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ON THE DIFFERENCE OF CONSECUTIVE EIGENVALUES OF UNIFORMLY ELLIPTIC OPERATORS OF HIGHER ORDERS**

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Abstract

This paper considers the upper bound for the difference of consecutive eigenvalues of a class of uniformly elliptic operators of higher orders. The upper bound of $\lambda_{n+1}-\lambda_n$ is dependent on the first $n-1$ eigenvalues and the coefficients in equations.

Keywords Uniformly elliptic operator, Eigenvalue, Eigenfunction.
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§1. Introduction

The estimates for the bound of $(n+1)$th eigenvalue as well as the difference of consecutive eigenvalues of the harmonic operator and its polynomials are well known (see [1-5]). These estimates are that the $(n+1)$th eigenvalue is bounded from above by an amount depending on the first n eigenvalues and being independent of the measure of the domain in which the problem is concerned. In this paper we generalize the same kind of problems to a certain uniformly elliptic operator of higher orders, and obtain the results similar to in form the ones for harmonic operator and its polynomials. The results in [1-5] are all corollaries of the theorems in this paper.

Suppose that Ω is a bounded domain in R^m $(m\geq 2)$ with the piecewise smooth boundary $\partial\Omega$ and $u(x)$ is a solution of the problem

$$(-1)^l\sum_{i_1,i_2,\cdots,i_l=1}^{m} D_{i_1\cdots i_l}(a_{i_1\cdots i_l}D_{i_1\cdots i_l}u)=\lambda u,\quad x\in\Omega,$$

$$u=\frac{\partial u}{\partial\nu}=\cdots=\frac{\partial^{l-1}u}{\partial\nu^{l-1}}=0,\quad l\geq 1,\quad x\in\partial\Omega,\tag{1.1}$$

where $\overrightarrow{\nu}$ is the unit outward normal to $\partial\Omega$. $a_{i_1i_2\cdots i_l}$ is a $C^l(\bar\Omega)$ function in x and satisfies the uniformly elliptic condition, i.e., there exists a constant $\mu>0$ such that

$$\min_{\bar\Omega} a_{i_1i_2\cdots i_l}\geq\mu,\quad i_1,i_2,\cdots,i_l=1,2,\cdots,m.\tag{1.2}$$

Our main result is as follows.

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Theorem 1.1. *Let λ_j be the eigenvalues of* (1.1). *Then we have*

$$\lambda_{n+1}-\lambda_n \le \frac{4l}{\mu m^2 n^2}\left[(2l+m-2)\max_{\bar\Omega} a_{i_1 i_2\cdots i_l}\left(\sum_{j=1}^n \lambda_j^{1-1/l}\right)+\right.$$
$$\left.+\,\mu^{1/2l}(l-1)\max_{\bar\Omega}|\nabla a_{i_1\cdots i_l}|\left(\sum_{j=1}^n \lambda_j^{1-3/2l}\right)\right]\left(\sum_{j=1}^n \lambda_j^{1/l}\right), \tag{1.3}$$

$$\lambda_{n+1}\le \frac{1}{\mu m^2}\left[\mu m^2+4l(2l+m-2)\max_{\bar\Omega} a_{i_1\cdots i_l}\right)\lambda_n+$$
$$+\,\frac{4}{\mu m^2}\mu^{1/2l}l(l-1)\max_{\bar\Omega}|\nabla a_{i_1\cdots i_l}|\lambda_n^{1-1/2l}. \tag{1.4}$$

Remark 1.1. If we take $a_{i_1\cdots i_l}=1$ in (1.1) and $\mu=1$ in (1.2), then $\max_{\bar\Omega} a_{i_1\cdots i_l}=1$ and $\max_{\bar\Omega}|\nabla a_{i_1 i_2\cdots i_l}|=0$. For this case, (1.3) yields

$$\lambda_{n+1}-\lambda_n \le \frac{4l}{m^2n^2}(2l+m-2)\left(\sum_{j=1}^n \lambda_j^{1-1/l}\right)\left(\sum_{j=1}^n \lambda_j^{1/l}\right), \tag{1.5}$$

which is just the result in [4] for polyharmonic operators. So the corresponding results in [1-5] are all included.

Remark 1.2. To see the sharpness of estimates (1.3) and (1.4), we consider, for simplicity, the following problem:

$$\begin{cases} -\Delta u=\lambda u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $\Omega\subset R^2$ is a unit square with $0<x_1<1$ and $0<x_2<1$. A straightforward calculation yields $\lambda_1=2\pi^2$, $\lambda_2=5\pi^2$, $\lambda_3=8\pi^2$ and so on. Therefore,

$$\lambda_2-\lambda_1=3\pi^2,\ \text{i.e.,}\ \lambda_2=2.5\lambda_1;\quad \lambda_3-\lambda_2=3\pi^2,\ \text{i.e.,}\ \lambda_3=\lambda_1+1.2\lambda_2.$$

In this case estimate (1.5) with $l=1$, $m=2$ and $n=1,2$ gives respectively

$$\lambda_2-\lambda_1\le 2\lambda_1(=4\pi^2),\ \text{i.e.,}\ \lambda_2\le 3\lambda_1;\quad \lambda_3-\lambda_2\le\lambda_1+\lambda_2(=7\pi^2),\ \text{i.e.,}\ \lambda_3\le\lambda_1+2\lambda_2.$$

It is usually very difficult to get the exact values of eigenvalues. But it is enough to obtain the bounds of eigenvalues in many cases.

§2. Proof of Theorem 1.1

In the sequel we will use the standard notations:

$$D_k=\frac{\partial}{\partial x_k},\quad k=1,2,\cdots,m,\quad \nabla=(D_1,D_2,\cdots,D_m),$$

$$D_{i_1 i_2\cdots i_l}=D_{i_1}D_{i_2}\cdots D_{i_l},\quad i_1,i_2,\cdots,i_l=1,2,\cdots,m,\ \text{and}\ D_{i_0}=D_{i_{l+1}}=I.$$

From (1.1) we know

$$\lambda_j=\sum_{i_1,i_2,\cdots,i_l=1}^m\int_\Omega a_{i_1 i_2\cdots i_l}\left|D_{i_1 i_2\cdots i_l}u_j\right|^2dx,\quad j=1,2,\cdots. \tag{2.1}$$

So we can order the eigenvalues of problem (1.1) so that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ with the corresponding eigenfunctions $u_1, u_2, \cdots, u_n, \cdots$ satisfying

$$\int_\Omega u_i u_j dx = \delta_{ij}, \quad i, j = 1, 2, \cdots. \tag{2.2}$$

By virtue of (1.2),

$$\sum_{i_1, i_2, \cdots, i_l = 1}^{m} \int_\Omega |D_{i_1 \cdots i_l} u_j|^2 \, dx \leq \lambda_j / \mu. \tag{2.3}$$

Let $\phi_{jk} (j = 1, 2, \cdots, n; k = 1, 2, \cdots, m)$ be the trial functions defined by

$$\phi_{jk} = \phi_{jk}(x) = x_k u_j - \sum_{p=1}^{n} b_{jp}^k u_p$$

where $b_{jp}^k = \int_\Omega x_k u_j u_p dx$, $j = 1, 2, \cdots, n$, $k = 1, 2, \cdots, m$. It is obvious that ϕ_{jk} are orthogonal to $u_1, u_2, \cdots, u_n$ and

$$\phi_{jk} = \frac{\partial \phi_{jk}}{\partial \nu} = \cdots = \frac{\partial^{l-1} \phi_{jk}}{\partial \nu^{l-1}} = 0, \quad x \in \partial\Omega. \tag{2.4}$$

Hence, we can use the well-known Rayleigh theorem to obtain

$$\lambda_{n+1} \leq \frac{(-1)^l \sum\limits_{i_1, \cdots, i_l = 1}^{m} \int_\Omega \phi_{jk} D_{i_1 \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} \phi_{jk}) dx}{\int_\Omega \phi_{jk}^2 dx}. \tag{2.5}$$

By a rather complicated calculus it yields

$$\begin{aligned} &(-1)^l \sum_{i_1, \cdots, i_l = 1}^{m} D_{i_1 \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} \phi_{jk}) \\ =& \lambda_j x_k u_j - \sum_{p=1}^{n} b_{jp}^k \lambda_p u_p + \sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \sum_{t=1}^{l} \delta_{k i_t} D_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} u_j) \\ &+ \sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \sum_{t=1}^{l} \delta_{k i_t} D_{i_1 \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_l} u_j), \end{aligned}$$

where

$$\delta_{k i_t} = \begin{cases} 1, & \text{when } i_t = k, \\ 0, & \text{when } i_t \neq k, \end{cases} \quad i_t = 1, 2, \cdots, m.$$

The orthogonality of ϕ_{jk} with $u_i (i = 1, 2, \cdots, n)$ gives the following result:

$$\begin{aligned} &\sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \int_\Omega \phi_{jk} D_{i_1 \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} \phi_{jk}) dx \\ =& \lambda_j \int_\Omega \phi_{jk}^2 dx + \sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \sum_{t=1}^{l} \delta_{k i_t} \int_\Omega \phi_{jk} D_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} u_j) dx \\ &+ \sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \sum_{t=1}^{l} \delta_{k i_t} \int_\Omega \phi_{jk} D_{i_1 \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_l} u_j) dx. \end{aligned} \tag{2.6}$$

For simplifying writing we define I_{jki_t}, J_{jki_t}, I and J as follows:

$$I_{jki_t} = \sum_{i_1, \cdots, i_l = 1}^{m} (-1)^l \delta_{k i_t} \int_\Omega \phi_{jk} D_{i_1 \cdots i_{t-1} i_{t+1} \cdots i_l} (a_{i_1 \cdots i_l} D_{i_1 \cdots i_l} u_j) dx,$$

$$J_{jki_t}=\sum_{i_1,\cdots,i_l=1}^{m}(-1)^l\delta_{ki_t}\int_\Omega\phi_{jk}D_{i_1\cdots i_l}(a_{i_1\cdots i_l}D_{i_1\cdots i_{t-1}i_{t+1}\cdots i_l}u_j)dx,$$

$$I=\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{t=1}^{l}I_{jki_t},\quad J=\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{t=1}^{l}J_{jki_t}.$$

Then, it follows from (2.6) that

$$\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i_1,\cdots i_l=1}^{m}(-1)^l\int_\Omega\phi_{jk}D_{i_1\cdots i_l}(a_{i_1\cdots i_l}D_{i_1\cdots i_l}\phi_{jk})dx$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}\lambda_j\int_\Omega\phi_{jk}^2dx+I+J. \tag{2.7}$$

By (2.5),

$$\lambda_{n+1}\left(\sum_{j=1}^{n}\sum_{k=1}^{m}\int_\Omega\phi_{jk}^2dx\right)\le\sum_{j=1}^{n}\sum_{k=1}^{m}\lambda_j\int_\Omega\phi_{jk}^2dx+I+J. \tag{2.8}$$

Replacing λ_j in (2.8) by λ_n yields

$$(\lambda_{n+1}-\lambda_n)\left(\sum_{j=1}^{n}\sum_{k=1}^{m}\int_\Omega\phi_{jk}^2dx\right)\le I+J. \tag{2.9}$$

Introducing an operator ∇^p such that

$$\nabla^p=\begin{cases}\Delta^{p/2}, & p=2k,\ k=1,2,\cdots,\\ \nabla(\Delta^{(p-1)/2}), & p=2k-1,\ k=1,2,\cdots,\end{cases}$$

we can easily prove, by the inductive method, that for an eigenfunction u_j of (1.1) it holds that

$$\sum_{i_1,\cdots,i_l=1}^{m}\int_\Omega|D_{i_1\cdots i_r}u_j|^2\,dx=\int_\Omega|\nabla^r u|^2\,dx,\quad 1\le r\le l \tag{2.10}$$

and

$$\left(\int_\Omega|\nabla^s u_j|^2\,dx\right)^{1/s}\le\left(\int_\Omega\left|\nabla^{s+1}u_j\right|^2dx\right)^{1/(s+1)},\quad 1\le s\le l. \tag{2.11}$$

(2.10) and (2.3) yield

$$\int_\Omega|\nabla^l u_j|^2dx\le\lambda_j/\mu. \tag{2.12}$$

Using (2.11) inductively and (2.12), we find

$$\int_\Omega|\nabla^s u_j|^2dx\le(\lambda_j/\mu)^{s/l},\quad 1\le s\le l. \tag{2.13}$$

Now, we have the following essential lemma which plays a key role in the proof of Theorem 1.1.

Lemma. *For $I+J$ the following estimate holds*

$$I+J \le l(2l+m-2)\max_{\bar\Omega} a_{i_1\cdots i_l}\sum_{j=1}^{n}(\lambda_j/\mu)^{1-1/l}+ \\ + l(l-1)\max_{\bar\Omega}|\nabla a_{i_1\cdots i_l}|\sum_{j=1}^{n}(\lambda_j/\mu)^{1-3/2l}.$$

Proof. In the sequel, we denote $i_1 i_2\cdots i_{t-1}i_{t+1}\cdots i_l$ and $i_1\cdots i_{t-1}i_{t+1}\cdots i_{s-1}i_{s+1}\cdots i_l$ by (i_t) and $(i_t i_s)$ respectively. Integrating by parts for I_{jki_t}, it yields

$$I_{jki_t} = -\sum_{i(l)=1}^{m}\delta_{ki_t}\int_\Omega x_k a_{i(l)}D_{(i_t)}u_j D_{i(l)}u_j dx - \sum_{i(l)=1}^{m}\sum_{\substack{s=1\\ s\neq t}}^{l}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{(i_t i_s)}u_j D_{i(l)}u_j dx \\ +\sum_{p=1}^{n}b_{jp}^{k}\sum_{i(l)=1}^{m}\delta_{ki_t}\int_\Omega a_{i(l)}D_{(i_t)}u_p D_{i(l)}u_j dx, \tag{2.14}$$

where $D_{i(l)}=D_{i_1\cdots i_l}$, $a_{i(l)}=a_{i_1\cdots i_l}$, and $\sum\limits_{i(l)=1}^{m} = \sum\limits_{i_1,\cdots,i_l=1}^{m}$.

Similarly,

$$J_{jki_t} = \sum_{i(l)=1}^{m}\delta_{ki_t}\int_\Omega x_k a_{i(l)}D_{i(l)}u_j D_{(i_t)}u_j dx + \sum_{i(l)=1}^{m}\sum_{s=1}^{l}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{(i_t)}u_j D_{(i_s)}u_j dx \\ -\sum_{i(l)=1}^{m}\sum_{p=1}^{n}b_{jp}^{k}\delta_{ki_t}\int_\Omega a_{i(l)}D_{i(l)}u_p D_{(i_t)}u_j dx. \tag{2.15}$$

Combining (2.14) with (2.15) yields

$$\sum_{j=1}^{n}(I_{jki_t}+J_{jki_t}) = -\sum_{j=1}^{n}\sum_{i(l)=1}^{m}\sum_{\substack{s=1\\ s\neq t}}^{l}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{i(l)}u_j D_{(i_t i_s)}u_j dx \\ +\sum_{j=1}^{n}\sum_{i(l)=1}^{m}\sum_{s=1}^{l}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{(i_t)}u_j D_{(i_s)}u_j dx.$$

Define

$$I^*_{jki_ti_s} = -\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{i(l)}u_j D_{(i_t i_s)}u_j dx, \tag{2.16}$$

$$J^*_{jki_ti_s} = \sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{(i_s)}u_j D_{(i_t)}u_j dx.$$

We then have

$$\sum_{j=1}^{n}(I_{jki_t}+J_{jki_t}) = \sum_{j=1}^{n}\left(\sum_{\substack{s=1\\ s\neq t}}^{l}I^*_{jki_ti_s}+\sum_{s=1}^{l}J^*_{jki_ti_s}\right).$$

By (2.16),

$$I^*_{jki_ti_s}=\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega a_{i(l)}D_{(i_t)}u_jD_{(i_s)}u_jdx+$$
$$+\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega(D_{i_t}a_{i(l)})D_{(i_t)}u_jD_{(i_ti_s)}u_jdx.$$

Using Schwartz inequality yields

$$\sum_{k=1}^{m}I^*_{jki_ti_s}\le\max_{\bar\Omega}a_{i(l)}\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{(i_t)}u_j||D_{(i_s)}u_j|dx$$
$$+\left(\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{(i_t)}u_j|^2dx\right)^{1/2}\left(\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{i_t}a_{i(l)}|^2|D_{(i_ti_s)}u_j|^2dx\right)^{1/2}. \tag{2.17}$$

For $t\neq s$, using (2.10) and (2.13) we find

$$\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{(i_t)}u_j||D_{(i_s)}u_j|dx=\sum_{k=1}^{m}\sum_{(i_ti_s)=1}^{m}\int_\Omega|D_kD_{(i_ti_s)}u_j|^2dx$$
$$=\int_\Omega|\nabla^{l-1}u_j|^2dx\le(\lambda_j/\mu)^{1-1/l}. \tag{2.18}$$

Similarly,

$$\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{(i_t)}u_j|^2dx=\int_\Omega|\nabla^{l-1}u_j|^2dx\le(\lambda_j/\mu)^{1-1/l}, \tag{2.19}$$

$$\sum_{k=1}^{m}\sum_{i(l)=1}^{m}\delta_{ki_t}\delta_{ki_s}\int_\Omega|D_{i_t}a_{i(l)}|^2|D_{(i_ti_s)}u_j|^2dx\le\max_{\bar\Omega}|\nabla a_{i(l)}|^2(\lambda_j/\mu)^{1-2/l}. \tag{2.20}$$

Substituting (2.18), (2.19) and (2.20) into (2.17), we have

$$\sum_{k=1}^{m}I^*_{jki_ti_s}\le\max_{\bar\Omega}a_{i(l)}(\lambda_j/\mu)^{1-1/l}+\max_{\bar\Omega}|\nabla a_{i(l)}|(\lambda_j/\mu)^{1-3/2l}.$$

Hence,

$$\sum_{\substack{s,t=1\\ s\neq t}}^{l}\sum_{k=1}^{m}I^*_{jki_ti_s}\le l(l-1)\left[\max_{\bar\Omega}a_{i(l)}(\lambda_j/\mu)^{1-1/l}+\max_{\bar\Omega}|\nabla a_{i(l)}|(\lambda_j/\mu)^{1-3/2l}\right]. \tag{2.21}$$

Similarly, for $J_{jki_ti_s}$ we have

$$\sum_{\substack{s,t=1\\ s\neq t}}^{l}\sum_{k=1}^{m}J^*_{jki_ti_s}\le l(l-1)\max_{\bar\Omega}a_{i(l)}(\lambda_j/\mu)^{1-1/l}, \tag{2.22}$$

$$\sum_{s=t=1}^{l}\sum_{k=1}^{m}J^*_{jki_ti_s}\le ml\max_{\bar\Omega}a_{i(l)}(\lambda_j/\mu)^{1-1/l}. \tag{2.23}$$

Combining (2.21), (2.22) and (2.23) the lemma then follows.

To finish the proof of Theorem 1.1, we need the estimate that

$$\sum_{j=1}^{n}\sum_{k=1}^{m}\int_{\Omega}\phi_{jk}^2 dx \geq \mu^{1/l}m^2n^2(\sum_{j=1}^{n}\lambda_j^{1/l})^{-1}/4. \tag{2.24}$$

This is true. In fact, from the definition of ϕ_{jk} we have

$$\sum_{j=1}^{n}\int_{\Omega}\phi_{jk}D_k u_j dx = \sum_{j=1}^{n}\int_{\Omega}x_k u_j D_k u_j dx - \sum_{p,j=1}^{n} b_{jp}^k \int_{\Omega} u_p D_k u_j dx. \tag{2.25}$$

Because of $b_{jp}^k = b_{pj}^k$ the second term on the right hand side of (2.25) is zero. From the identity of $\int_{\Omega} x_k u_j D_k u_j dx = -\frac{1}{2}$, we get

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\int_{\Omega}\phi_{jk}D_k u_j dx = -mn/2.$$

By Schwartz inequality,

$$m^2n^2/4 \leq \left(\sum_{k=1}^{m}\sum_{j=1}^{n}\int_{\Omega}\phi_{jk}^2 dx\right)\sum_{j=1}^{n}\int_{\Omega}|\nabla u_j|^2 dx. \tag{2.26}$$

From (2.13) we know that $\int_{\Omega}|\nabla u_j|^2 dx \leq (\lambda_j/\mu)^{1/l}$, which with (2.26) gives (2.24). Substituting (2.24) into (2.9), and using our lemma, assertion (1.3) then follows. Replacing λ_j in (1.3) by λ_n yields (1.4). This completes the proof.

Remark 2.1. Similar to the method in [3], we can get for problem (1.1) the following estimate

$$\sum_{j=1}^{n}\frac{\lambda_j^{1/l}}{\lambda_{n+1}-\lambda_j} \geq \frac{\mu m^2n^2}{4}\left[l(2l+m-2)\max_{\bar{\Omega}} a_{i_1\cdots i_l}\left(\sum_{j=1}^{n}\lambda_j^{1-1/l}\right)\right.$$
$$\left.+l(l-1)\mu^{1/2l}\max_{\bar{\Omega}}|\nabla a_{i_1\cdots i_l}|\left(\sum_{j=1}^{n}\lambda_j^{1-3/2l}\right)\right]^{-1}, \tag{2.27}$$

where $m \geq 2$, $n \geq 1$ and $l \geq 1$. If $a_{i_1\cdots i_l}$ is as in Remark 1.1 and $\mu = 1$, then (2.27) takes the form of

$$\sum_{j=1}^{n}\frac{\lambda_j^{1/l}}{\lambda_{n+1}-\lambda_j} \geq \frac{m^2n^2}{4l(2l+m-2)}\left(\sum_{j=1}^{n}\lambda_j^{1-1/l}\right)^{-1},$$

which is the result obtained in [4] for polyharmonic operator. Moreover, let $l = 1, 2$ respectively. Then the results in [1] and [3] follow.

Remark 2.2. By the same method as above we can deduce the upper bounds of eigenvalues for the more general uniformly elliptic operators as follows

$$\begin{cases} \sum_{r=1}^{l}\sum_{i_1,\cdots,i_r=1}^{m}(-1)^r D_{i_1\cdots i_r}(a_{i_1\cdots i_r}D_{i_1\cdots i_r}u) = \lambda u, & x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1}u}{\partial \nu^{l-1}} = 0, & x \in \partial\Omega, \end{cases} \tag{2.28}$$

where $a_{i_1\cdots i_l} \in C^l(\bar{\Omega})$ and $\min_{\bar{\Omega}} a_{i_1\cdots i_l} \geq \mu > 0$, $a_{i_1\cdots i_r} \geq 0$, $a_{i_1\cdots i_r} \in C^r(\bar{\Omega})$, $r = 1, 2, \cdots, l-1$,

$i_1, i_2, \cdots, i_l = 1, 2, \cdots, m$. The results are

$$\sum_{j=1}^{n} \frac{\lambda_j^{1/l}}{\lambda_{n+1}-\lambda_j} \geq \frac{\mu^{1/l} m^2 n^2}{4} \Bigg[\sum_{r=1}^{l} \sum_{j=1}^{n} r(2r+m-2) \max_{\bar{\Omega}} a_{i_1 \cdots i_r} (\lambda_j/\mu)^{(r-1)/l} + \sum_{r=1}^{l} \sum_{j=1}^{n} r(r-1) \max_{\bar{\Omega}} |\nabla a_{i_1 \cdots i_r}| (\lambda_j/\mu)^{(2r-3)/2l} \Bigg]^{-1}, \tag{2.29}$$

$$\lambda_{n+1} - \lambda_n \leq \frac{4}{m^2 n^2} \Bigg[\sum_{r=1}^{l} \sum_{j=1}^{n} r(2r+m-2) \max_{\bar{\Omega}} a_{i_1 \cdots i_r} (\lambda_j/\mu)^{(r-1)/l} + \sum_{r=1}^{l} \sum_{j=1}^{n} r(r-1) \max_{\bar{\Omega}} |\nabla a_{i_1 \cdots i_r}| (\lambda_j/\mu)^{(2r-3)/2l} \Bigg] \sum_{j=1}^{n} (\lambda_j/\mu)^{1/l}. \tag{2.30}$$

These are the generalization of the corresponding results in [4]. Therefore, [1], [2], [3] and [5] are special cases of this paper since [4] is a generalization of them.

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