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BIFURCATION THEORY OF QUADRATIC DIFFERENTIAL SYSTEMS**

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Abstract

This paper deals with the number of limit cycles and bifurcation problem of quadratic differential systems. Under conditions $a < 0$, $b + 2l > 0$, $l + 1 < 0$, the author draws three bifurcation diagrams of the system (1.18) below in the (δ, m) plane, which show that the maximum number of limit cycles around a focus is two in this case.

Keywords Limit cycles, Quadratic differential systems, Bifurcation theory.
1991 MR Subject Classification 34C05.

In this paper we study firstly the number of limit cycles (LC, for abbreviation) around a weak focus $O(0,0)$ of the system:

$$\dot{x} = -y + lx^2 + mxy + y^2, \quad \dot{y} = x(1 + ax + by), \tag{0.1}$$

where $mab \neq 0$, and then the same problem for the system (1.18) below, where a term δx is added to the right hand side of the first equation of (0.1), and draw the bifurcation diagrams under certain conditions.

§1.

It is well-known that when $m = 0$ but $b + 2l \neq 0$, $O(0,0)$ is a weak focus of (0.1), $N(0,1)$ is a weak focus or saddle, and there is no LC. If $m = b + 2l = 0$, then O is a center, and the system (0.1) is integrable; if N is of index $+1$, it is a center, too.

In the following we will assume $am(b + 2l)(l + 1) \neq 0$ and investigate the generation and the number of LC's of (0.1) chiefly around O. Without loss of generality, we may assume $a < 0$. Since the first focal quantity of (0.1) at O is

$$W_1 = m(l + 1) - a(b + 2l), \tag{1.1}$$

we can always let m increase or decrease from zero to

$$m^* = a(b + 2l)/(l + 1) \tag{1.2}$$

so that $W_1 = 0$. In order to determine the sign of m^*, the number of critical points on $1 + ax + by = 0$ and the behavior of the line of divergence:

$$\operatorname{div}(P, Q) = (2l + b)x + my = 0, \tag{1.3}$$

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we have to distinguish the following four cases:

1. $b+2l>0$, $l+1<0$. This leads to

$$l<-1, \quad b>-2l>2 \quad \text{and} \quad m^*>0. \tag{1.4}$$

2. $b+2l<0$, $l+1<0$. There are two sub-cases:

$$\text{i) } l<0,\ b<0 \quad \text{and} \quad \text{ii) } l<0,\ b\geq 0, \tag{1.4'}$$

we have now $m^*<0$.

3. $b+2l>0$, $l+1>0$. There are three sub-cases:

$$\text{i) } b>0,\ l\geq 0; \quad \text{ii) } b\leq 0, l>0 \quad \text{and} \quad \text{iii) } b>0,\ l<0. \tag{1.4''}$$

4. $b+2l<0$, $l+1>0$. There are also three sub-cases:

$$\text{i) } b\geq 0,\ l<0; \quad \text{ii) } b<0, l<0 \quad \text{and} \quad \text{iii) } b<0,\ l\geq 0. \tag{1.4'''}$$

We will discuss the bifurcation problem in this paper only under condition (1.4); the discussion about other cases will be studied later on. The above program shows that we here use three of the coefficients: a, b, l as classifying variables, and then for each set of fixed values of a, b, l, we will try to study the bifurcation problem in the (δ, m) parameter plane.

When m increases from zero and passes through m^*, O changes from unstable focus to stable focus; hence an unstable LC generates from O or a stable LC disappears at O, this depends on $W_2>$ or <0, i.e., $(1+l)^2(1+b)-a^2(b+2l+1)>0$ or <0. Here W_2 is the second focal quantity of (0.1) at O:

$$W_2=ma(5a-m)[(n+l)^2(n+b)-a^2(b+2l+n)], \quad (\text{here } m>0, a<0, n=1).$$

Now, the x-coordinates of the critical points $S_i(x_i,y_i)$ on the line $1+ax+by=0$ satisfy the equation:

$$F(x)=(b^2l+a^2-abm)x^2+(ab+2a-bm)x+1+b=0. \tag{1.5}$$

Since $1+b>0$ and $F(-1/a)=b^2l/a^2<0$, we have:

1) If $b^2+a^2-abm>0$, then $0<x_1<-1/a<x_2$. From Fig.1 we know that O and S_2 are anti-saddles, N and S_1 are saddles. Solving (1.5) gives:

$$x_i=[bm-2a-ab\pm b\sqrt{(a+m)^2-4l(1+b)}]/2(b^2l+a^2-abm). \tag{1.6}$$

Since $(a+m)^2-4l(1+b)>0$, it is impossible to get $x_1=x_2$. Actually, from the figure of $F(x)$ in Fig.2 and (1.6) we see that $x_2\downarrow -1/a$, $x_1\downarrow 0$ only when $m\to+\infty$. At the limiting case $F(x)=0$ decomposes into two vertical lines $x=0$ and $1+ax=0$.

2) If $b^2l+a^2-abm<0$, then $x_2<0<x_1<-1/a$, and we have Figs.3 and 4. From Fig.3 we see that S_1, S_2 and N are all saddles.

Let

$$\widetilde{m}=(b^2l+a^2)/ab. \tag{1.7}$$

$\widetilde{m}$ may be ≥ 0 or <0; when $-l>>1$ ($<<1$), it is positive (negative). When $m<\widetilde{m}$ ($>\widetilde{m}$), S_2 lies on the left (right) hand side of the y-axis; when $m=\widetilde{m}$, S_2 goes to infinity.

The divergence of (0.1) at S_1 is

$$\operatorname{div}(P,Q)|_{S_1}=(2l+b)x_1+my_1=[(b^2+2lb-ma)x_1-m]/b. \tag{1.8}$$

Let $\overline{m}_1$ be the value of m such that $S_1(x_1, y_1)$ lies on div$=0$. Since the intersection point of div$=0$ and $1+ax+by=0$ is

$$(m/[b(2l+b)-ma], \quad -(2l+b)/[b(2l+b)-ma]), \tag{1.9}$$

$\overline{m}_1/[b(2l+b)-\overline{m}_1 a]$ must be a root of $F(x)=0$. This gives:

$$(l+b)\overline{m}_1^2 + a(b+2l)\overline{m}_1 - (1+b)(b+2l)^2 = 0; \tag{1.10}$$

we have to take

$$\overline{m}_1 = (b+2l)(-a+\sqrt{a^2+4(1+b)(l+b)})/2(l+b) > 0. \tag{1.11}$$

From (1.6) and (1.8) we see that the values of x_1 and div$|_{S_1}$ both depend on m. We will now study the values of $\partial x_1/\partial m$ and $\partial \mathrm{div}_{S_1}/\partial m$ as m varies. First, differentiating (1.5) with respect to m gives:

$$\partial F/\partial m|_{x_1} = -abx_1^2 - bx_1 + [(b^2 l + a^2 - abm)2x_1 + (ab+2a-bm)]\partial x_1/\partial m = 0; \tag{1.12}$$

hence

$$\frac{\partial x_1}{\partial m} = \frac{bx_1(1+ax_1)}{2(b^2 l + a^2 - abm)x_1 + ab + 2a - bm} = \frac{bx_1(1+ax_1)}{\partial F/\partial x|_{x_1}}. \tag{1.13}$$

From Fig.2 we see that when $b^2 l + a^2 - abm > 0$, $(1+ax_1)\partial F/\partial x|_{x_1} < 0$, so

$$\partial x_1/\partial m < 0. \tag{1.14}$$

When $b^2 l + a^2 - abm < 0$, from Fig.4 we have $\partial F/\partial x|_{x_1} < 0$, so (1.14) still holds. Therefore, from Fig.1 we see that div$|_{S_1}$ changes sign only when S_1 moves leftward and passes through the line div$=0$.

Now, differentiating (1.8) with respect to m gives:

$$\frac{\partial \mathrm{div}|_{S_1}}{\partial m} = \frac{-(1+ax_1) + (b^2+2lb-ma)\partial x_1/\partial m}{b}. \tag{1.15}$$

Since $b^2 + 2lm - ma > 0$, $1+ax_1 > 0$ and $\partial x_1/\partial m < 0$, we have:

$$\partial \mathrm{div}|_{S_1}/\partial m < 0. \tag{1.16}$$

It is easy to prove that in Figs.1 and 3 trajectories cross the segment $\overline{NS_1}$ from right to left, and in Fig.3 trajectories cross the segment $\overline{S_1 S_2}$ from left to right. So in both figures separatrices surrounding O always come from (and go to) S_1.

According to the magnitudes of $\overline{m}_1$ and m^*, there are three different cases:

1. $\overline{m}_1 = m^*$, that is, m^* is a root of (1.10); this gives:

$$(1+b)(1+l)^2 - a^2(b+2l+1) = 0. \tag{1.17}$$

From the formulae of focal quantities of a focus $O(0,0)$ of (0.1) we have $W_1 = W_2 = W_3 = 0$. This means that $(0.1)_{m^*}$ has a center O, the two separatrices l_1 and l_2 passing through S_1 also coincide at $m = m^*$ and become a separatrix cycle, which is the boundary of the center region around O.

We strongly believe that (0.1) has no closed orbit around O for all other values of $m \neq m^*$, but we can not give a strict proof now. Assume this conjecture to be true, then the bifurcation diagram will be as shown in Fig.5.

2. $\overline{m}_1 > m^*$.

This means that as $m = m^*$, we have $W_1 = 0$, $W_2 > 0$. When m increases from m^* we have $W_1 < 0$, O becomes stable, an unstable LC Γ_1 generates from O, but still we have $\mathrm{div}|_{S_1} > 0$. Γ_1 expands with the increasing of m, it must become ultimately an inner unstable separatrix cycle passing through S_1 at $m_1' < \overline{m}_1$; and when $m > m_1'$ no LC around O exists. For, if $m_1' \geq \overline{m}_1$, then this inequality means: when $m = \overline{m}_1$, both O and S_1 are weak critical points, but Γ_1 still exists (maybe as a separatrix cycle). This is impossible by a well-known theorem (see [5], §15, Theorem 15.13). So there is at most one LC around O.

3. $\overline{m}_1 < m^*$.

This means that when m increases and O is still an unstable focus $\mathrm{div}|_{S_1}$ has already changed from positive to negative, l_1 and l_2 coincide and then change their relative position at a certain value of m, say m_1', with $\overline{m}_1 \leq m_1' < m^*$, and generate a stable LC Γ_1'. As m increases from m_1', Γ_1' contracts. Since at $m = m^*$ we have $W_1 = 0$, $W_2 < 0$ so $\Gamma_1' = 0$. When $m > m^*$, W_1 becomes negative, no semi-stable LC generates between O and S_1. So we have one LC Γ_1' around O for a certain interval (m_1', m^*) of m.

To sum up, we have the following diagrams for these three different cases (see Fig. 5–Fig. 8):

no LC	center	no LC
$W_1 > 0$	$m^* = \overline{m}_1$	$W_1 < 0$
0 unstable	$W_1 = W_2 = 0$	0 stable

Fig. 5

no LC		Γ_1 unstable		no LC
$W_1 > 0$	m^*	$W_1 < 0$	m_1'	$\overline{m}_1$
	$W_1 = 0$ $W_2 > 0$			

Fig. 7

no LC		Γ_1' unstable		no LC
$\overline{m}_1$	m_1'		m^*	$W_1 < 0$
$W_1 > 0$			$W_1 = 0$ $W_2 > 0$	

Fig. 8

There are two points in the above statement to which the reader will think to be questionable.

I. As m varies, system (0.1) does not form a family of rotated vector fields even at one side of the straight line $1 + ax + by = 0$, for

$$\partial\theta/\partial m = -x^2 y(1 + ax + by)/[P^2 + Q^2].$$

But from [1] we know: to add a term μxy to the right hand side of the first equation in (0.1) can be understood as: to add firstly the term $\mu x(y + 1/b)$ and then the term $-\mu x/b$; these applications of two different families of half plane rotated vector fields were denoted by F_3 and F_2 respectively in [1]. Therefore, even if LC is not monotonously expand or contract when m varies, its deformation can be obtained by a monotonous expansion in one manner and then a monotonous contraction in another manner. Hence the properties of the coincidence and change position of separatrices (and thus the generation of a LC) as well as the generation of a LC through the change of stability of the focus O keep unchanged.[1]

II. Does semi-stable LC appear suddenly between O and S_1 when m varies? This still remains open even under the variation of a parameter in a family of rotated vector fields. But from our experience we believe that the following Proposition holds true:

Proposition. *In the vector field defined by a quadratic system:*

$$\dot{x} = -y + \delta x + lx^2 + mxy + ny^2, \quad \dot{y} = x(1 + ax + by) \tag{1.18}$$

when any one of the independent coefficients varies in one direction, around any focus semi-stable LC may appear suddenly at most for one value of this coefficient; moreover, the number of semi-stable LC appeared is at most one.[2]

Remark 1.1. This Proposition does not prevent the appearance of three LC's, but it prevents the appearance of four or more LC's around the same focus. Thus, in Figs.7 and 8 the sudden appearance of a semi-stable LC for (0.1) when m varies between m^* and m_1' is impossible. Otherwise, semi-stable LC will appear two times around O when in (1.18) δ decreases from a certain positive value to zero.

Remark 1.2. This Proposition is a natural generalization of the following property of a real cubic curve (it must be a solution curve of a quadratic system) to the case of the intersection curve (i.e., LC) of the real plane with a certain complex solution of the complex quadratic system (0.1) when one of the independent coefficient varies in one direction:

When a cubic curve moves upward or downward, the number of contact points of this curve with a horizontal line which appears suddenly is at most one (See Fig.9), the number of times of sudden appearance is at most two (See Fig.10).

We conjecture that Fig.10 cooresponds to the appearance of LC's around two foci; it also suggests that as an independent coefficient varies in one direction, a (1,3) distribution of LC's may become a (3,1) distribution.

Remark 1.3. The assertion we made in this Proposition concerns with the general quadratic system (1.18). For the special case:

$$\dot{x} = -y + \delta x + lx^2 + mxy + ny^2, \quad \dot{y} = x \tag{1.19}$$

with $m(l + n) < 0$, $\delta > 0$, when δ increases from zero, the stable LC Γ generated from O expands until it becomes a finite or infinite separatrix cycle and then disappears. In the whole process semi-stable LC appears neither within nor outside Γ.

§2.

Now, let us consider system (1.18) under condition (1.4) and study the signs of $\partial x_1/\partial\delta$ and $\partial \mathrm{div}|_{S_1}/\partial\delta$ as δ varies. Notice that the x-coordinate of the saddle point $S_1(x_1, y_1)$ on $1 + ax + by = 0$ will satisfy the equation:

$$G(x) = (b^2l + a^2 - abm)x^2 + (ab + 2a - bm + \delta b^2)x + 1 + b = 0. \tag{2.1}$$

[1]The behavior of Γ_1 (or Γ_1') when $\delta = 0$ but m increases can also be explained by the increase of the stability of O (or the decrease of the value of $\mathrm{div}|_{S_1}$).

[2]By "an independent coefficient" we mean that in the space (a, b, δ, l, m) there is an open curve $a = a(\tau), \cdots, m = m(\tau)$, and as τ increases or decreases, $a(\tau), \cdots, m(\tau)$ all vary monotonously. Of course, some of the five variables may not depend on τ and remain fixed.

Since $G(-1/a) = b^2(l - a\delta)/a^2 < 0$ when $0 < |\delta| << 1$, and > 0 when $\delta >> 1$, the figures of $G(x)$ for δ small when $b^2l + a^2 - abm > 0$ and $b^2l + a^2 - abm < 0$ are shown in Figs.11 and 12 respectively; and those for δ large are shown in Figs.13 and 14, respectively.

It is easily seen that

$$\partial x_1/\partial\delta = -b^2x_1/\partial G/\partial x_1 > 0, \tag{2.2}$$

since $\partial G/\partial x_1 < 0$ in any case.

Now, $\mathrm{div}(P,Q) = \delta + (2l+b)x + my = 0$, so $\mathrm{div}|_{S_1} = [(b^2 + 2lb - ma)x_1 - m + b\delta]/b$, and

$$\partial\mathrm{div}/\partial\delta|_{S_1} = [(b^2 + 2lb - ma)\partial x_1/\partial\delta + 1]/b > 0, \tag{2.3}$$

since $b^2 + 2lb - ma > 0$.

For the x-coordinate of the intersection point of div$=0$ and $1 + ax + by = 0$: $S_1((m - b\delta)/(b^2 + 2bl - ma), (a\delta - b - 2l)/(b^2 + 2bl - ma))$ to be a root of $G(x) = 0$, δ must satisfy the equation:

$$\begin{aligned}&[a^2 - b^2(b+l)]\delta^2 + [2mb(b+l) - a(b+2l)(b+2)]\delta\\ &+(1+b)(b+2l)^2 - a(b+2l)m - (l+b)m^2 = 0.\end{aligned} \tag{2.4}$$

It is easy to prove that (2.4) is a hyperbola in the (δ, m) plane when $l + b > 0$. The upper branch of it passes through and has positive slope at the point $G(O, \overline{m}_1)$, since when $\delta = 0$ (2.4) reduces to (1.10). The two asymptotic directions of (2.4) are

$$\delta = [-b(b+l) \pm a\sqrt{b+l}]/[a^2 - b^2(b+l)]m = K_{1,2}m$$

with $K_1 > 0$ and $K_2 < 0$.

Now, similar to Fig.6 let us take the straight line in Fig.7 as the m-axis in the (δ, m) plane, and let δ increase or decrease from zero. Notice that if $O(0,0)$ is an unstable weak focus of (0.1) then an unstable LC Γ_2' will be generated from O as δ decreases, Γ_2' expands with the decreasing of δ; if $O(0,0)$ is a stable weak focus of (0.1) then a stable LC Γ_2 will be generated from O as δ increases, Γ_2 expands with the increasing of δ. On the other hand, Γ_1 contracts (expands) with the increasing (decreasing) of δ. Hence in the (δ, m) plane we will have a semi-stable LC bifurcation curve C_1 with its left end point $\bar{O}(0, m^*)$, which represents the coincidence of Γ_1 and Γ_2; and a separatrix bifurcation curve C_2 with its end point $B(0, m_1')$, which represents the fact that Γ_2 becomes a separatrix loop passing through S_1. C_1 and C_2 must meet at a point A on the hyperbola L (div$=0$). Otherwise, the two regions with different number of LC's will not be separated from each other. Corresponding to A, (1.18) has a separatrix loop passing through the weak saddle S_1. There are also bifurcation curves $C_3(C_4)$, which represent the fact that $\Gamma_1(\Gamma_2')$ becomes a separatrix loop passing through S_1. C_2 and C_4 both extend to infinity as $|m| \to +\infty$; maybe, they will have vertical asymptotics $\delta = 2$ and $\delta = -2$ since, when $|\delta| \geq 2$, (1.18) can have no LC. C_3 and C_4 joint together at a point on the horizontal line $m = m^*$. So we have the bifurcation diagram as shown in Fig.15.

Similarly, using the straight line in Fig.8 as the m-axis in the (δ, m) plane, we can get a bifurcation diagram as shown in Fig.16. In each figure we have a curvilinear triangle with two tails.

Finally we get the following

Theorem. *Under conditions*

$$a < 0, \quad b + 2l > 0 \quad \text{and} \quad l + 1 < 0 \tag{2.5}$$

the system (1.18) *has a bifurcation diagram* Fig.5 *when*

$$(1+l)^2(1+b) - a^2(b+2l+1) = 0; \tag{2.6}$$

a bifurcation diagram Fig.15 (Fig.16), *when*

$$(1+l)^2(1+b) - a^2(b+2l+1) > 0(< 0). \tag{2.7}$$

So (1.18) *can have at most* 2 *LC's.*

Notice that Fig.5 can be taken as the limiting case of Fig.15 or Fig.16 as $(1+l)^2(1+b) - a^2(b+2l+1) \to 0$. Moreover, the curvilinear triangle in the bifurcation diagram of [2] is similar to the curvilinear triangle $\bar{O}AB$ in Fig.16.

Remark 2.1. For another special case of (1.18):

$$\dot{x} = -y + \delta x + lx^2 + y^2, \quad \dot{y} = x(1+ax), \quad a > 0, \ l < 0 \tag{2.8}$$

we can use a Dulac function e^{-2ly} to prove that when $\delta = 0$ (2.8) has no closed orbit around $O(0,0)$, and O is unstable. If δ decreases from zero, an unstable LC Γ generates from O. Now, $N(0,1)$ is a fixed saddle point. The other two critical points are on $1 + ax = 0$: $R(-1/a, y_1 > 0)$ with index +1 and $S(-1/a, y_2 < 0)$ with index -1. They come nearer and nearer and finally disappear when $\delta < (4l - a^2)/4a$. By inspecting the direction of trajectories on the segment $\overline{NS}$, we see that separatrices comes from and goes to the neighbourhood of O passing through N. Since $\text{div}|_N = \delta < 0$, we conclude that: if no semi-stable LC appears suddenly outside Γ (another form of the Proposition) the number of LC's arounds O is at most two. Actually, for δ in a certain interval $\subset (l/a, 0)$ the number of LC's around O is two. For (2.8), the appearance of two LC's around O was first discovered in [3] for $a < 0$, $l \geq 2a^2$, $\delta < 0$; and rediscovered by [4] in the case of a Poincaré bifurcation when $|\delta|$, $|l|$ are small.

We have finished in this paper the analysis of the bifurcation problem only for one of the many (maybe more than 100) cases, and only around one focus O. Later we will try to find a certain case such that system (0.1) may have two LC's around O and at the same time, aside from foci there exists also finite saddle point, although this seems not an easy task.

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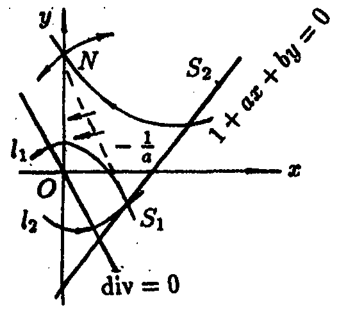


Fig. 1

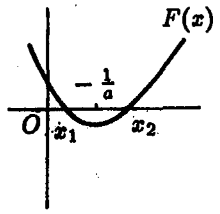


Fig. 2

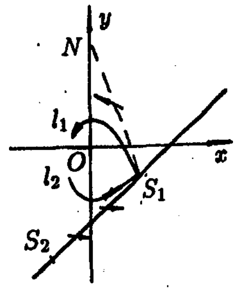


Fig. 3

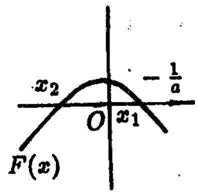


Fig. 4

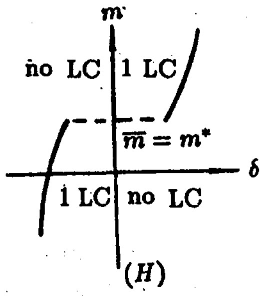


Fig. 6

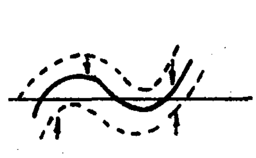


Fig. 9

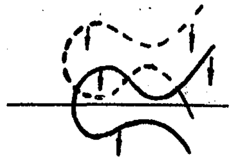
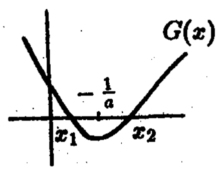
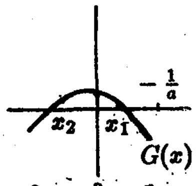


Fig. 10



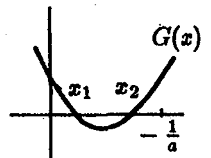
$b^2l + a^2 - abm > 0$

Fig. 11



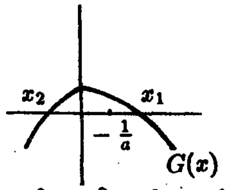
$b^2l + a^2 - abm < 0$

Fig. 12



$b^2l + a^2 - abm > 0$

Fig. 13



$b^2l + a^2 - abm < 0$

Fig. 14

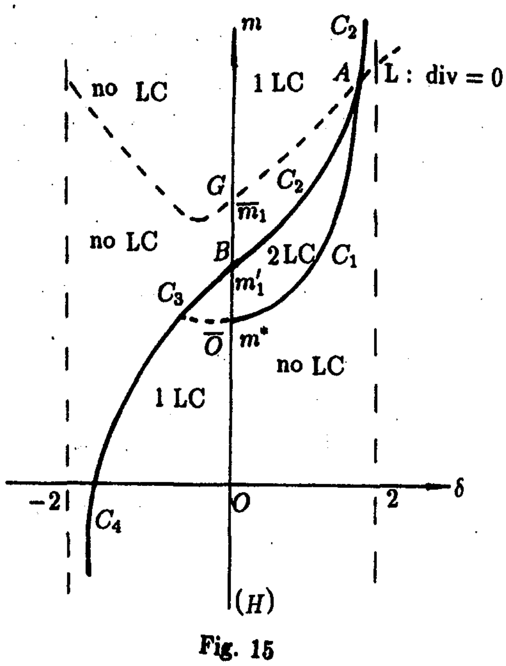


Fig. 15

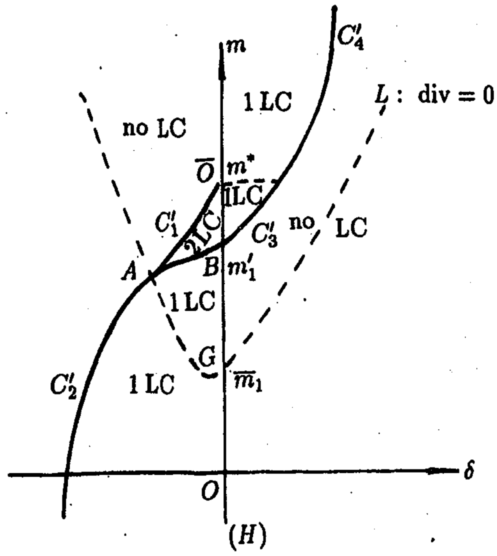


Fig. 16