bounded operators from $(Z^*, \langle\cdot,\cdot\rangle)$ to $(\Pi_u, (\cdot,\cdot), (\Pi_s, (\cdot,\cdot)$ and $(Z\langle\cdot,\cdot\rangle)$ respectively. And

$$F = -SC^\dagger U, \quad G = -SC^\dagger V_s,$$

$$B = \frac{1}{2} S(-C^\dagger C - D^\dagger D + 2Q), \quad Q = -Q^*.$$

Then $V = \{S, U, V_s, C, D, B\}$. Now we prove that Π_s is a Hilbert space and V_s is a unilateral operator on Π_s. As $L = \Pi_K \ominus V\Pi_K$ is a positive subspace, $M_+(L)$ is a semipositive closed subspace. Then $\Pi_s = [M_+(L)]_0$ is a Hibert subspace. Suppose that V_s is not a unilateral operator on Π_s. Then there is a Wold decomposition in the case of Hibert space, i.e. $\Pi_s = \Pi_{s'} \oplus \Pi_{s''}$ and $V_s = V_{s'} \oplus V_{s''}$, where $V_{s'}$ is a unitary on $\Pi_{s'}$, $V_{s''}$ is a unilateral operator on $\Pi_{s''}$. And $\Pi_{s'} = \bigcap_{n=0}^{\infty} V_s^n \Pi_s$. Then $\Pi_{s'} \neq \{0\}$, i.e. there exists nonzero vector $x \in \Pi_{s'}$ and $x = V_s^n x_0^{(n)}$, $n = 0, 1, 2, \ldots$,

$$V^n x_0^{(n)} = \sum_{j=0}^{n-1} S^{n-1-j} G V_s^j x_0^{(n)} + V_s^n x_0^{(n)},$$

where $x_0^{(n)} \in \Pi_s, n = 0, 1, 2, \ldots$. Then there exists $z_n \in Z$ such that

$$\sum_{j=0}^{n-1} S^{n-1-j} G V_s^j x_0^{(n)} = S^n z_n.$$

So $x = V^n(x_0^{(n)} - z_n), n = 0, 1, 2, \ldots$, i.e. $x \in \bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K = Z \oplus \Pi_u$. This contradicts $x \in \Pi_s$. So V_s is a unilateral operator on Π_s. Hence $V = \{S, U, V_s, C, D, B\}$ is Wold decomposition of V.

Let us prove the residual part of the theorem.

Let $\Pi_K = \Pi_{u'} \oplus \{Z' \dot{+} Z'^*\} \oplus \Pi_{s'}$ be another generalized standard decomposition such that $V = \{S', U', V_{s'}, C', D', D'\}$ is another Wold decomposition of V. According to Definition 2.2, $\Pi_{u'} \oplus Z' = \bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K$. Then

$$Z' = [\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K] \cap [\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]^\perp = Z$$

and $S = V|Z = S'$. We define quotient spaces $[\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]/Z$ and $[M_+(L)]/Z$, equipped with the indefinite inner product $([x],[y])_Z = (x,y)$. Then $[\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]/Z$ is a Pontryagin space and $[M_+(L)]/Z$ is a Hibert space. Let $U_1 : \Pi_{u'} \to [\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]/Z$ and $U_1 x = [x]$, $\forall x \in \Pi_{u'}$. It is the same for U_2. Then U_1 and U_2 are both unitary operators. We define a linear operator $[V|\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]$ on $[\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K]/Z$ such that $[V|\bigcap_{n=0}^{\infty} \mathrm{V}^n \Pi_K][x] = [Vx]$. This operator is unitary on $[\bigcap_{n=0}^{\infty} V^n\Pi_K]/Z$ and $U_1U'U_1^* = [V|\bigcap_{n=0}^{\infty} V^n\Pi_K]$. It is the same for $[V|M_+(L)]$ and $U_2V_{s'}U_2^* = [V|M_+(L)]$. Then U' and $V_{s'}$ are unitarily equivalent to $[V|\bigcap_{n=0}^{\infty} V^n\Pi_K]$ and $[V|M_+(L)]$ respectively. This concludes the proof.

For Wold decomposition of V on Π_K, we ask $Z\oplus\Pi_u = \bigcap_{n=0}^{\infty} V^n\Pi_K$. It assures that unitary part and unilateral part of V associated with Wold decomposition are uniquely determined up to unitary equivalence. However, for isometric operator on Π_K there are many generalized Wold decompositions. We ask if the unilateral parts of V associated with the generalized Wold decompositions are unitarily equivalent. In the following, we solve the problem and compare the generalized Wold decomposition with Wold decomposition.

Let $V = \{S, U, V_s, C, D, B\}$ associated with $\Pi_K = \Pi_u \oplus \{Z\dot{+}Z^*\} \oplus \Pi_s$ be Wold decomposition. We obtain the following theorem.

Theorem 2.2. *Let V be an isometric operator on Π_K. If there is a generalized standard decomposition $\Pi_K = \Pi_{u'} \oplus \{Z' + Z'^*\} \oplus \Pi_{s'}$ such that $V = \{S', U', V_{s'}, C', D', B'\}$ is a generalized Wold decomposition of V, then*

1. $\Pi_{u'} \oplus Z' \subset \Pi_u \oplus Z, Z \subset Z', \Pi_{u'} \subset \Pi_u$, and $S = S'|Z$;

2. $V_{s'}$ and V_s are unitarily equivalent, where V_s is the unilateral part of V associated with its Wold decomposition.

Proof. It is obvious that

$$\bigcap_{n=0}^{\infty} V^n\Pi_K = \{x|(V^{\dagger n}x, V^{\dagger n}x) = (x,x), n = 1, 2, \ldots,\}.$$

Then

$$\Pi_{u'} \oplus Z' \subset \Pi_u \oplus Z = \bigcap_{n=0}^{\infty} V^n\Pi_K.$$

We choose a suitable Hilbert pair $\{Z', Z''^*\}$ such that

$$\Pi_{u'} \oplus \{Z' + Z''^*\} \subset \Pi_u \oplus \{Z\dot{+}Z^*\}.$$

Then there is another generalized triangle model $V = \{S', U', V_{s''}, C'', D'', B''\}$ associated with

$$\Pi_K = \Pi_{v'} \oplus \{Z' + Z''^*\} \oplus \Pi_{s''}.$$

Then $Z' \oplus \Pi_{s''} = Z' \oplus \Pi_{s'}$ and $V_{s''}$ is unitarily equivalent to $V_{s'}$. Hence $V_{s''}$ is a unilateral operator on $\Pi_{s''}$ either.

$$\Pi_s = \Pi_K \ominus [\Pi_u \oplus \{Z\dot{+}Z^*\}] \subset \Pi_K \ominus [\Pi_{u'} \oplus \{Z' + Z''^*\}] = \Pi_{s''}.$$

If $\Pi_s \neq \Pi_{s''}$, then $\Pi_{s''} \ominus \Pi_s$ is an infinite dismensional Hilbert space. We choose orthogonal bases $\{e_n\}_{n=1}^{\infty}$ for $\Pi_{s''} \ominus \Pi_s$. Then

$$e_n = x_n + z_n + z_n'^* + p_n,$$

where $x_n \in \Pi_u, z_n \in Z$, $z_n'^* \in Z^*$ and $p_n \in \Pi_s$. Since $e_n \perp \Pi_s$, $p_n = 0$. Let $y_n = z_n + z_n'^*$ and $[\cdot,\cdot]$ is the definite inner product associated with $\Pi_K = \Pi_u \oplus \{Z\dot{+}Z^*\} \oplus \Pi_s$. Then x_n is orthogonal to y_n with respect to $(\cdot,\cdot)$ or $[\cdot,\cdot]$. As $[e_n, e_m] = 0$, $[y_n, y_m] = 0$, $n \neq m$. However, the number of dimensions of $\{Z + Z'^*\}$ at most is $2K$. So there exists ℓ, $2 \le \ell \le 2K+1$, such that $y_\ell = 0$. Then $e_\ell \perp \{Z\dot{+}Z^*\}$ and $e_\ell \in \Pi_u$. Hence

$$(V^{\dagger^n} e_\ell, V^{\dagger^n} e_\ell) = (U^{\dagger^n} e_\ell, U^{\dagger^n} e_\ell) = (e_\ell, e_\ell) = 1.$$

In another way according to the generalized Wold decomposition

$$V = \{S', U', V_{s''}, C'', D'', B''\},$$

we have

$$(V^{\dagger^n} e_\ell, V^{\dagger^n} e_\ell) = (V_{s''}^{*^n} e_\ell, V_{s''}^{*^n} e_\ell) \to 0, \quad n \to \infty,$$

which is impossible. Hence $\Pi_{s''} = \Pi_s$. So

$$Z \oplus \Pi_s = (Z \oplus \Pi_u)^\perp \subset (Z' \oplus \Pi_{u'})^\perp = Z' \oplus \Pi_{s''} = Z' \oplus \Pi_s.$$

Then $Z \subset Z'$. $Z' \oplus \Pi_{u'} \subset Z \oplus \Pi_u$ implies that $\Pi_{u'} \subset \Pi_u$. For any $p \in \Pi_s$,

$$Vp = V_s p + Gp = V_s'' p + G'' p.$$

As $G''p$ and $Gp \in Z'$ and $p' \perp Z'$ for any $p' \in Z'$, then

$$(Vp, p') = (V_s p, p') = (V_{s''} p, p'),$$

so $V_s = V_{s''}$. Hence $V_{s'}$ is unitary equivalent to V_s. This concludes the proof.

Theroem 2.2 sufficiently shows that the unilatural part of isometric operator on Π_K is its intrinsic feature.

Let $L_s = \Pi_s \ominus V_s \Pi_s$ be a wandering subspace of V_s and

$$\Pi_s = M_+^s(L_s) = \overset{\infty}{\underset{0}{\oplus}} V_s^n L_s.$$

Obviously L_s is a wandering subspace of V either. We define $M_+(L_s) = \bigvee_{n=0}^{\infty} V^n Ls$ and have the following corollaries.

Corollary 2.1. (i) $\dim L_s = \dim L$, (ii) $M_+(L_s) = M_+(L)$.

Corollary 2.2. *Let V be an isometric operator on Π_K, $V = \{S, U, V_s, C, D, B\}$ is its Wold decomposition. Then V has regular Wold decomposition if and only if $D = 0$ or $G = 0$.*

The proofs are obvious.

Below, we will discuss the regular Wold decomposition.

§3. Regular Wold Decompositions of Isometric Operators on Π_K

In his doctoral dissertation, B. W. McEnnis showed that the necessary and sufficient condition for an isometry V on Π to be of Wold decomposition is $\Pi = \bigvee_{n=0}^{\infty} V^n L \oplus \bigvee_{n=0}^{\infty} V^n \Pi$. At the beginning of this paper, we have pointed out that it is difficult to verify this condition. Below, we will prove another necessary and sufficient condition for V to be of regular Wold decomposition.

Theorem 3.1. *Suppose that V is an isometry on Π_K (its generalized triangle model is (1.1)) and $(G_0, G_1, G_2, \ldots, G_n, \ldots)$ is a linear operator from $\oplus(I_n = I)$ to Z, where $G_n = \sum_{k=0}^{n} S^{k+1} F^\dagger V_m^{n-k}$ and $I = \Pi_m \ominus V_m \Pi_m$. Then, the necessary and sufficient condition for V to be of regular Wold decomposition is that $(G_0, G_1, G_2, \ldots, G_n, \ldots)$ is bounded. In that case there exists a generalized standard decomposition*

$$\Pi_K = \Pi_{\ell'} \oplus \{Z' + Z'^*\} \oplus \{\Pi_{m'} \oplus P\}$$

such that

$$V = \begin{bmatrix} S' & A' & (B' & 0) & C' \\ & U' & 0 & 0 & E' \\ & & U_{m'} & & F' \\ & & & V_{p'} & 0 \\ & & & & S'^{-1*} \end{bmatrix} \begin{matrix} Z' \\ \Pi_{\ell'} \\ \Pi_{m'} \\ P \\ Z'^{*} \end{matrix},$$

where $V_{m''} = \begin{bmatrix} U_{m'} & \\ & V_{p'} \end{bmatrix}$ *is Wold decomposition on the Hilbert suace* $\Pi_{m'} \oplus P$, *i.e.* $U_{m'}$ *is a unitary on* $\Pi_{m'}$ *and* $V_{p'}$ *is a unilateral shift on* P.

Proof. Using the generalized triangle model (1.1), we calculate $(V\Pi_K)^{\perp}$ and $V^n(V\Pi_K)^{\perp}$. Assume $(z_1', x', y', z_2'^*) \in (V\Pi_K)^{\perp}$; for any vector $(z_1, x, y, z_2^*) \in \Pi_K$ we have

$$(Sz_1, z_2'^*) + (Ax, z_2'^*) + (By, z_2'^*) + (Cz_2^*, z_2'^*) + (S^{*-1}z_2^*, Z_1')$$
$$+ (U_\ell x, x') + (Ez_2^*, x') + (V_m y, y') + (Fz_1^*, y') = 0.$$

Let $x = y = z_2^* = 0$. Hence $(SZ_1, z_2'^*) = 0$, which implies $z_2'^* = 0$. Again let $y = z_2^* = z_1 = 0$. Hence $(U_\ell x, x') = 0$, which implies $x' = 0$. Therefore

$$(S^{*-1}z_2^*, z_1') + V_m y, y') + (Fz_2^*, y') = 0.$$

Obviously, $y' \in (\Pi_m \ominus V_m\Pi_m)$ and

$$(S^{*-1}z_2^*, z_1') + (Fz_2^*, y') = 0.$$

This is equivalent to $z' = -SF^{\dagger}y'$. Thus

$$L = (V\Pi_K)^{\perp} = \{-SF^{\dagger}y' + y' | y' \in (\Pi_m \ominus V_m\Pi_m)\}.$$

By the induction, we obtain

$$V^n L = \{(-SF^{\dagger} - SF^{\dagger}V_m - \cdots - SF^{\dagger}V_m)y' + V_m^n y | y' \in I\}.$$

It is clear that the necessary and sufficient condition for $\bigvee_{n=0}^{\infty} V^n L$ to be a regular subspace is that $\bigvee_{n=0}^{\infty} V^n L$ is non-degenerate.

Now we prove the sufficiency. Let the norm $\| \quad \|$ be associated with a generalized standard decomposition

$$\Pi_K = \Pi_\ell \oplus \{Z + Z^*\} \oplus \Pi_m.$$

Suppose that $z_n + y_n \in \bigvee_{n=0}^{\infty} V^m L$ is a convergence sequence. Since dim $Z < \infty$ and Π_m is closed, we have $y_n \to y \in \Pi_m$, $z_n \to z \in Z$. Assume that $y = 0$. Because $\bigvee_{n=0}^{\infty} V^n I$ is a closed subspace of Π_m, we can write

$$y_n = \sum_{k=0}^{\infty} V_m^k y_k^{(n)}, \ y_k^{(n)} \in I, \ k, n = 0, 1, 2, \ldots.$$

From the isometry of V_m it follows that

$$\|y_n\|^2 = \sum_{k=0}^{\infty} \|y_k^{(n)}\|^2 \to 0 \ (n \to +\infty).$$

Since $(G_0, G_1, G_2, \ldots, G_n, \ldots)$ is bounded, we have $\sum_{k=0}^{\infty} G_k y_k^{(n)} \to 0 \ (n \to +\infty)$. By the expression of $V^n L$, we know $z \to 0$, which means $z = 0$. Therefore, $\bigvee_{n=0}^{\infty} V^n L$ is nondegenerate.

Second we prove the necessity. In the case, we conclude that $\sum\limits_{k=0}^{\infty} G_k y_k$ is convergent for any $\{y_k\} \in \oplus I_n$. If it is not true, then for some $\delta_0 > 0$ there exist $m_N, I_N \leq N$ for any positive integer N such that

$$\|\sum_{m_N}^{m_N+\ell_N} G_k y_k\| > \delta_0 \quad \text{for some } \{y_k\}_0^\infty \in \overset{\infty}{\underset{0}{\oplus}} I_n.$$

Set

$$(y_0'^{(N)}, y_1'^{(N)}, \dots, y_k'^{(N)}, \dots,) = \Bigg(0\dots, 0, \overset{(m_N)}{y_{m_N}} \Big/ \|\sum_{m_N}^{m_N+\ell_N} G_k y_k\|, \dots,$$
$$\overset{(m_N+l_N)}{y_{m_N}+l_N} \Big/ \|\sum_{m_N}^{m_N+\ell_N} G_k y_k\|, 0\dots\Bigg).$$

Since $\left\|\sum\limits_{k=0}^{\infty} G_k y_k'^{(N)}\right\| = 1$ and $\dim Z < \infty$, there exists a convergent subsequence $\{\sum\limits_{k=0}^{\infty} G_k y_k'^{(N_\ell)}\}_{\ell=0}^\infty$ of $\{\sum\limits_{k=0}^{\infty} G_k y_k'^{(N)}\}_{N=0}^\infty$ such taht

$$\sum_{k=0}^{\infty} G_k y_k'^{(N_\ell)} \to z' \neq 0 \ (\ell \to +\infty).$$

But

$$\begin{aligned}\sum_{k=0}^{\infty} \|y_k'^{(N)}\|^2 &= (\sum_{m_N}^{m_N+\ell_N} \|y_k\|^2) \Big/ \|\sum_{m_N}^{m_N+\ell_N} G_k y_k\|^2 \\ &\leq (\sum_{m_N}^{m_N+\ell_N} \|y_k\|^2)/\delta_0^2 \to 0 (N \to +\infty),\end{aligned}$$

so

$$\sum_{k=0}^{\infty} G_k y_k'^{(N_\ell)} + \sum_{k=0}^{\infty} V_m^k y_k'^{(N_\ell)} \to z' \neq 0,$$

which is impossible.

Similar to the above proof, we can show $\sum\limits_{k=0}^{\infty} G_k y_k^{(n)} \to 0$ for any $\{y_k^{(n)}\}_0^\infty \to 0$, which means that $\bigvee\limits_{n=0}^{\infty} V^n L$ is non-degenerate.

If V is of regular Wold decomposition, then

$$V = \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{matrix} H_0 \\ H_1 \end{matrix},$$

where H_0 and H_1 are Pontryagin spaces, V_0 is a unitary on H_0, and V_1 is a unilateral shift on H_1. By the generalized triangle models of V_0 and V_1, we obtain

$$V=\begin{bmatrix}\begin{bmatrix} S_0 & A_0 & B_0 & C_0\\ & U_{\ell_0} & & E_0\\ & & V_{m_0} & F_0\\ & & & S_0^{*-1}\end{bmatrix}\begin{matrix} Z_0\\ \Pi_{\ell_0}\\ \Pi_{m_0}\\ Z_0^*\end{matrix} & \\ & \begin{bmatrix} S_1 & A_1 & B_1 & C_1\\ & U_{\ell_1} & & E_1\\ & & V_{m_1} & F_1\\ & & & S_1^{*-1}\end{bmatrix}\begin{matrix} Z_1\\ \Pi_{\ell_1}\\ \Pi_{m_1}\\ Z_1^*\end{matrix}\end{bmatrix}$$

$$=\begin{array}{c}\begin{bmatrix}\begin{bmatrix} S_0 & \\ & S_1\end{bmatrix} & \begin{bmatrix} A_0 & \\ & A_1\end{bmatrix} & \begin{bmatrix} B_0 & \\ & B_1\end{bmatrix} & \begin{bmatrix} C_0 & \\ & C_1\end{bmatrix}\\ & \begin{bmatrix} U_{\ell_0} & \\ & U_{\ell_1}\end{bmatrix} & & \begin{bmatrix} E_0 & \\ & E_1\end{bmatrix}\\ & & \begin{bmatrix} V_{m_0} & \\ & V_{m_1}\end{bmatrix} & \begin{bmatrix} F_0 & \\ & F_1\end{bmatrix}\\ & & & \begin{bmatrix} S_0^{*-2} & \\ & S_1^{*-2}\end{bmatrix}\end{bmatrix}\end{array}\begin{matrix} Z_0\\ Z_1\\ \Pi_{\ell_0}\\ \Pi_{\ell_1}\\ \Pi_{m_0}\\ \Pi_{m_1}\\ Z_0^*\\ Z_1^*\end{matrix}.$$

Because $V\Pi_K$ contains a maximal semi-negative subspace, $z_0=0$, $\Pi_{\ell_0}=0$, and Π_{m_0} is a Hilbert space. The corresponding decomposition of operator is

$$V=\begin{bmatrix} V_{m_0} & & & & \\ & S_1 & A_1 & B_1 & C_1\\ & & U_{\ell_1} & & E_1\\ & & & V_{m_1} & F_1\\ & & & & S_1^{*-1}\end{bmatrix}\begin{matrix}\Pi_{m_0}\\ Z_1\\ \Pi_{\ell_1}\\ \Pi_{m_1}\\ Z_1^*\end{matrix}.$$

Thus all conclusions of Theroem 3.1 are proved.

Corollary 3.1. *There is no pure unilateral shift on any Pontryagin space Π_K, $0<k<+\infty$.*

Theorem 3.2. *Suppose that V is an isometry on Π_K. Then, the necessary and sufficient condition for V to be of regular Wold decomposition is that $\|P_n\|$, $n=1,2,\ldots$, are uniformly bounded, where P_n, $n=1,2,\ldots$, are projections from Π_K onto $\bigcap_{m=1}^{n} V^m\Pi_K$, $n=1,2,\ldots$.*

Proof. This theorem is only a direct corollary of Theorem 4.5 in [4].

Remark 3.1. Using the structure of the proof of Theorem 3.1, we can express the projections $P_n=1,2,\ldots$ as follows

$$P_n=\begin{bmatrix} I & A_n & B_n & C_n\\ & I & & E_n\\ & & (I-P_{(V_p^n p)^\perp}) & F_n\\ & & & I\end{bmatrix},$$

where $A_n=E_n=0$, $C_n=S^nF_n^*P_{(V_p^n p)^\perp}F_nS^{*n}$, and

$$F_n=(V_p^{n-1}F+V_p^{n-2}FS^{n-2}FS^{*-1}+\cdots+FS^{*(n-1)})S^{*-n}P_{(V_p^n p)^\perp},\quad n=1,2,\ldots.$$

Below, construct an example to show that regular Wold decomposition does not hold for some isometric operators on Π_K.

Example 3.1. Set $\Pi_\ell=\ell^2$ and $(x,y)=-x_0\bar{y}_0+\sum_{k=0}^{\infty}x_n\bar{y}_n$, where $x=(x_0,x_1,\cdots),y=$

$(y_0, y_1, \cdots)$. Let

$$
\begin{aligned}
Z =&\operatorname{span}\{\frac{1}{\sqrt{2}}(e_0+e_1)\},\\
Z^* =&\operatorname{span}\{\frac{1}{\sqrt{2}}(e_0-e_1)\},\\
P =&\operatorname{span}\{e_n|n=2,3,\cdots\},
\end{aligned}
$$

where $e_n=(0,\cdots,0,1,0,\cdots)$.

Obviously, $\Pi_\ell = P\oplus\{Z\dot{+}Z^*\}$. For any $\lambda>0$, construct an isometry on Π_K as follows

$$
V_\lambda = \begin{bmatrix} \lambda & B & C \\ & V_P & F \\ & & 1/\lambda \end{bmatrix},
$$

where $V_Pe_n=e_{n+1},\ n=2,3,\cdots$ (clearly $(V_PP)^\perp=\{e_2\}$), $F((e_0-e_1)/\sqrt{2})=e_2$ (which implies $F^*(e_2)=(e_0+e_1)/\sqrt{2}$ and $F^*(e_k)=0, k\neq 2$). Consequentially,

$$
\begin{aligned}
G_ne_2 =&\sum_{k=0}^{n} S^{k+1}F^*V_P^{n-k}e_2\\
=&S^{n+1}F^*e_2=\lambda^{n+1}(e_0+e_1)/\sqrt{2}
\end{aligned}
$$

and $G_ne_k=0,\ k\neq 2$. Hence, the necessary and sufficient condition for $(G_0,G_1,\cdots,G_n,\cdots)$ to be bounded is $\lambda\leq 1$.

We can also use the method of structure in the proof of theorem 3.1 to clarify $\bigcap\limits_{n=0}^{\infty}V^nL$. We have

$$
\begin{aligned}
L =&\operatorname{span}\{-\lambda(e_0+e_1)/\sqrt{2}+e_2\},\\
V_\lambda^nL =&\operatorname{span}\{-\lambda^{n+1}(e_0+e_1)/\sqrt{2}+e_{n+2}\},
\end{aligned}
$$

$$
\bigvee_{n=0}^{\infty}V_\lambda^nL=\operatorname{span}\{(-\sum_{k=0}^{\infty}\lambda^{k+1}a_k)(e_0+e_1)/\sqrt{2}+\sum_{k=0}^{\infty}a_ke_{k+2}|\sum_{0}^{\infty}|a_k|^2<\infty\}.
$$

It is clear that $\sum\limits_{k=0}^{\infty}|\lambda^{k+1}||a_k^{(n)}|\to 0$, when $\lambda\leq 1$ and $\sum\limits_{k=0}^{\infty}|a_k^{(n)}|^2\to 0\,(n\to+\infty)$, which shows that $\bigvee\limits_{n=0}^{\infty}V^nL$ is non-degenerate.

When $\lambda>1$, for example $\lambda=2$, let

$$
(a_0^{(n)},a_1^{(n)},\cdots,a_k^{(n)},\cdots)=(0,\cdots,0,\overset{(n)}{1/2^{n+1}},0,\cdots)\to 0.
$$

But $\sum\limits_{k=0}^{\infty}\lambda^{k+1}a_k^{(n)}=1$, which implies $\bigvee\limits_{n=0}^{\infty}V^nL=Z\oplus(\oplus V_P^nP)$.

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