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"MEANED" EXACT CONTROLLABILITY FOR WAVE EQUATIONS

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Abstract

With a control of Dirichlet type the author wants to bring the wave system to a "meaned" state in the time interval $[T-h, T]$. With the HUM Method the optimal control is got and its limit behavior when h goes to zero is studied.

Keywords Wave equation, Control of Dirichlet type, Optimal control.
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§1. Introduction and the Principal Results

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega = \Gamma$. Following the notations of J. L. Lions[1,2], we set

$$\Gamma(x^0) = \{x \in \Gamma; \quad \nu(x) \cdot m(x) > 0\}$$

for some point $x^0 \in \mathbf{R}^n$, $\nu(x)$ is the outunit normal to the boundary at the point x and $m(x) = x - x^0$. Then we denote:

$$\Sigma = \Gamma \times (0,T),\ \Sigma(x^0) = \Gamma(x^0) \times (0,T),\ R(x^0) = \max\{|x - x^0|; x \in \bar{\Omega}\},\ T(x^0) = 2R(x^0).$$

We consider the following problem:

$$\begin{cases} y'' - \Delta y = 0 & \text{in } Q = \Omega \times (0,T), \\ y = \begin{vmatrix} v\,(\text{control}) \\ 0 \end{vmatrix} & \begin{matrix} \text{on } \Sigma(x^0), \\ \text{on } \Sigma \setminus \Sigma(x^0), \end{matrix} \\ y(0) = y^0, y'(0) = y^1 & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

For any given initial data $\{y^0, y^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, we will denote for simplicity:

$$y := y(v) := y(t; v) := y(x, t; v),$$

the weak solution of (1.1) associated with the control v.

By the results of J. L. Lions[1,2], we know that if $T > T(x^0)$, then for any $\{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a v_0 such that

$$\begin{aligned} & v_0 \in \mathcal{U}_{ad} = \{v \in L^2(\Sigma(x^0)); \quad \text{s.t. } y(T; v) = z^0, y'(T; v) = z^1\}, \\ & J(v_0) = \min_{v \in \mathcal{U}_{ad}} J(v), \text{ where } J(v) = \frac{1}{2}\int_{\Sigma(x^0)} v^2 d\Sigma, \\ & v_0 = \frac{\partial \Phi}{\partial \nu} \qquad \text{on } \Sigma(x^0), \end{aligned} \tag{1.2}$$

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where Φ is the solution of the following problem for a certain $\{\Phi^0, \Phi^1\} \in H^1(\Omega) \times L^2(\Omega)$:

$$\begin{cases} \Phi'' - \Delta\Phi = 0 & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(T) = \Phi^0, \Phi'(T) = \Phi^1 & \text{on } \Omega \times \{T\}. \end{cases} \tag{1.3}$$

Now we are going to search $\{z_h^0, z_h^1\}$ and v_h for any $h > 0$, such that

$$v_h \in \mathcal{U}_{ad,h} = \left\{ \begin{array}{l} v \in L^2(\Sigma(x^0)); \quad \text{s.t.} \\ \dfrac{1}{h}\displaystyle\int_{T-h}^{T} y(t;v)dt = z_h^0, \quad \dfrac{1}{h}\displaystyle\int_{T-h}^{T} y'(t;v)dt = z_h^1 \end{array} \right\}, \tag{1.4}$$
$$J(v_h) = \min_{v \in \mathcal{U}_{ad,h}} J(v).$$

And when $h \to 0$, if we have

$$\{z_h^0, z_h^1\} \to \{z^0, z^1\} \tag{1.5}$$

by a convenient topology, we would like to have

$$v_h \to v_0 \tag{1.6}$$

by a certain topology.

Remark 1.1. We can see that for any $\{y^0, y^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ and any $v \in L^2(\Sigma(x^0))$, we have

$$\Delta(\frac{1}{h}\int_{T-h}^{T} y(t;v)dt) \in H^{-1}(\Omega), \quad \frac{1}{h}\int_{T-h}^{T} y'(t;v)dt \in L^2(\Omega).$$

So, it is necessary to introduce a process like (1.4) and (1.5), since we only have $\{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$.

In what follows, we will suppose that there is a certain $h_0 > 0$, such that

$$0 < h \leq h_0, \quad T(x^0) < T - h_0 \tag{1.7}$$

and observing the linearity of the system (1.1), we can suppose without restricting the generality that

$$y^0 = y^1 = 0 \qquad \text{in } \Omega. \tag{1.8}$$

Theorem 1.1. *For any $\{z_h^0, z_h^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique v_h which verifies (1.4), and we have*

$$\|v_h\|_{L^2(\Sigma(x^0))} \leq C\left(\|z_h^0\|^2_{H_0^1(\Omega)} + |z_h^1|^2_{L^2(\Omega)}\right)^{1/2}, \tag{1.9}$$

where the constant C is independent of h.

Lemma 1.1. *When $h \to 0$, if $\{z_h^0, z_h^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies*

$$\{z_h^0, z_h^1\} \rightharpoonup \{z^0, z^1\} \qquad \textit{weakly in } L^2(\Omega) \times H^{-1}(\Omega), \tag{1.10}$$

and if $\{v_h\}_{h>0}$ is bounded in $L^2(\Sigma(x^0))$, then

$$\begin{array}{ll} v_h \rightharpoonup v_0 & \textit{weakly in } L^2(\Sigma(x^0)), \\ y_h \overset{*}{\rightharpoonup} y_0 & \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega)), \end{array} \tag{1.11}$$

where $y_h = y(v_h)$, $\quad y_0 = y(v_0)$.

For the strong convergence of $\{v_h\}_{h>0}$, here, we give a construction of the series

$$\{z_h^0, z_h^1\}_{h>0} \subset H_0^1(\Omega) \times L^2(\Omega)$$

for any $\{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$. In fact, we can choose

$$\bar{z}_h^0 = \frac{1}{h}\int_{T-h}^{T} z_h(t)dt, \quad \bar{z}_h^1 = \frac{1}{h}\int_{T-h}^{T} z_h'(t)dt, \tag{1.12}$$

where $z_h = z_h(x,t)$ satisfies

$$\begin{cases} z_h'' - \Delta z_h = 0 & \text{in } \Omega \times (0,h), \\ z_h = 0 & \text{on } (\Gamma \setminus \Gamma(x^0)) \times (0,h), \\ \int_0^h z_h dt = 0 & \text{on } \Gamma(x^0), \\ \int_{\Gamma(x^0)\times(0,h)} z_h^2 d\Gamma dt \to 0 & \text{when } h \to 0, \\ \{z_h(0), z_h'(0)\} \to \{z^0, z^1\} & \text{strongly in } L^2(\Omega) \times H^{-1}(\Omega), \text{ as } h \to 0. \end{cases} \tag{1.13}$$

We have the following results.

Theorem 1.2. *If $\{z_h^0, z_h^1\}$ satisfies*

$$\{z_h^0 - \bar{z}_h^0, z_h^1 - \bar{z}_h^1\} \to \{0,0\} \qquad \textit{strongly in } H_0^1(\Omega) \times L^2(\Omega) \tag{1.14}$$

for a certain $\{z_h\}_{h>0}$, which satisfies (1.13), then we have

$$\|v_h\|_{L^2(\Sigma(x^0))} \le \|v_0\|_{L^2(\Sigma(x^0))} + o(1), \tag{1.15}$$

$$\{z_h^0, z_h^1\} \to \{z^0, z^1\} \qquad \textit{strongly in } L^2(\Omega) \times H^{-1}(\Omega). \tag{1.16}$$

Hence

$$\begin{aligned} v_h &\to v_0 && \textit{strongly in } L^2(\Sigma(x^0)), \\ y_h &\to y_0 && \textit{strongly in } L^\infty(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega)). \end{aligned} \tag{1.17}$$

Remark 1.2. Particularly, we can choose

$$z_h \equiv z \qquad \text{in } \Omega \times (0,h), \tag{1.18}$$

where z satisfies

$$\begin{cases} z'' - \Delta z = 0 & \text{in } \Omega \times (0,h_0), \\ z = 0 & \text{on } \Gamma \times (0,h_0), \\ z(0) = z^0, z'(0) = z^1 & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.19}$$

and we take

$$\{z_h^0, z_h^1\} = \{\bar{z}_h^0, \bar{z}_h^1\}, \tag{1.20}$$

or, when $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, we can simply take for any $h > 0$,

$$\{z_h^0, z_h^1\} \equiv \{z^0, z^1\}. \tag{1.21}$$

§2 Proof of Theorem 1.1

Following the HUM method of J. L. Lions[1,2], we consider the optimal control problem:

$$\min_{v\in\mathcal{U}_{ad,h}} J(v), \qquad J(v) = \frac{1}{2}\int_{\Sigma(x^0)} v^2 d\Sigma. \tag{2.1}$$

If

$$\mathcal{U}_{ad,h} \neq \emptyset, \tag{2.2}$$

then (2.1) admits a unique solution $v_h \in \mathcal{U}_{ad,h}$:

$$J(v_h) = \min_{v\in\mathcal{U}_{ad,h}} J(v). \tag{2.3}$$

Then in order to characterize v_h by an optimal system, we use the penalization method

$$J_\epsilon(v,z)=\frac{1}{2}\int_{\Sigma(x^0)}v^2d\Sigma+\frac{1}{2\epsilon}|z''-\Delta z|^2_{L^2(Q)}, \tag{2.4}$$

where v and z satisfy

$$\begin{array}{l} v\in L^2(\Sigma(x^0)), z''-\Delta z\in L^2(Q),\\ z=\begin{cases} v & \text{on } \Sigma(x^0),\\ 0 & \text{on } \Sigma\setminus\Sigma(x^0),\end{cases}\\ \frac{1}{h}\int_{T-h}^T z(t)dt=z_h^0,\quad \frac{1}{h}\int_{T-h}^T z'(t)dt=z_h^1,\\ z(0)=z'(0)=0\quad \text{on } \Omega\times\{0\}.\end{array} \tag{2.5}$$

We consider the following problem:

$$\min J_\epsilon(v,z), \tag{2.6}$$

where v and z satisfy (2.5).

According to (2.2), this problem admits a unique solution:

$$v_{h\epsilon},\quad y_{h\epsilon}. \tag{2.7}$$

When $\epsilon\to 0$, we can prove that (cf. [1])

$$\begin{array}{ll} v_{h\epsilon}\to v_h & \text{strongly in } L^2(\Sigma(x^0)),\\ y_{h\epsilon}\to y_h & \text{strongly in } L^\infty(0,T;L^2(\Omega))\cap W^{1,\infty}(0,T;H^{-1}(\Omega)).\end{array} \tag{2.8}$$

We define

$$P_{h\epsilon}=\begin{cases} P_{1h\epsilon}=-\frac{1}{\epsilon}(y''_{h\epsilon}-\Delta y_{h\epsilon}) & \text{in } \Omega\times(T-h,T),\\ P_{2h\epsilon}=-\frac{1}{\epsilon}(y''_{h\epsilon}-\Delta y_{h\epsilon}) & \text{in } \Omega\times(0,T-h).\end{cases} \tag{2.9}$$

Then $P_{h\epsilon}\in L^2(Q)$, and for any ζ, z which satisfy

$$\begin{array}{l} \zeta\in L^2(\Sigma(x^0)), z''-\Delta z\in L^2(Q),\\ z=\begin{cases} \zeta & \text{on } \Sigma(x^0),\\ 0 & \text{on } \Sigma\setminus\Sigma(x^0),\end{cases}\\ \frac{1}{h}\int_{T-h}^T z(t)dt=0,\ \frac{1}{h}\int_{T-h}^T z'(t)dt=0,\\ z(0)=z'(0)=0\quad \text{on } \Omega\times\{0\},\end{array} \tag{2.10}$$

we have

$$\int_Q P_{h\epsilon}(z''-\Delta z)dxdt-\int_{\Sigma(x^0)}v_{h\epsilon}\zeta d\Sigma=0. \tag{2.11}$$

If $P_{1h\epsilon}$, $P_{2h\epsilon}$ are regular enough, then $P_{1h\epsilon}$, $P_{2h\epsilon}$ will satisfy the following equation:

$$\begin{cases} \rho_1''-\Delta\rho_1=-\phi^1(x) & \text{in } \Omega\times(T-h,T),\\ \rho_1=0 & \text{on } \Gamma\times(T-h,T),\\ \rho_1(T)=0,\ \rho_1'(T)=\phi^0(x) & \text{on } \Omega\times\{T\},\end{cases} \tag{2.12}$$

$$\begin{cases} \rho_2''-\Delta\rho_2=0 & \text{in } \Omega\times(0,T-h),\\ \rho_2=0 & \text{on } \Gamma\times(0,T-h)\\ \rho_2(T-h)=\rho_1(T-h) & \text{on } \Omega\times\{T-h\},\\ \rho_2'(T-h)=\rho_1'(T-h)-\phi^0(x) & \text{on } \Omega\times\{T-h\}.\end{cases} \tag{2.13}$$

with $\{\phi^0,\phi^1\}=\{\phi^0_{h\epsilon},\phi^1_{h\epsilon}\}$ for a certain $\{\phi^0_{h\epsilon},\phi^1_{h\epsilon}\}\in L^2(\Omega)\times H^{-1}(\Omega)$.

Remark 2.1. We will denote the solutions ρ_1, ρ_2 of (2.12) and (2.13) by $\rho_1=\rho_1(\phi^0,\phi^1)$, $\rho_2=\rho_2(\phi^0,\phi^1)$, when we want to indicate their relations with $\{\phi^0,\phi^1\}$.

We have the following result, which says that $P_{1h\epsilon}$, $P_{2h\epsilon}$ are, in fact, regular enough, and which will be proved in §5.

Lemma 2.1. *For any* $w \in L^2(\Sigma(x^0)), P \in L^2(Q)$, *if* w, P *satisfy*

$$\begin{aligned} &\int_{\Omega\times(0,T)} P(z''-\Delta z)dxdt - \int_{\Sigma(x^0)} w\zeta d\Sigma = 0, \\ &\int_{T-h}^{T} w(t)dt = 0 \qquad on\ \Gamma(x^0) \end{aligned} \tag{2.14}$$

for any ζ, z *which satisfy (2.10), then there exists a* $\{\phi^0,\phi^1\}\in L^2(\Omega)\times H^{-1}(\Omega)$, *such that*

$$P = \begin{cases} \rho_1(\phi^0,\phi^1) & in\ \Omega\times(T-h,T), \\ \rho_2(\phi^0,\phi^1) & in\ \Omega\times(0,T-h), \end{cases} \tag{2.15}$$

$$w = \bar{v}(\phi^0,\phi^1) \overset{def}{=} \begin{cases} \frac{\partial\rho_1}{\partial\nu} - \frac{1}{h}\int_{T-h}^{T}\frac{\partial\rho_1}{\partial\nu}dt & on\ \Gamma(x^0)\times(T-h,T), \\ \frac{\partial\rho_2}{\partial\nu} & on\ \Gamma(x^0)\times(0,T-h). \end{cases} \tag{2.16}$$

So, there really exists a $\{\phi^0_{h\epsilon},\phi^1_{h\epsilon}\} \in L^2(\Omega)\times H^{-1}(\Omega)$, such that

$$P_{1h\epsilon} = \rho_1(\phi^0_{h\epsilon},\phi^1_{h\epsilon}),\ P_{2h\epsilon} = \rho_2(\phi^0_{h\epsilon},\phi^1_{h\epsilon}),\ v_{h\epsilon} = \bar{v}(\phi^0_{h\epsilon},\phi^1_{h\epsilon}). \tag{2.17}$$

By the results of §4, we have

$$\begin{aligned} C_h\|\bar{v}(\phi^0,\phi^1)\|^2_{L^2(\Sigma(x^0))} &\le |\phi^0|^2_{L^2(\Omega)} + \|\phi\|^2_{H^{-1}(\Omega)} \\ &\le C(\frac{1}{h}+\frac{1}{h^2})\|\bar{v}(\phi^0,\phi^1)\|^2_{L^2(\Sigma(x^0))}, \end{aligned} \tag{2.18}$$

where C_h is a constant dependent of h.

Combining (2.8), (2.17) and (2.18), we can deduce that there exists a $\{\bar{\phi}^0_h,\bar{\phi}^1_h\}\in L^2(\Omega)\times H^{-1}(\Omega)$, such that

$$\begin{aligned} &\phi^0_{h\epsilon}\to\bar{\phi}^0_h && \text{strongly in} L^2(\Omega), \\ &\phi^1_{h\epsilon}\to\bar{\phi}^1_h && \text{strongly in } H^{-1}(\Omega), \\ &v_h = \bar{v}(\bar{\phi}^0_h,\bar{\phi}^1_h). \end{aligned} \tag{2.19}$$

We have proved that if v_h exists, that is to say, if (2.2) is true, then v_h will verifies (2.19). Now we are going to verify (2.2) by the HUM Method.

Firstly, for any $\{\phi^0,\phi^1\} \in L^2(\Omega)\times H^{-1}(\Omega)$, we solve

$$\begin{aligned} &\psi''-\Delta\psi = 0 && \text{in } Q, \\ &\psi = \bar{v}(\phi^0,\phi^1) = \begin{cases} \frac{\partial\rho_1}{\partial\nu} - \frac{1}{h}\int_{T-h}^{T}\frac{\partial\rho_1}{\partial\nu}dt \\ \frac{\partial\rho_2}{\partial\nu} \end{cases} && \begin{matrix}\text{on } \Gamma(x^0)\times(T-h,T)\ , \\ \text{on } \Gamma(x^0)\times(0,T-h)\ ,\end{matrix} \\ &\psi = 0 && \text{on } \Sigma\setminus\Sigma(x^0), \\ &\psi(0) = \psi'(0) = 0 && \text{on } \Omega\times\{0\}, \end{aligned} \tag{2.20}$$

where $\rho_1 = \rho_1(\phi^0,\phi^1)$, $\rho_2 = \rho_2(\phi^0,\phi^1)$. We set

$$\begin{aligned} &u^0(x) = -\Delta^{-1}\phi^1(x) \in H^1_0(\Omega), \qquad u^1(x) = \phi^0(x) \in L^2(\Omega), \\ &u(x,t) = \rho_1(x,t) + u^0(x), \qquad \text{in } \Omega\times(T-h,T). \end{aligned} \tag{2.21}$$

u will verify

$$\begin{cases} u''-\Delta u = 0 & \text{in } \Omega\times(T-h,T), \\ u = 0 & \text{on } \Gamma\times(T-h,T), \\ u(T) = u^0, u'(T) = u^1 & \text{on } \Omega\times\{T\}. \end{cases} \tag{2.22}$$

By the results of J. L. Lions[1,2], we know that

$$\frac{\partial u}{\partial \nu} \in L^2(\Gamma \times (T-h,T)), \qquad \frac{\partial \rho_2}{\partial \nu} \in L^2(\Gamma \times (0,T-h)).$$

Hence

$$\frac{\partial \rho_1}{\partial \nu} - \frac{1}{h}\int_{T-h}^{T} \frac{\partial \rho_1}{\partial \nu} dt = \frac{\partial u}{\partial \nu} - \frac{1}{h}\int_{T-h}^{T} \frac{\partial u}{\partial \nu} dt \in L^2(\Gamma \times (T-h,T)). \tag{2.23}$$

Then, we can define

$$\begin{aligned} \Lambda_h :\ & L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega), \\ & \{\phi^0, \phi^1\} \mapsto \{\frac{1}{h}\int_{T-h}^{T} \psi' dt,\ \frac{1}{h}\int_{T-h}^{T} \psi dt\}, \end{aligned} \tag{2.24}$$

where $\psi = \psi(\phi^0, \phi^1)$ is the solution of (2.20) associated with the initial data $\{\phi^0, \phi^1\}$.

Now, we consider another couple $\{\bar{\phi}^0, \bar{\phi}^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, and $\bar{\rho}_1 = \rho_1(\bar{\phi}^0, \bar{\phi}^1)$, $\bar{\rho}_2 = \rho_2(\bar{\phi}^0, \bar{\phi}^1)$; we multiply (2.20) by $\bar{\rho}_1$, $\bar{\rho}_2$; at the end, we will get

$$\begin{aligned} &(\frac{\psi(T) - \psi(T-h)}{h}, \bar{\phi}^0)_{L^2(\Omega), L^2(\Omega)} + \langle \frac{1}{h}\int_{T-h}^{T} \psi dt,\ \bar{\phi}^1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ =& \frac{1}{h}\int_{\Gamma(x^0)\times(T-h,T)} \left(\frac{\partial \rho_1}{\partial \nu} - \frac{1}{h}\int_{T-h}^{T} \frac{\partial \rho_1}{\partial \nu} dt \right) \frac{\partial \bar{\rho}_1}{\partial \nu} d\Sigma + \frac{1}{h}\int_{\Gamma(x^0)\times(0,T-h)} \frac{\partial \rho_2}{\partial \nu}\frac{\partial \bar{\rho}_2}{\partial \nu} d\Sigma \\ =& \frac{1}{h}\int_{\Gamma(x^0)\times(T-h,T)} \left(\frac{\partial \rho_1}{\partial \nu} - \frac{1}{h}\int_{T-h}^{T} \frac{\partial \rho_1}{\partial \nu} dt \right) \left(\frac{\partial \bar{\rho}_1}{\partial \nu} - \frac{1}{h}\int_{T-h}^{T} \frac{\partial \bar{\rho}_1}{\partial \nu} dt \right) d\Sigma \\ &+ \frac{1}{h}\int_{\Gamma(x^0)\times(0,T-h)} \frac{\partial \rho_2}{\partial \nu}\frac{\partial \bar{\rho}_2}{\partial \nu} d\Sigma. \end{aligned} \tag{2.25}$$

This means

$$\begin{aligned} &\langle \Lambda_h\{\phi^0, \phi^1\}, \{\bar{\phi}^0, \bar{\phi}^1\} \rangle_{L^2(\Omega)\times H_0^1(\Omega), L^2(\Omega)\times H^{-1}(\Omega)} \\ =& \frac{1}{h}\int_{\Sigma(x^0)} \bar{v}(\phi^0, \phi^1)\bar{v}(\bar{\phi}^0, \bar{\phi}^1) d\Sigma, \end{aligned} \tag{2.26}$$

hence

$$\Lambda_h = \Lambda_h^*. \tag{2.27}$$

By observing (2.18), we can deduce

$$\begin{aligned} &\langle \Lambda_h\{\phi^0, \phi^1\}, \{\phi^0, \phi^1\} \rangle_{L^2(\Omega)\times H_0^1(\Omega), L^2(\Omega)\times H^{-1}(\Omega)} \\ =& \frac{1}{h}\int_{\Sigma(x^0)} \bar{v}(\phi^0, \phi^1)\bar{v}(\phi^0, \phi^1) d\Sigma \ge C_h \|\{\phi^0, \phi^1\}\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}. \end{aligned} \tag{2.28}$$

Hence, Λ_h is an isomorphism.

So for any $\{z_h^1, z_h^0\} \in L^2(\Omega) \times H_0^1(\Omega)$ we can find a $\{\phi_h^0, \phi_h^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, such that

$$\Lambda_h\{\phi_h^0, \phi_h^1\} = \{z_h^1, z_h^0\}. \tag{2.29}$$

we can deduce that

$$\bar{v}(\phi_h^0, \phi_h^1) \in \mathcal{U}_{ad,h}. \tag{2.30}$$

So, (2.2) is verified.

Since Λ_h is an isomorphism, we can see from (2.19), (2.30) that

$$\begin{aligned} &\{\bar{\phi}_h^0, \bar{\phi}_h^1\} = \{\phi_h^0, \phi_h^1\}, \\ &v_h = \bar{v}(\phi_h^0, \phi_h^1). \end{aligned} \tag{2.31}$$

From (2.18), we can deduce

$$\begin{aligned}\left(\|z_h^0\|^2_{H_0^1(\Omega)} + |z_h^1|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} &= \|\Lambda_h\{\phi_h^0,\phi_h^1\}\|_{L^2(\Omega)\times H_0^1(\Omega)} \\ &\geq \frac{\langle \Lambda_h\{\phi_h^0,\phi_h^1\},\{\phi_h^0,\phi_h^1\}\rangle_{L^2(\Omega)\times H_0^1(\Omega),L^2(\Omega)\times H^{-1}(\Omega)}}{\|\{\phi_h^0,\phi_h^1\}\|_{L^2(\Omega)\times H^{-1}(\Omega)}} \\ &\geq C\frac{\frac{1}{h}\|\bar{v}(\phi_h^0,\phi_h^1)\|^2_{L^2(\Sigma(x^0))}}{\left(\frac{1}{h}+\frac{1}{h^2}\right)^{\frac{1}{2}}\|\bar{v}(\phi_h^0,\phi_h^1)\|_{L^2(\Sigma(x^0))}}.\end{aligned} \tag{2.32}$$

So we get (1.9).

§3. Proofs of Lemma 1.1 and Theorem 1.2

Proof of Lemma 1.1. If for a certain $\bar{v}_0 \in L^2(\Sigma(x^0))$ we have a subseries such that

$$v_h \rightharpoonup \bar{v}_0 \qquad \text{weakly in } L^2(\Sigma(x^0)), \tag{3.1}$$

by the results of J. L. Lions[1,2] we know that the series $\{\rho_{2h}(T-h),\rho'_{2h}(T-h)\}$ remains bounded in $H_0^1(\Omega)\times L^2(\Omega)$, where $\rho_{2h} = \rho_2(\phi_h^0,\phi_h^1)$. So we can choose a subseries (still denoted by h) such that

$$\begin{aligned}\rho_{2h}(T-h) &\rightharpoonup \bar{\Phi}^0 \qquad \text{weakly in } H_0^1(\Omega), \\ \rho'_{2h}(T-h) &\rightharpoonup \bar{\Phi}^1 \qquad \text{weakly in } L^2(\Omega)\end{aligned} \tag{3.2}$$

for a certain $\{\bar{\Phi}^0,\bar{\Phi}^1\}\in H_0^1(\Omega)\times L^2(\Omega)$, and

$$\begin{aligned} y_h &\overset{*}{\rightharpoonup} \bar{y}_0 \quad \text{weak} * \text{ in } L^\infty(0,T;L^2(\Omega))\cap W^{1,\infty}(0,T;H^{-1}(\Omega)), \\ \rho_{2h} &\overset{*}{\rightharpoonup} \bar{\Phi}_0 \quad \text{weak} * \text{ in } L^\infty(0,T;H_0^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega)),\end{aligned} \tag{3.3}$$

where ρ_{2h} is extended into $\Omega\times(T-h,T)$ by

$$\begin{aligned}\rho''_{2h} - \Delta\rho_{2h} = 0 \quad & \text{in } \Omega\times(T-h,T), \\ \rho_{2h} = 0 \quad & \text{on } \Gamma\times(T-h,T),\end{aligned} \tag{3.4}$$

$\bar{y}_0 = y(\bar{v}_0)$, $\bar{\Phi}_0$ satisfies the following equation:

$$\begin{cases}\Phi'' - \Delta\Phi = 0 & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(T) = \Phi^0, \Phi'(T) = \Phi^1 & \text{on } \Omega\times\{T\},\end{cases} \tag{3.5}$$

with the initial data $\{\Phi^0,\Phi^1\} = \{\bar{\Phi}^0,\bar{\Phi}^1\}$, and still more, we have

$$\frac{\partial\bar{\Phi}_0}{\partial\nu} = \bar{v}_0 \qquad \text{on } \Sigma(x^0). \tag{3.6}$$

It still remains to prove that

$$\bar{y}_0(T) = z^0, \quad \bar{y}'_0(T) = z^1. \tag{3.7}$$

For any $\{\Phi^0,\Phi^1\}\in H_0^1(\Omega)\times L^2(\Omega)$, and $\Phi = \Phi(\Phi^0,\Phi^1)$ the associated solution of (3.5), and for any $s\in(T-h,T)$, we have

$$\begin{aligned}&(y_h(s),\Phi^1)_{L^2(\Omega),L^2(\Omega)} - \langle y'_h(s),\Phi^0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &= \int_{\Gamma(x^0)\times(0,s)} v_h(t)\frac{\partial\Phi}{\partial\nu}(t+T-s)d\Sigma.\end{aligned} \tag{3.8}$$

Integrating (3.8) in $(T-h,T)$, we can get

$$\begin{aligned}&(z_h^0,\Phi^1)_{L^2(\Omega),L^2(\Omega)}-\langle z_h^1,\Phi^0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\&=\frac{1}{h}\int_{T-h}^{T}\int_{\Gamma(x^0)\times(0,s)}v_h(t)\frac{\partial\Phi}{\partial\nu}(t+T-s)d\Sigma ds.\end{aligned}\tag{3.9}$$

In (3.9) letting $h\to 0$, we will get

$$\begin{aligned}&(z^0,\Phi^1)_{L^2(\Omega),L^2(\Omega)}-\langle z^1,\Phi^0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\&=\int_{\Sigma(x^0)\times(0,T)}\bar{v}_0(t)\frac{\partial\Phi}{\partial\nu}d\Sigma=\int_{\Sigma(x^0)\times(0,T)}\frac{\partial\bar{\Phi}_0}{\partial\nu}\frac{\partial\Phi}{\partial\nu}d\Sigma.\end{aligned}\tag{3.10}$$

So we have (3.7). By the results of J. L. Lions[1,2] we can deduce from (1.2), (3.6) and (3.7) that

$$\begin{aligned}\bar{y}_0&=y_0\quad\text{in } Q,\\\bar{v}_0&=v_0\quad\text{on }\Sigma(x^0).\end{aligned}\tag{3.11}$$

Proof of Theorem 1.2. It suffices to prove (1.15). If $\bar{v}_h$ is such that

$$\frac{1}{h}\int_{T-h}^{T}\bar{y}_h(t)dt=\bar{z}_h^0,\qquad\frac{1}{h}\int_{T-h}^{T}\bar{y}_h'(t)dt=\bar{z}_h^1,\tag{3.12}$$

where $\bar{y}_h=y(\bar{v}_h)$, and $\bar{v}_h$ is optimal in the following sense:

$$\begin{aligned}&\|\bar{v}_h\|_{L^2(\Sigma(x^0))}=\min\|v\|_{L^2(\Sigma(x^0))},\\&\text{for any } v\in L^2(\Sigma(x^0))\quad\text{which makes (3.12) true with}\\&y=y(v)\quad\text{being in the place of }\bar{y}_h,\end{aligned}\tag{3.13}$$

According to (1.9) we have

$$\|v_h-\bar{v}_h\|_{L^2(\Sigma(x^0))}\le C\left(\|z_h^0-\bar{z}_h^0\|^2_{H_0^1(\Omega)}+|z_h^1-\bar{z}_h^1|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}=o(1).\tag{3.14}$$

If $\bar{v}_0\in L^2(\Sigma(x^0)\times(0,T-h))$ is such that:

$$\bar{y}_0(T-h)=z_h(0),\qquad\bar{y}_0'(T-h)=z_h'(0),\tag{3.15}$$

where $\bar{y}_0=y(\bar{v}_0)$ is the solution of (1.1) in $\Omega\times(0,T-h)$ with $v=\bar{v}_0$ on $\Gamma(x^0)\times(0,T-h)$, and $\bar{v}_0$ is optimal in the following sense:

$$\begin{aligned}&\|\bar{v}_0\|_{L^2(\Gamma(x^0)\times(0,T-h))}=\min\|v\|_{L^2(\Gamma(x^0)\times(0,T-h))},\\&\text{for any } v\in L^2(\Gamma(x^0)\times(0,T-h))\quad\text{which makes (3.15) true with}\\&y=y(v)\quad\text{being in the place of }\bar{y}_0,\end{aligned}\tag{3.16}$$

we extend $\bar{v}_0$ into $\Gamma(x^0)\times(T-h,T)$ by:

$$\bar{v}_0(t)=z_h(t-T+h)\qquad\text{on }\Gamma(x^0)\times(T-h,T).\tag{3.17}$$

Then, by the results of J. L. Lions[1,2], we have

$$\begin{aligned}\|v_0-\bar{v}_0\|_{L^2(\Sigma(x^0))}\le&\|v_0-\bar{v}_0\|_{L^2(\Gamma(x^0)\times(0,T-h))}+\|v_0-\bar{v}_0\|_{L^2(\Gamma(x^0)\times(T-h,T))}\\\le&C\left(|y(T-h;v_0)-z_h(0)|^2_{L^2(\Omega)}+\|y'(T-h;v_0)-z_h'(0)\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}}\\&+\|v_0\|_{L^2(\Gamma(x^0)\times(T-h,T))}+\|z_h\|_{L^2(\Gamma(x^0)\times(0,h))}\\=&o(1).\end{aligned}\tag{3.18}$$

Since $\bar{v}_0$ verifies (3.12), we have from (3.13)

$$\|\bar{v}_h\|_{L^2(\Sigma(x^0))} \leq \|\bar{v}_0\|_{L^2(\Sigma(x^0))}. \tag{3.19}$$

So we have

$$\begin{aligned}\|v_h\|_{L^2(\Sigma(x^0))} \leq& \|v_h - \bar{v}_h\|_{L^2(\Sigma(x^0))} + \|\bar{v}_h\|_{L^2(\Sigma(x^0))} \leq o(1) + \|\bar{v}_0\|_{L^2(\Sigma(x^0))}\\ \leq& \|v_0\|_{L^2(\Sigma(x^0))} + \|v_0 - \bar{v}_0\|_{L^2(\Sigma(x^0))} + o(1)\\ \leq& \|v_0\|_{L^2(\Sigma(x^0))} + o(1).\end{aligned} \tag{3.20}$$

§4. The Inverse Inequality

In this section we are going to verify (2.18). Since

$$\begin{aligned}\|\bar{v}(\phi^0,\phi^1)\|^2_{L^2(\Sigma(x^0))} =& \int_{\Gamma(x^0)\times(T-h,T)} \left(\frac{\partial\rho_1}{\partial\nu} - \frac{1}{h}\int_{T-h}^{T}\frac{\partial\rho_1}{\partial\nu}dt\right)^2 d\Sigma\\ &+ \int_{\Gamma(x^0)\times(0,T-h)} \left(\frac{\partial\rho_2}{\partial\nu}\right)^2 d\Sigma\end{aligned} \tag{4.1}$$

by the results of J. L. Lions[1,2], we know that (cf.(2.21))

$$\begin{aligned}\int_{\Gamma(x^0)\times(0,T-h)} (\frac{\partial\rho_2}{\partial\nu})^2 d\Sigma \geq& C\left(\|\rho_2(T-h)\|^2_{H_0^1(\Omega)} + |\rho_2'(T-h)|^2_{L^2(\Omega)}\right)\\ =& C\left(\|\rho_1(T-h)\|^2_{H_0^1(\Omega)} + |\rho_1'(T-h) - \phi^0|^2_{L^2(\Omega)}\right)\\ =& C\left(\|u(T)-u(T-h)\|^2_{H_0^1(\Omega)} + |u'(T)-u'(T-h)|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.2}$$

Hence (cf. (2.23))

$$\begin{aligned}\|\bar{v}(\phi^0,\phi^1)\|^2_{L^2(\Sigma(x^0))} \geq& \int_{\Gamma(x^0)\times(T-h,T)} \left(\frac{\partial u}{\partial\nu} - \frac{1}{h}\int_{T-h}^{T}\frac{\partial u}{\partial\nu}dt\right)^2 d\Sigma\\ &+ C\left(\|u(T)-u(T-h)\|^2_{H_0^1(\Omega)} + |u'(T)-u'(T-h)|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.3}$$

So, it suffices to prove the following results.

Theorem 4.1. *For any* $\{u^0,u^1\} \in H_0^1(\Omega)\times L^2(\Omega)$, *we have*

$$\begin{aligned}\|u^0\|^2_{H_0^1(\Omega)} + |u^1|^2_{L^2(\Omega)} \leq& C\left(\frac{1}{h}\int_{\Gamma(x^0)\times(0,h)} (\frac{\partial u}{\partial\nu} - \frac{1}{h}\int_0^h \frac{\partial u}{\partial\nu}dt)^2 d\Sigma\right.\\ &\left.+\frac{1}{h^2}\|u(h)-u(0)\|^2_{H_0^1(\Omega)} + \frac{1}{h^2}|u'(h)-u'(0)|^2_{L^2(\Omega)}\right),\end{aligned} \tag{4.4}$$

where $u = u(x,t)$ *satisfies*

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega\times(0,h),\\ u = 0 & \text{on } \Gamma\times(0,h),\\ u(0) = u^0, u'(0) = u^1 & \text{on } \Omega\times\{0\}.\end{cases} \tag{4.5}$$

Proof. By the results of J. L. Lions[1,2], we have

$$\frac{\partial u}{\partial\nu} \in L^2(\Gamma\times(0,h)), \tag{4.6}$$

$$Z(t)\,|_0^h + hE(0) \leq \frac{R(x^0)}{2}\int_{\Gamma(x^0)\times(0,h)} \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma, \tag{4.7}$$

where

$$E(0) = \frac{1}{2}\int_\Omega \left(|\nabla u^0|^2 + |u^1|^2\right) dx,$$
$$Z(t) = (u'(t), m\cdot\nabla u(t) + \frac{n-1}{2}u(t))_{L^2(\Omega),L^2(\Omega)}. \tag{4.8}$$

For any $0<\epsilon<<1$ we will denote by C_ϵ some corresponding sufficiently small constants.

$$\begin{aligned}\left|Z(t)|_0^h\right| =&|(u'(h)-u(0),\, m\cdot\nabla u(h)+\frac{n-1}{2}u(h))_{L^2(\Omega),L^2(\Omega)}\\ &+(u'(0),\, m\cdot\nabla u(h)+\frac{n-1}{2}u(h)-m\cdot\nabla u(0)-\frac{n-1}{2}u(0))_{L^2(\Omega),L^2(\Omega)}|\\ \le&\epsilon h E(0)+C_\epsilon\frac{1}{h}\left(\|u(h)-u(0)\|^2_{H_0^1(\Omega)}+|u'(h)-u'(0)|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.9}$$

Hence

$$\begin{aligned}E(0)\le& C\Big(\frac{1}{h}\int_{\Gamma(x^0)\times(0,h)}(\frac{\partial u}{\partial\nu})^2 d\Sigma+\frac{1}{h^2}\|u(h)-u(0)\|^2_{H_0^1(\Omega)}\\ &+\frac{1}{h^2}|u'(h)-u'(0)|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{4.10}$$

Since we have

$$\begin{aligned}&\frac{1}{h}\int_{\Gamma(x^0)\times(0,h)}\left(\frac{\partial u}{\partial\nu}\right)^2 d\Sigma\\ =&\frac{1}{h}\int_{\Gamma(x^0)\times(0,h)}\left(\frac{\partial u}{\partial\nu}-\frac{1}{h}\int_0^h\frac{\partial u}{\partial\nu}dt\right)^2 d\Sigma+\frac{1}{h^2}\int_{\Gamma(x^0)}\left(\int_0^h\frac{\partial u}{\partial\nu}dt\right)^2 d\Gamma,\end{aligned} \tag{4.11}$$

it suffices to verify

$$\int_{\Gamma(x^0)}\left(\int_0^h\frac{\partial u}{\partial\nu}dt\right)^2 d\Gamma\le\epsilon h^2E(0)+C_\epsilon(\|u(h)-u(0)\|^2_{H_0^1(\Omega)}+|u'(h)-u'(0)|^2_{L^2(\Omega)}). \tag{4.12}$$

There exists (cf. Lions [1,2]) a vector field $q(x)=(q_1(x),q_2(x),...,q_n(x))\in(C^2(\bar\Omega))^n$, such that

$$q(x)=\nu(x)\qquad\text{on }\Gamma. \tag{4.13}$$

We multiply (4.5) by $q_k\int_0^h\frac{\partial u}{\partial x_k}dt(:=\sum\limits_{k=1}^n q_k\int_0^h\frac{\partial u}{\partial x_k}dt$, (we will always omit the $\sum$ notation for simplicity)

$$\begin{aligned}0=&\int_{\Omega\times(0,h)}q_k\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)(u''-\triangle u)dxdt\\ =&\int_\Omega q_k\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)u'(t)dx\Big|_0^h-\int_{\Gamma\times(0,h)}q_k\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)\frac{\partial u}{\partial\nu}d\Gamma dt\\ &+\int_{\Omega\times(0,h)}\frac{\partial q_k}{\partial x_j}\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)\frac{\partial u}{\partial x_j}dxdt+\int_{\Omega\times(0,h)}q_k\left(\int_0^h\frac{\partial^2u}{\partial x_j\partial x_k}dt\right)\frac{\partial u}{\partial x_j}dxdt.\end{aligned} \tag{4.14}$$

For the last term, we have

$$\begin{aligned}&\int_{\Omega\times(0,h)} q_k\left(\int_0^h \frac{\partial^2 u}{\partial x_j\partial x_k}dt\right)\frac{\partial u}{\partial x_j}dxdt=\int_\Omega \frac{1}{2}q_k\frac{\partial}{\partial x_k}\left(\int_0^h\frac{\partial u}{\partial x_j}dt\right)^2 dx\\ &=\int_\Omega \frac{1}{2}q_k\nu_k\left(\int_0^h\frac{\partial u}{\partial x_j}dt\right)^2dx-\int_\Omega\frac{1}{2}\frac{\partial q_k}{\partial x_k}\left(\int_0^h\frac{\partial u}{\partial x_j}dt\right)^2dx.\end{aligned}\tag{4.15}$$

Since $u=0$, on $\Gamma\times(0,h)$, we have

$$\nabla u=\frac{\partial u}{\partial\nu}\nu \qquad \text{on } \Gamma\times(0,h).\tag{4.16}$$

By combining (4.14), (4.15) and (4.16), we can see that

$$\begin{aligned}&\frac{1}{2}\int_\Gamma\left(\int_0^h\frac{\partial u}{\partial\nu}dt\right)^2 d\Gamma\\ &=\int_\Omega q_k\left(\int_0^h\frac{\partial u}{\partial x}dt\right)u'(t)dx\Big|_0^h+\int_\Omega\frac{\partial q_k}{\partial x_j}\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)\left(\int_0^h\frac{\partial u}{\partial x_j}dt\right)dx\\ &\quad-\int_\Omega\frac{1}{2}\frac{\partial q_k}{\partial x_k}\left(\int_0^h\frac{\partial u}{\partial x_j}dt\right)^2dx\\ &\le C(|u'(h)-u'(0)|^2_{L^2(\Omega)}+\int_\Omega\left(\int_0^h\frac{\partial u}{\partial x_k}dt\right)^2dx).\end{aligned}\tag{4.17}$$

But, we have

$$\Delta(\int_0^h udt)=u'(h)-u'(0).\tag{4.18}$$

Hence

$$\|\int_0^h udt\|^2_{H_0^1(\Omega)}\le C|u'(h)-u'(0)|^2_{L^2(\Omega)}.\tag{4.19}$$

Then, by combining (4.17), (4.19), we can obtain (4.12).

Corollary 4.1 (Uniqueness Theorem). *If u is a solution of (4.5), which satisfies*

$$u(h)=u(0),\quad u'(h)=u'(0)\ \ in\ \Omega,\tag{2.20}$$

$$\frac{\partial u}{\partial\nu}\quad is\ independent\ of\ t\ on\ \Gamma(x^0)\times(0,h,),\tag{4.21}$$

then

$$u\equiv 0\qquad in\ \Omega\times(0,h).\tag{4.22}$$

§5. Proof of Lemma 2.1

Firstly, we can observe that the propriety that Λ_h is an isomorphism is independent of this proof, so we can use it here.

Using this $w\in L^2(\Sigma(x^0))$ in (1.1), we can get $y_w=y(w)$, the solution of (1.1) associated with w, and

$$\left\{\frac{1}{h}\int_{T-h}^T y_w'(t)dt,\qquad \frac{1}{h}\int_{T-h}^T y_w(t)dt\right\}\in L^2(\Omega)\times H_0^1(\Omega).\tag{5.1}$$

So, there exists a unique $\{\phi^0, \phi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, such that

$$\Lambda_h\{\phi^0, \phi^1\} = \left\{\frac{1}{h}\int_{T-h}^{T} y'_w(t)dt, \qquad \frac{1}{h}\int_{T-h}^{T} y_w(t)dt\right\}. \tag{5.2}$$

We set

$$\begin{aligned}\rho = \rho(x,t) =&\rho(x,t;\phi^0,\phi^1)\\ =&\rho(\phi^0,\phi^1) = \begin{cases} \rho_1(\phi^0,\phi^1) & \text{on } \Omega\times(T-h,T),\\ \rho_2(\phi^0,\phi^1) & \text{on } \Omega\times(0,T-h).\end{cases}\end{aligned} \tag{5.3}$$

Then, we have

$$\int_Q (P-\rho)(z''-\Delta z)dxdt - \int_{\Sigma(x^0)} (w-\bar{v}(\phi^0,\phi^1))\zeta d\Sigma = 0, \tag{5.4}$$

for any ζ, z which satisfy (2.10).

We can take in (5.4)

$$\begin{aligned} z &= y_w - \psi(\phi^0,\phi^1),\\ \zeta &= w - \bar{v}(\phi^0,\phi^1).\end{aligned} \tag{5.5}$$

We can deduce that

$$w = \bar{v}(\phi^0,\phi^1) \qquad \text{on } \Sigma(x^0). \tag{5.6}$$

So we have

$$\int_Q (P-\rho)(z''-\Delta z)dxdt = 0, \tag{5.7}$$

for any z which satisfies (2.10). For any $f \in L^2(Q)$, we solve the following problem:

$$\begin{cases} z_1'' - \Delta z_1 = f & \text{in } Q,\\ z_1 = 0 & \text{on } \Sigma,\\ z_1(0) = 0, z_1'(0) = 0 & \text{on } \Omega\times\{0\}.\end{cases} \tag{5.8}$$

There exists a $\{\phi_f^0, \phi_f^1\} \in L^2(\Omega)\times H^{-1}(\Omega)$, such that

$$\Lambda_h\{\phi_f^0,\phi_f^1\} = \{\frac{1}{h}\int_{T-h}^{T} z_1'(t)dt, \quad \frac{1}{h}\int_{T-h}^{T} z_1(t)dt\}. \tag{5.9}$$

We can see that

$$z_f = z_1 - \psi(\phi^0,\phi^1) \tag{5.10}$$

verifies (2.10). By submitting z_f into (5.7), we get

$$\int_Q (P-\rho) f dxdt = 0, \tag{5.11}$$

for any $f \in L^2(Q)$. So, we obtain

$$P \equiv \rho \qquad \text{in } Q. \tag{5.12}$$

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