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A FREE BOUNDARY PROBLEM FOR ONE-DIMENSIONAL EQUATIONS OF A VISCOUS GAS

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Abstract

Utilizing the unique solution $(w(s), z(s))$ to the singular nonlinear two-point boundary value problem (1.11), the authors construct a unique self-similar solution $(\varphi(t), v(x,t), u(x,t)) = (Yt, v(\frac{x}{t}), u(\frac{x}{t}))$ to the free boundary problem (1.1)-(1.6), in which (1.1) and (1.2) are one-dimensional equations of a viscous gas. The arguments are elementary which involve only the use of the shooting method and the integral representations for $(w(s), z(s))$.

Keywords Equations of a viscous gas, Free boundary problem, Self-similar solution, Shooting method.
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§1. Introduction and Main Results

In this paper we consider a free boundary problem of the form

$$v_t - u_x = 0 \text{ in } D, \tag{1.1}$$

$$u_t + (p(v))_x = (t^N k(v)|u_x|^{N-1}u_x)_x \text{ in } D, \tag{1.2}$$

$$v(x,0) = B > A \text{ for } x > \varphi(0) = 0, \tag{1.3}$$

$$u(x,0) = 0 \text{ for } x > \varphi(0) = 0, \tag{1.4}$$

$$v|_{x=\varphi(t)} = A \text{ for } t > 0, \tag{1.5}$$

$$g_N(M, -t^N k(v)|u_x|^{N-1}u_x, \alpha)|_{x=\varphi(t)} = (\varphi'(t))^2 \text{ for } t > 0, \tag{1.6}$$

in which $D := \{(x,t);\, x > \varphi(t),\, t > 0\}$, $\varphi(t)$ is unknown a priori and must be determined as part of the solution,

$$g_N(M,\beta,\alpha) = \begin{cases} M - \dfrac{2N}{N-1}k^{1/N}(A)\beta^{(N-1)N} - \alpha\beta, & \text{if } N < 1, \\ M - 2k(A)\log\beta - \alpha\beta, & \text{if } N = 1, \\ M_+ - \dfrac{2N}{N-1}k^{1/N}(A)\beta^{(N-1)N} - \alpha\beta, & \text{if } N > 1, \end{cases} \tag{1.7}$$

and $M_+ = \max\{M, 0\}$. The system (1.1)–(1.2) represents the one-dimensional flow of a viscous gas in the Lagrange coordinates x and t. In this system, $v, u, p(v)$, and $k(v)$ denote the specific volume, the velosity, the pressure, and what we call the coefficient of viscosity of

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the gas, respectively, and the right hand side of (1.2) stands for the Von Neumann-Richtmyer viscosity (see [1, p. 232]).

Here we make the following hypotheses:

(H_1) $p(v) > 0$, $k(v) > 0$, and $p'(v)$ are all continuous functions defined in $(0, +\infty)$; moreover, $p(A) - p(s) \geq 0$ for all $s \in [A, B]$.

(H_2) $N > 0$, and $\alpha \geq 0$, M is an arbitrary real number when $N \leq 1$ and $M \geq \frac{p(A)-p(B)}{(B-A)}$ when $N > 1$.

Obviously, the hypothesis (H_1) is satisfied by an ideal polytropic gas where

$$p(v) = Pv^{-\gamma},\ P > 0,\ \gamma > 0;\ k(v) = Kv^{-\delta},\ K > 0,\ \delta > 0,$$

and by any non-ideal viscous gas. Problems on motion of a compressible viscous fluid (usually, gas) have been studied by a number of authors. For details, see [2, 3, 4] and their references. However, problems such as the free boundary problem (1.1)–(1.6) have not been, to our knowledge, condsidered before.

Our main result is as follows.

Theorem 1.1. *The free boundary problem* (1.1)–(1.6) *has a unique self-similar solution of the fom*

$$(\varphi(t), v(x,t), u(x,t)) = (Y_A t, v(y), u(y)),\ y = \frac{x}{t}, \tag{1.8}$$

where the pair $(Y_A, v(y))$ *is a unique solution to the free boundary problem for an ordinary differential equation*

$$\left.\begin{aligned} &(k(v)|yv'|^{N-1}yv')' = -y^2v' - p'(v)v',\quad y > Y_A,\\ &v(+\infty) = B > A,\ v(Y_A) = A > 0,\\ &g_N(M, k(v)|yv'|^{N-1}yv',\ \alpha)|_{y=Y_A} = Y_A^2, \end{aligned}\right\} \tag{1.9}$$

and the function $u(y)$ *is defined by*

$$u(y) := \int_y^{+\infty} sv'(s)ds \quad for \quad y \geq Y_A. \tag{1.10}$$

Moreover, $Y_A > 0$, *both* $v(y)$ *and* $-u(y)$ *are strictly increasing in* $y \geq Y_A$ *if* $N \leq 1$; *if* $N > 1$ *there is a number* $Y_B > Y_A$ *such that* $v(y)$ *and* $-u(y)$ *are both strictly increasing in* $y \in [Y_A, Y_B]$ *and* $v(y) = B$, $u(y) = 0$ *for* $y \geq Y_B$. *In order to express explicitly the dependence of the unique self-similar solution upon* M *and* α, *we denote it by* $(Y_A(M,\alpha)t, v(y; M.\alpha), u(y; M, \alpha))$, $y = \frac{x}{t}$. *Then* $Y_A(M,\alpha)$, $-v(y; M, \alpha)$, *and* $u(y; M, \alpha)$ *are all increasing in* M *and decreasing in* α.

To establish the above-mentioned theorem, we need to explore the two-point boundary value problem

$$\left.\begin{aligned} &w'(s) = -p'(s) - z^2(s),\ s \in [A, B),\\ &z'(s) = \left(\frac{k(s)}{w(s)}\right)^{1/N} z(s),\ s \in [A, B),\\ &z(A) = \sqrt{g_N(M, w(A), \alpha)},\ w(B) = 0. \end{aligned}\right\} \tag{1.11}$$

In section 2 we shall prove that a somewhat more general two-point boundary problem has a unique solution, employing the shooting method, and exhaust the continuous dependence of the solution upon M and α. In the last section we shall construct a self-similar solution to the free boundary problem (1.1)–(1.6), utilizing the unique solution to the two-point boundary value problem (1.11).

§2. Two-Point Boundary Value Problem

In this section we consider a two-point boundary value problem of the form

$$w'(s) = -p'(s) - z^2(s), \ s \in [A, B), \tag{2.1}$$

$$z'(s) = f(s, w(s))z(s), \ s \in [A, B), \tag{2.2}$$

$$z(A) = \sqrt{g(M, w(A), \alpha)}, \ w(B) = 0, \tag{2.3}$$

where the following hypotheses are adopted:

(H_3) $f(s, w)$ is a positive continous function defined in $[A, B) \times (0, +\infty)$ such that it is (strictly) decreasing and locally Lipschitz continuous in w, and $p(s)$ satisfies hypothesis (H_1).

(H_4) $g(M, \beta, \alpha)$ is a continuous function defined in $(-\infty, +\infty) \times (0, +\infty) \times [0, +\infty)$ such that it is increasing in M for each fixed β and α, and it is strictly decreasing in β and in α, respectively, when the other two variable are fixed. Moreover $\lim_{\beta \downarrow 0} g(M, \beta, \alpha) \geq \frac{p(A)-p(B)}{B-A}$ for some M and each fixed α, and there is a positive number β_0 such that $g(M, \beta_0, \alpha) = 0$.

Obviously, the two-point boundary value problem (1.11) is a particular case of the problem (2.1)–(2.3), since in the particular case the functions $f(s, w) = \left(\frac{k(s)}{w}\right)^{1/N}$ and $g(M, \beta, \alpha) = g_N(M, \beta, \alpha)$ satisfy the hypotheses (H_3) and (H_4), respectively.

Lemma 2.1. *For each fixed $S \in [A, B]$, $W > 0$, and $Z \geq 0$, the initial value problem*

$$w'(s) = -p'(s) - z^2(s), \ s \in [A, B), \tag{2.1}$$

$$z'(s) = f(s, w(s))z(s), \ s \in [A, B), \tag{2.2}$$

$$w|_{s=S} = W, \quad z|_{s=S} = Z \tag{2.3S}$$

has a unique solution $(w(s), z(s))$, which can be represented by

$$w(s) = W + p(S) - p(s) - \int_S^s z^2(t)dt, \tag{2.4}$$

$$z(s) = Z \cdot \exp\left\{\int_S^s f(t, w(t))dt\right\}, \tag{2.5}$$

and depends continuously on S, W, Z. If the maximal interval of existence for the solution is denoted by (S_1, S_2), then either $S_1 = A (S_2 = B)$ or $\lim_{s \downarrow S_1} w(s) = 0 \, (\lim_{s \uparrow S_2} w(s) = 0)$.

Proof. The proof can be found in book [5].

Lemma 2.2. *Let $(w(s; \beta), z(s; \beta))$ be a (unique) solution to the initial value problem*

$$w'(s) = -p'(s) - z^2(s), \ s \in [A, B), \tag{2.1}$$

$$z'(s) = f(s, w(s))z(s), \ s \in [A, B), \tag{2.2}$$

$$w|_{s=A} = \beta > 0, \quad z|_{s=A} = \sqrt{g(M, \beta, \alpha)}. \tag{2.3A}$$

If $\beta_1 > \beta_2 > 0$, then $w(s; \beta_1) > w(s; \beta_2)$ and $z(s; \beta_1) < z(s, \beta_2)$ for all s in the maximal interval of existence for the solution $(w(s; \beta_2), z(s; \beta_2))$.

Proof. If the first assertion is not true, then there will be a point $s = S$ in the maximal interval of existence for the solution $(w(s; \beta_2), z(s; \beta_2))$ such that $w(s; \beta_1) > w(s; \beta_2)$ in

$[A, S)$ and $w(S;\beta_1) = w(S;\beta_2)$, since $w(A;\beta_1) = \beta_1 > \beta_2 = w(A;\beta_2)$. Whence it follows by (2.4) and (2.5) that

$$z(s;\beta_1) = \sqrt{g(M;\beta_1,\alpha)} \cdot \exp\left\{\int_A^s f(t,w(t;\beta_1))dt\right\}$$
$$< \sqrt{g(M;\beta_2,\alpha)} \cdot \exp\left\{\int_A^s f(t,w(t;\beta_2))dt\right\} = z(s;\beta_2) \qquad (2.6)$$

for all $[A, S)$ (since $g(M,\beta_1,\alpha) < g(M,\beta_2,\alpha)$), and

$$0 < \beta_1 - \beta_2 = \int_A^S [z^2(s;\beta_1) - z^2(s;\beta_2)]ds \le 0.$$

This is absurd and hence the first assertion is true.

The second assertion is an immediate consequence of the first one, as has been pointed out by (2.6).

Lemma 2.3. *The two-point boundary value problem* (2.1)–(2.3) *has a positive solution, denoted by* $(w(s;M,\alpha), z(s;M,\alpha))$, *under the hypotheses* (H_3) *and* (H_4).

Proof. Define the set $E := \{\beta > 0;\ w(B,\beta) > 0\}$. Let β_0 be a positive number such that $g(M,\beta_0,\alpha) = 0$. Then $\beta_0 \in E$, i.e., E is nonempty, since, according to (2.4) and (2.5), $z(s;\beta_0) \equiv 0$ and $w(s;\beta_0) = p(A) - p(s) + \beta_0 > 0$ for $s \in [A, B]$. Now we claim that $\beta^* := \inf E > 0$. If not so, then $\beta^* = 0$ and hence $w(s;0) \ge 0$, by Lemma 2.1. Thus it follows by (2.4) and (2.5) that for all $s \in [A, B]$

$$z^2(s;0) > \frac{p(A) - p(B)}{B - A},$$

since $g(M,\beta,\alpha)$ satisfies (H_6) for some M and α, i.e.,

$$\lim_{\beta\downarrow 0} g(M,\beta,\alpha) \ge \frac{p(A) - p(B)}{B - A},$$

and hence

$$0 \le w(B;0) = p(A) - p(B) - \int_A^B z^2(s;0)ds < 0,$$

which is impossible.

we prove that the solution $(w(s;\beta^*), z(s;\beta^*))$ is a positive solution to the two-point boundary value problem (2.1)–(2.3). Clearly, it is enough to show that $w(B;\beta^*) = 0$. If $w(B;\beta^*) > 0$, then there will be a positive number $\beta < \beta^*$ such that $w(B;\beta) = \frac{1}{2}w(B;\beta^*)$, by Lemmas 2.1 and 2.2, i.e., $\beta \in E$, which contradicts the definition of β^*. This thus completes the proof the lemma.

Lemma 2.4. *Let both* $(w_1(s), z_1(s))$ *and* $(w_2(s), z_2(s))$ *be solutions defined on* $[a, b] \subset [A, B]$, *to the equations* (2.1) *and* (2.2). *If* $w_1(a) = w_2(a)$ *and* $w_1(b) = w_2(b)$, *then* $w_2(s) \equiv w_2(s)$ *for all* $s \in [a, b]$.

Proof. If this is not the case, we may without loss of generality assume that $w_1(s) < w_2(s)$ in (a, b). Whence it follows by (2.1) and (2.2) that

$$w_1'(a) \le w_2'(a),\ z_1(a) \le z_2(a),\ z_1(s) \ge z_2(s) \text{ in } (a, b),$$

and

$$0 > w_1(s) - w_2(s) = \int_s^b [z_1^2(t) - z_2^2(t)]dt \ge 0 \text{ in } (a, b),$$

which is absurd. The proof is complete.

Lemma 2.4 implies the uniqueness of the solution $(w(s;M,\alpha),z(s;M,\alpha))$ to the two-point boundary value problem (2.1)–(2.3).

Lemma 2.5. *If* $w(A;M_1,\alpha) > w(A;M_2,\alpha)$, *then*

(i) $w(s;M_1,\alpha) > w(s;M_2,\alpha)$ *for* $s \in [A,B)$,

(ii) $z(s;M_1,\alpha) > z(s;M_2,\alpha)$ *for* $s \in [A,B)$,

(iii) $M_1 > M_2$.

Proof. If the assertion (i) is not ture, then there will be a point $S \in (A,B)$ such that $w_1(S) = w_2(S)$ and $w_1(s) > w_2(s)$ in $[A,S)$, and hence $w_1(s) \equiv w_2(s)$ and $z_1(s) = z_2(s)$ in $[S,B)$, by Lemma 2.4, where $(w_j(s),z_j(s)) := (w(s;M_j,\alpha),z(s;M_j,\alpha))$, $j=1,2$. Whence it follows by Lemma 2.1 that $w_1(s) \equiv w_2(s)$ on $[A,B]$, which contradicts the assumption $w_1(A) > w_2(A)$. Thus, the assertion (i) is true.

(2.4) and (2.5) give

$$w_j(A) = p(B) - p(A) + g(M_j,w_j(A),\alpha)\int_A^B \exp\{2\int_A^s f(t,w_j(t))dt\}ds, \quad j=1,2.$$

Whence it follows by (i) that

$$\begin{aligned} &z_1^2(A) = g(M_1,w_1(A),\alpha) > g(M_2,w_2(A),\alpha) = z_2^2(A),\\ &g(M_1,w_1(A),\alpha) - g(M_2,w_1(A),\alpha)\\ >&g(M_2,w_2(A),\alpha) - g(M_2,w_1(A),\alpha) > 0 \end{aligned} \tag{2.7}$$

by the hypothesis (H_4) . This shows that $M_1 > M_2$, i.e., the assertion (iii) is valid.

If the assertion (ii) is false, then there will be a point $s = S \in (A,B)$ such that $z_1(S) = z_2(S)$ in $[A,S)$, by (2.7). Hence it follows by (i) and (2.5) that

$$\begin{aligned} z_2^2(s) - z_2^2(s) =&z_1^2(S)\left[\exp\left\{2\int_S^s f(t,w_1(t))dt\right\} - \exp\left\{2\int_S^s f(t,w_2(t))dt\right\}\right]\\ \le&0 \quad \text{for all} \;\; s \ge S, \end{aligned}$$

i.e., $-[w_1'(s) - w_2'(s)] \le 0$ for all $s \ge S$. Integrating the above over $[S,B]$ yeilds $w_1(S) - w_2(S) \le 0$, which contradicts the assertion (i). This shows that the assertion (ii) is true.

Lemma 2.6. *If* $M_1 \ge M_2$, *then*

(i)′ $w(s;M_1,\alpha) \ge w(s;M_2,\alpha)$ *for* $s \in [A,B)$,

(ii)′ $z(s;M_1,\alpha) \ge z(s;M_2,\alpha)$ *for* $s \in [A,B)$.

Proof. Using the reductio ad absurdum, we conclude that $w(A;M_1,\alpha) \ge w(A;M_2,\alpha)$, by Lemmas 2.4 and 2.5. Whence it follows again by Lemmas 2.4 and 2.5 that the assertion (i)′ and (ii)′ are true.

In the same way, we can prove the following two lemmas.

Lemma 2.7. *If* $w(A;M,\alpha_1) > w(A;M,\alpha_2)$, *then*

(iv) $w(s;M,\alpha_1) > w(s;M,\alpha_2)$ *for* $s \in [A,B)$,

(v) $z(s;M,\alpha_1) > z(s;M,\alpha_2)$ *for* $s \in [A,B)$,

(vi) $\alpha_1 < \alpha_2$.

Lemma 2.8. *If* $\alpha_1 \le \alpha_2$, *then*

(iv)' $w(s;M,\alpha_1) \ge w(s;M,\alpha_2)$ *for* $s \in [A,B)$,

(v)' $z(s;M,\alpha_1) \ge z(s;M,\alpha_2)$ *for* $s \in [A,B)$.

We can summarize the above results in the following statement.

Theorem 2.1. *Suppose that the hypotheses* (H_3) *and* (H_4) *hold. Then the two-point boundary value problem* (2.1)–(2.3) *has a unique positive solution* $(w(s;M,\alpha),\ z(s;M,\alpha))$. *Moreover, both* $w(s;M,\alpha)$ *and* $z(s;M,\alpha)$ *are increasing in* M *and decreasing in* α.

Finally, we prove the following statement.

Theorem 2.2. *Suppose further that* $f(s,w)=\left(\frac{k(s)}{w}\right)^{1/N}$. *Then* $z(B-0,M,\alpha)=+\infty$ *when* $N\le 1$ *and* $z(B-0;M,\alpha)<+\infty$ *when* $N>1$.

Proof. We first consider the case $N\le 1$. If $z(B-0;M,\alpha)<+\infty$, then $w'(B-0;M,\alpha)$ is finite, by virtue of (2.1). Choose a number θ such that $\theta>|w'(s;M,\alpha)|$ for all $s\in[A,B)$. Then $\theta(B-s)\ge w(s;M,\alpha)$ for all $s\in[A,B]$. Hence it follows by (2.5) that

$$\int_A^B\left(\frac{k(s)}{\theta(B-s)}\right)^{1/N}ds\le\int_A^B\left(\frac{k(s)}{w(s;M,\alpha)}\right)^{1/N}ds<+\infty.$$

This is impossible, because $k(s)>0$ on $[A,B]$ and $N\le 1$.

We now consider the case $N>1$. If $z(B-0;M,\alpha)=+\infty$, then $w'(B-0;M,\alpha)=-\infty$, and thus there exists a point $s=S\in[A,B)$ such that $w'(s;M,\alpha)+1<0$ in $[S,B)$, i.e., $w(s;M,\alpha)\ge B-s$ for all $s\ge S$.

$$\begin{aligned}z(s;M,\alpha)=&z(S;M,\alpha)\exp\left\{\int_S^s\left(\frac{k(s)}{w(s;M,\alpha)}\right)^{1/N}ds\right\}\\ \le&z(S;M,\alpha)\exp\left\{\int_S^B\left(\frac{k(s)}{(B-s)}\right)^{1/N}ds\right\}<+\infty,\end{aligned}$$

which contradicts the assertion $z(B-0;M,\alpha)=+\infty$.

Up to now the proof is complete.

§3. Proof of Theorem 1.1

In this section we contruct a self-similar solution to the free boundary problem (1.1)–(1.6), utilizing the unique positive solution to the two-point boundary value problem (1.11).

By a solution to the free boundary problem (1.1)–(1.6), we mean the triple $(\varphi(t),v(x,t),u(x,t))$ satisfying the following conditions:

(a) $\varphi(t)$ is a continuously differentiable function defined on $[0,+\infty)$ with $\varphi(0)=0$.

(b) both $v(x,t)$ and $u(x,t)$ are continuously differentiable functions defined in $\overline{D}\setminus\{(0,0)\}$ and $k(v(x,t))|u_x(x,t)|^{N-1}u_x(x,t)$ admits a continuous derivative with respect to x in D, and

(c) the triple itself satisfies (1.1)–(1.6).

Similarly, we call the pair $(Y_A,v(y))$ a solution to the free boundary problem (1.9), if it satisfies the conditions:

(a) Y_A is a positive number,

(b) $v(y)$ is an increasing, continuously differentiable function defined on $[Y_A,+\infty)$ and $k(v(y))|yv'(y)|^{N-1}yv'(y)$ admits a continuous derivative in $(Y_A,+\infty)$, and

(c) the pair itself satisfies (1.9).

Theorem 2.1 asserts that the under the hypotheses (H_1) and (H_2) the two-point boundary value problem

$$w'(s) = -p'(s) - z^2(s) \quad \text{for } s \in [A, B), \tag{3.1}$$

$$z'(s) = \left(\frac{k(s)}{w(s)}\right)^{1/N} z(s) \quad \text{for } s \in [A, B), \tag{3.2}$$

$$z(A) = \sqrt{g_N(M, w(A), \alpha)}, \quad w(B) = 0 \tag{3.3}$$

has a unique positive solution $(w(s; M, \alpha), z(s; M, \alpha))$, in which $z(s; M, \alpha)$ is strictly increasing in s and in M, and strictly decreasing in α. Consequently, the function $s = v(y; M, \alpha)$, inverse to $y = z(s; M, \alpha)$, exists in $(Y_A(M, \alpha), Y_B(M, \alpha))$, where $Y_A(M, \alpha) := z(A; M, \alpha)$ and $Y_B(M, \alpha) := z(B - 0; M, \alpha)$. Theorem 2.2 tells us that $Y_B(M, \alpha) = +\infty$ when $N \le 1$ and $Y_B(M, \alpha) < +\infty$ when $N > 1$. When $N > 1$, it is stipulated that $v(y; M, \alpha) = B$ for all $y \ge Y_B(M, \alpha)$. Clearly,

$$\begin{aligned} &v(Y_A(M, \alpha); M, \alpha) = A, \\ &\lim_{y \uparrow Y_B} v(y; M, \alpha) = B \quad \text{and} \quad \lim_{y \uparrow Y_B} v'(y; M, \alpha) = 0 \end{aligned} \tag{3.4}$$

since for all $y \in [Y_A(M, \alpha),\ Y_B(M, \alpha))$

$$y = z(v(y; M, \alpha); M, \alpha) \quad \text{and} \quad v'(y; M, \alpha) = \frac{1}{z'(s; M, \alpha); M, \alpha)} > 0. \tag{3.5}$$

This shows that $v(y; M, \alpha)$ is continuously differentiable on $[y_A(M, \alpha), +\infty)$ and strictly increasing in $y \le Y_B(M, \alpha)$.

Define

$$u(y; M, \alpha) = \int_{v(y;M,\alpha)}^{B} z(s; M, \alpha) ds \qquad (\le (B - A)^{1/2}(w(A) + p(A) - p(B))^{1/2}). \tag{3.6}$$

Then $-u(y; M, \alpha)$ is also continuously differentiable on $[Y_A(M, \alpha), +\infty)$ and strictly increasing in $y \le Y_B(M, \alpha)$, and $u(+\infty; M, \alpha) = 0$; when $N > 1$, $u(y; M, \alpha) = 0$ for all $y \ge Y_B(M, \alpha)$.

We now study the dependence of $Y_A(M, \alpha)$, $v(y; M, \alpha)$, $u(y; M, \alpha)$ upon M, α.

Clearly, $Y_A(M, \alpha) = z(A; M, \alpha)$ is strictly increasing in M and strictly decreasing in α.

If $M_1 > M_2$, then

$$\begin{aligned} &z(v(y; M_2, \alpha); M_2, \alpha) - z(v(y; M_1, \alpha); M_2, \alpha) \\ =&z(v(y; M_1, \alpha); M_1, \alpha) - z(v(y; M_1, \alpha); M_2, \alpha) > 0 \quad \text{for all} \quad y \ge Y_A(M, \alpha) \end{aligned}$$

by (3.5), i.e., $v(y; M_2, \alpha) > v(y; M_1, \alpha)$ for all $y \ge Y_A(M_1, \alpha)$, and hence,

$$\begin{aligned} &u(y; M_1, \alpha) - u(y; M_2, \alpha) \\ =&\int_{v(y;M_1,\alpha)}^{v(y;M_2,\alpha)} z(s; M_1, \alpha) ds + \int_{v(y;M_2,\alpha)}^{B} [z(s; M_1, \alpha) - z(s; M_2, \alpha)] ds \\ >&0 \quad \text{for all} \quad y \ge Y_A(M_1, \alpha). \end{aligned}$$

If $\alpha_1 > \alpha_2$, then

$$\begin{aligned} &z(v(y; M, \alpha_1); M, \alpha_1) - z(v(y; M, \alpha_2); M, \alpha_1) \\ =&z(v(y; M, \alpha_2); M, \alpha_2) - z(v(y; M, \alpha_2); M, \alpha_1) > 0 \quad \text{for all} \quad y \ge Y_A(M, \alpha_2) \end{aligned}$$

by (3.5) again, i.e., $v(y;M,\alpha_1) > v(y;M,\alpha_2)$ for $y \geq Y_A(M,\alpha)$, and hence

$$\begin{aligned} &u(y;M,\alpha_2) - u(y;M,\alpha_1) \\ &= \int_{v(y;M,\alpha_2)}^{v(y;M,\alpha_1)} z(s;M,\alpha_1)ds + \int_{v(y;M,\alpha_1)}^{B} [z(s;M,\alpha_2) - z(s;M,\alpha_1)]ds \\ &>0 \quad \text{for all} \quad y \geq Y_A(M,\alpha_2). \end{aligned}$$

Next, we prove that the pair $(Y_A, v(y)) := (Y_A(M,\alpha), v(y;M,\alpha))$ is a solution to the free boundary problem (1.9).

Substituting $s = v(y)$ into (3.1) and (3.2) yields

$$w'(v(y)) = -p'(v(y)) - y^2 \quad \text{for} \quad y \in [Y_A, Y_B), \tag{3.7}$$

$$w(v(y)) = k(v(y))|yv'(y)|^{1/N} yv'(y) \quad \text{for} \quad y \in [Y_A, Y_B), \tag{3.8}$$

and hence,

$$\begin{aligned} &(k(v(y))|yv'(y)|^{1/N} yv'(y))' = w'(v(y))v'(y) \\ &= -(y^2 + p'(v(y)))v'(y) \quad \text{for} \quad y \in [Y_A, Y_B); \end{aligned} \tag{3.9}$$

when $N > 1$, the above equations read all $0 = 0$ for all $y \geq Y_B$. The equations (3.8) and (3.9) show that the function $k(v(y))|yv'(y)|^{N-1} yv'(y)$ is continuously differentiable on $[Y_A, +\infty)$.

From (3.3), (3.7), and (3.8), it follows that

$$g_N(M, k(v(y))|yv'(y)|^{N-1} yv'(y), \alpha)|_{y=Y_A} = Y_A^2.$$

To sum up, the pair $(Y_A, v(y))$ is a solution to the free boundary problem (1.9).

Finally, let us define the triple given by

$$(\varphi(t), v(x,t), u(x,t)) = (Y_A t, v(\frac{x}{t}), u(\frac{x}{t})), \tag{3.10}$$

where $(Y_A, v(y))$ is a solution to the free boundary problem (1.9) and $u(y)$ is defined by (3.6) or (1.5). It is easy to verify that the triple (3.10) is a self-similar solution to the free boundary problem (1.1)–(1.6).

Up to now, the proof of Theorem 1.1 is complete. Evidently, Theorem 1.1 is still valid, if the function $g_N(M,\beta,\alpha)$ is replaced by the function $g(M,\beta,\alpha)$ which satisfies the hypothesis (H_4).

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