Chin. Ann. of Math.
14B: 4(1993),455-464.

EXISTENCE OF INDECOMPOSABLE AND SIMPLE BLOCK DESIGN $B(5,4;v)$**

DING CHANGMIN* SHEN HAO*

Abstract

It is proved that there exists an indecomposable and simple block design $B(5,4;v)$ if and only if $v \equiv 0$ or $1(\mathrm{mod}5)$ and $v \geq 6$.

Keywords Indecomposable, Simple block design, Subdesign.
1991 MR Subject Classification 05B05.

§1. Introduction

Let v, k and λ be positive integers. A balanced incomplete block design, denoted by $B(k,\lambda;v)$, is an ordered pair $(X,\mathcal{A})$, where X is a finite set containing v elements and $\mathcal{A}$ is a collection of k-subsets (called blocks) of X such that each pair of distinct elements of X is contained in exactly λ blocks.

A $B(k,\lambda;v)$ is called simple and denoted by $NB(k,\lambda;v)$ if it contains no repeated blocks. Let $(X,\mathcal{A})$ be a $B(k,\lambda;v)$. If there is a subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $(X,\mathcal{B})$ is a $B(k,\lambda_1;v)$ for some λ_1 with $1 \leq \lambda_1 < \lambda$, then $(X,\mathcal{A})$ is called decomposable. Otherwise it is called indecomposable.

The existence of simple and indecomposable designs has been widely studied. The necessary conditions for the existence of an $NB(k,\lambda;v)$ or an indecomposable $NB(k,\lambda;v)$ are

$$\begin{gathered}\lambda(v-1) \equiv 0(\mathrm{mod}(k-1)),\\ \lambda v(v-1) \equiv 0(\mathrm{mod}(k(k-1))),\\ \lambda \leq \binom{v-2}{k-2}.\end{gathered} \tag{1}$$

For $k=3$, M. Dehon[4] proved that the necessary conditions for the existence of an $NB(3,\lambda;v)$ are also sufficient. For $k=4$, it is proved in [6] that there exists an $NB(4,2;v)$ if and only if $v \equiv 1(\mathrm{mod}3)$ and $v \geq 7$ and that there exists an $NB(4,3;v)$ if and only if $v \equiv 0$ or $1(\mathrm{mod}\ 4)$ and $v \geq 5$. It is also proved in [10] that there exists an $NB(4,4;v)$ if and only if $v \equiv 1$ (mod 3), $v \geq 7$, and that there exists an $NB(4,5;v)$ if and only if $v \equiv 1$ or $4(\mathrm{mod}\ 12)$, $v \geq 13$.

For the existence of indecomposable $NB(k,\lambda;v)$, we list some known results below:

Manuscript received February 25, 1991.
*Department of Applied Mathematics, Shanghai Jiao Tong University, Shanghai 200030, China.
**Project supported by the National Natural Science Foundation of China.

(1)[5] An indecomposable $NB(3,2;v)$ exists if and only if

$$v \equiv 0 \text{ or } 1(\text{mod}3), \quad v \geq 4 \quad \text{and} \quad v \neq 7.$$

An indecomposable $NB(3,3;v)$ exists if and only if

$$v \equiv 1 \ (\text{mod}2), \quad v \geq 5.$$

(2)[8] An indecomposable $NB(3,4;v)$ exists if and only if

$$v \equiv 0 \text{ or } 1(\text{mod}3), \quad v \geq 10.$$

(3)[7] An indecomposable $NB(4,2;v)$ exists if and only if

$$v \equiv 1(\text{mod}3), \quad v \geq 7.$$

(4)[9] An indecomposable $NB(4,3;v)$ exists if and only if

$$v \equiv 0 \text{ or } 1(\text{mod}4), \quad v \geq 5.$$

(5)[10] There exists an indecomposable $NB(4,4;v)$ if and only if $v \geq 10$, $v \equiv 1(\text{mod}3)$, with six possible exceptions. There exists an indecomposable $NB(4,5;v)$ if and only if $v \equiv 1$ or $4(\text{mod } 12)$, $v \geq 13$, with four possible exceptions.

For $k \geq 5$, there is less knowing for the existence of indecomposable and simple $B(k, \lambda; v)$. The purpose of this paper is to prove the following theorem:

Theorem. *An indecomposable $NB(5,4;v)$ exists if and only if $v \equiv 0$ or $1(\text{mod}5)$, $v \geq 6$.*

§2. Recursive Constructions

In this section we will give several recursive constructions for indecomposable $NB(k, \lambda; v)$. First we give some notions which are important to our work.

A group-divisible design denoted by $GD(K, \lambda, M; v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a v-set of points, $\mathcal{G}$ is a partition of X into groups with sizes in M and $\mathcal{B}$ is a collection of subset of X (blocks) with sizes in K, such that

(i) $|B \cap G| \leq 1$ for all $B \in \mathcal{B}$ and $G \in \mathcal{G}$,

(ii) any pair of points of X from distinct groups occures in exaxtly λ blocks.

A group-divisible design $GD(K, \lambda, M; v)$ is called simple and denoted by $NGD(K, \lambda, M; v)$ if it contains no repeated blocks. We denote $GD(k, \lambda, m, v)$ for $K = \{k\}$, $M = \{m\}$. A transversal design $TD(k, \lambda, m)$ is a $GD(k, \lambda, m; v)$ with $v = km$.

Let K be a set of positive integers. A pairwise balanced design $B(K, 1; v)$ is a pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a collection of subsets (called blocks) of V such that $|B| \in K$ for every $B \in \mathcal{B}$ and each pair of distinct elements is contained in a unique block.

Let $(X, \mathcal{A})$ be a $B(k, \lambda; v)$, $(Y, \mathcal{B})$ be a $B(k, \lambda; u)$. If $v < u$, $X \subset Y$ and $\mathcal{A} \subset \mathcal{B}$, then we say that $(X, \mathcal{A})$ is a subdesign of $(Y, \mathcal{B})$.

The following lemma is obviously true.

Lemma 2.1. *If a $B(k, \lambda; v)$ contains an indecomposable $B(k, \lambda; v)$ as a subdesign, then the $B(k, \lambda; v)$ is also indecomposable.*

We suppose that the reader is familiar with the concepts of combinatorial theory such as difference families, mutually orthogonal Latin squares. Let K be a set of positive integers. Let

$$B(K) = \{v : \text{there is a } B(K, 1; v)\}.$$

If $B(K) = K$, then K is called PBD-closed.

Let

$$NB(5,4) = \{v : \text{there is a simple } B(5,4;v)\}.$$

and

$$INB(5,4) = \{v : \text{there is an indecomposable } B(5,4;v)\}.$$

Then we have the following theorem.

Theorem 2.1. (i) $NB(5,4)$ *is* PBD-*closed.*

(ii) $INB(5,4)$ *is* PBD-*closed.*

Proof. (i) Let $K = NB(5,4)$. We only need to prove $B(K) \subset K$.

Let $(X, \mathcal{B})$ be a $B(K,1;v)$. For each block $B \in \mathcal{B}$, $|B| = k \in K$, we can construct a simple $B(5,4;k)$, denoted by $(B, \mathcal{B}_B)$. Let $\mathcal{A} = \bigcup_{B\in\mathcal{B}} \mathcal{B}_B$. We get an $NB(5,4;v)$. So $v \in K$; then $NB(5,4)$ is PBD-Closed.

(ii) can be proved similarly.

The following lemmas can be found in [1].

Lemma 2.2. *Let m,n,k,λ be integers, S,R be two sets of integers. If $n \in GD(S,1,R)$, $mR \subset B(k,\lambda)$ and $mS \subset GD(k,\lambda,m)$, then $mn \in B(k,\lambda)$.*

Lemma 2.3. *If $n \in GD(S,1,R)$, $mR+1 \subset B(k,\lambda)$ and $mS \subset GD(k,\lambda,m)$, then $mn+1 \in B(k,\lambda)$.*

We improve these results and obtain the following two results:

Theorem 2.2. *Let S,R be two sets of integers, m,n,k,λ be integers. If $n \in GD(S,1,R)$, $mR \subset NB(k,\lambda)$, $mS \subset NGD(k,\lambda,m)$, then there is an $NB(k,la;mn)$ containing a subdesign $NB(k,\lambda;mv)$ for some $v \in R$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be a $GD(S,1,R;n)$ where $X = \{x, \cdots, z\}$. Give each $x \in X$ weight m, $x = \{x_1, \cdots, x_m\}$. Since $mR \subset NB(k,\lambda)$, construct an $NB(k,\lambda;m|G|)$ denoted by $(G, \mathcal{B}_G)$ for each $G \in \mathcal{G}$. Let

$$\mathcal{B}_1 := \bigcup_{G\in\mathcal{G}} \mathcal{B}_G.$$

Since $mS \subset NGD(k,\lambda,m)$, construct an $NGD(k,\lambda,m;m|B|)$ denoted by $(B, \mathcal{G}', \mathcal{B}_B)$ on each $B \in \mathcal{B}$ with groups $\{\{x_1, \cdots, x_m\} : x \in B\}$. Let

$$\mathcal{B}_2 := \bigcup_{B\in\mathcal{B}} \mathcal{B}_B.$$

Let $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$. It is obvious that there are no repeated blocks in $\mathcal{A}$. So $(X, \mathcal{A})$ is an $NB(k,\lambda;mn)$ containing a subdesign $NB(k,\lambda;m|G|)$.

Similarly we have the following theorem.

Theorem 2.3. *If $n \in GD(S,1,R)$, $mR+1 \subset NB(k,\lambda)$, $mS \subset NGD(k,\lambda;m)$, then there exists an $NB(k,\lambda;mn+1)$ containing a subdesign $NB(k,\lambda;mv+1)$ for some $v \in R$.*

The following theorem can be found in [1].

Theorem 2.4. *For all $u \geq 2$, there exists a group-divisible design $GD(\{5,6,7\},1,M_5;u)$ where*

$$M_5 = \{2,3,\cdots,24,26,31,32,33,34,36\}.$$

Thus, by the above three theorems, to prove our maim theorem, it is sufficient to prove the existence of an indecomposable $NB(5,4;v)$ for each $v \in 5M_5$ or $v \in 5M_5+1$.

§3. Proof of the Main Theorem

In this section we will determine the existence of indecomposable $NB(5,4;v)s$.

Lemma 3.1. *If there exists an $NB(5,4;v)$ and $v \equiv 0,6,10$ or 16 (mod 20), then the design is indecomposable.*

Proof. For $v \equiv 0,6,10$ or 16(mod 20), by (1) there does not exist an $NB(5,\lambda;v)$ for $1 \le \lambda \le 3$; so for such v's, any $NB(5,4;v)$ must be indecomposable.

Lemma 3.2. *There exists an indecomposable $NB(5,4;v)$ for*

$$v \in \{6,10,11,15,16,21,20,25,26,35,40,41,45,50,51,56,60,61,70,71,81,86,95\}.$$

Proof. Taking all the 5-subsets of a 6-set as blocks gives an $NB(5,4;6)$. For $v = 10,16,20$, the design $B(5,4;v)$s are constructed in [1]. We form a $B(5,4;v)$ for each of the remaining v's in the Appendix. Checking these designs, we know that they are simple, and that the design $B(5,4;v)$s for $v = 11,21,25,35$ are indecomposable. Then by Lemma 3.1 and Lemma 2.1, they are also indecomposable.

Lemma 3.3. *If there is a $TD(5,v)$, $v > 6$, then there is an $NTD_4(5,v)$.*

Proof. Let $(X,\mathcal{G},\mathcal{B})$ be a $TD(5,v)$ where $\mathcal{G} = \{A,B,C,D,E\}$ and $A = Z_5$.

Let

$$\mathcal{B}_i = \{(a+i,b,c,d,e) : (a,b,c,d,e) \in \mathcal{B}\}, i = 1,2,3.$$

Then $(X,\mathcal{G},\mathcal{B}\cup\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}3)$ is an $NTD_4(5,v)$.

Lemma 3.4. (i) *If $v \in NB(5,4)$, $N(v) \ge 3$, then $5v \in NB(5,4)$.*

(ii) *If $v+1 \in NB(5,4)$, $N(v) \ge 3$, then $5v+1 \in NB(5,4)$.*

Proof. (i) Since $N(v) \ge 3$, we have $v \in TD(5)$. By Lemma 3.3, $v \in NTD_4(5)$. Let $(X,\mathcal{G},\mathcal{B})$ be the $NTD_4(v)$, construct an $NB(5,4;v)$ denoted by $(G,\mathcal{B}_G)$ on each $G \in \mathcal{G}$, and let $\mathcal{A} := \bigcup_{G\in\mathcal{G}} \mathcal{B}_G$. Then $(X,\mathcal{B}\cup\mathcal{A})$ be an $NB(5,4;5v)$. So $5v \in NB(5,4)$.

(ii) can be proved similarly.

Lemma 3.5. *There exists an indecomposable $NB(5,4;v)$ for*

$$v \in \{46,55,75,76,80,100,101,105,130,155\}.$$

Proof. Apply Lemma 3.4 with $v = 9,11,15,16,20,21,26,31$, then we get the result by Lemmas 3.2 and 2.1.

Lemms 3.6. *There exists an indecomposable $NB(5,4;v)$ for*

$$v \in \{66,90,96,110,111,115,116,120,121,160,161,166,170,171\}.$$

Proof. If there exists a $GD(K,1,M;v)$, $k \subset INB(5,4)$ and $M \subset INB(5,4)$, then by Theorem 2.1 $v \in INB(5,4)$. For each v, we construct a $GD(K,1,M;v)$ in Table 1 such that $K \subset INB(5,4)$ and $M \subset INB(5,4)$; the conclusion then follows.

Lemma 3.7. $\{31,91,156,181\} \in INB(5,4)$.

Proof. $31,91,156,18 \in B(6)$ (see [14]). The conclusion then follows from Theorem 2.1.

Table 1:

v	K	M	Remark
66	6	11	$N(11) \geq 4$ $TD(6,11)$
90	6	15	$N(15) \geq 4$ $TD(6,15)$
96	6	16	$N(16) \geq 4$ $TD(6,16)$
110	10	11	$N(11) \geq 8$ $TD(10,11)$
111	10,11	11,1	Delete ten points from a group of the $TD(11,11)$
116	10,11	11,6	Delete five points from a group of the $TD(11,11)$
120	10,11	11,10	Delete one point from a group of the $TD(11,11)$
121	11	11	$N(11) \geq 9$ $TD(11,11)$
160	10	16	$N(16) \geq 8$ $TD(10,16)$
161	10,11	16,1	Delete fifteen points from a group of the $TD(11,16)$
166	10,11	16,6	Delete ten points from a group of the $TD(11,16)$
170	10,11	16,10	Delete six points from a group of the $TD(11,16)$
171	10,11	16,11	Delete five points from a group of the $TD(11,16)$
115			Since $19 \in TD(6)$, then $115 \in B(6,20)$. By Theorem 2.1, 115 $INB(5,4)$

Theorem 3.1. *If there exist a $B(5,1;u)$ and a simple $B(5,4;v)$ containing an indecomposable $NB(5,4;w)$ as a subdesign, and $N(v-w) \geq 3$, then*

$$w + u(v-w) \in INB(5,4).$$

Proof. Let $(X_1, \mathcal{B})$ be a $B(5,1;u)$ and give each point $x \in X_1$ weight $(v-w)$. Since $N(v-w) \geq 3$, by Lemma 3.3 we can construct an $NTD_4(5, v-w)$ on each $B \in \mathcal{B}$, with block set $\mathcal{B}_B$, group set $\mathcal{G} = \{\{x_1, \cdots, x_{v-w}\} : x \in B\}$. Let

$$\mathcal{A}_1 := \bigcup_{B \in \mathcal{B}} \mathcal{B}_B.$$

Let $X_0 = \{\infty_1, \cdots, \infty_w\}$ such that $X_1 \cap X_0 = \emptyset_0$. For each $G \in \mathcal{G}$, let $(G \cup X_0, \mathcal{B}_G)$ be an $NB(5,4;v)$ on $G \cup X_0$ containing $(X_0, \mathcal{A}_0)$ as a subdesign. Let

$$\mathcal{A}_2 := \bigcup_{G \in \mathcal{G}} (\mathcal{B}_G \setminus \mathcal{A}_0).$$

Let

$$X = (v-w)X_1 \cup X_0, \ \mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_0.$$

Then $(X, \mathcal{A})$ is an $NB(5,4; u(v-w)+w)$ and by Lemma 2.1 it is indecomposable.

Lemma 3.8. $\{106, 131\} \subset INB(5,4)$.

Proof. For 106, apply Theorem 3.1 with $u = 21, v = 6, w = 1$. For 131, apply Theorem 3.1 with $u = 5, v = 31, w = 6$.

Difference families is a useful tool in the construction of design (see [13]). Let $S = (B_1, \cdots, B_s)$ be a family of subsets on a finite group $G = V$ with $|B_i| \in K$. Then S is a

(v, K, λ)-difference family if and only if each $x \in G \setminus \{0\}$ occures exactly λ times in the list of differences $b_{ij} - b_{il}$ $(i \in \{1, \cdots, s\},\ j \neq l)$.

Lemma 3.9. *Let $S = (B_1, \cdots, B_l)$ be a $B(4,4,v)$-difference family, and $G_{B_i} = 1$ for $i = 1, \cdots, l$, and there exists an indecomposable $NB(5,4;l)$, then there exists an indecomposable $NB(5,4)$.*

Proof. Since S is a $(4,4,v)$-difference family and $G_{B_i} = 1$ for $i = 1, \cdots, l$, the $B(4,4;v)$ has l 4-factors denoted by P_i, $i = 1, \cdots, l$. Let $B'_{ij} = B_{ij} \cup \{\infty_i\}$ for each $B_{ij} \in P_i$, $i = 1, \cdots, l$; and let $(X_0, \mathcal{A}_0)$ be an indecomposable $NB(5,4;l)$ where $X_0 = \bigcup_{i=1}^{I} \{\infty_i\}$. Let

$$\mathcal{B} := \bigcup_{i=1}^{I} (\bigcup_{j=1}^{v} B'_{ij}) \cup \mathcal{A}_0.$$

Then $(X \cup X_0, \mathcal{B})$ is an indecomposable $NB(5,4;v+1)$ by Lemma 2.1.

R. M. Wilson[11] proved the following lemmas in 1972.

Lemma 3.10. *If $q \in P^*$ and if k or $k-1$ divides 2λ, then the necessary existence condition $\lambda(q-1) \equiv 0(\mathrm{mod} k(k-1))$ for a (q,k,λ)-difference family is sufficient too.*

Lemma 3.11. *Let k be the order of an affine plane. If there exists a (v,k,λ)-difference family in the group G and if there exists a (v',k',λ')-difference family in the group G', then there exists a $(vv', kk', \lambda\lambda')$-difference family in $G \oplus G'$.*

Now we can prove the following lemma.

Lemms 3.12. $\{65, 68, 165\} \subset INB(5,4)$.

Proof. For $v = 49, 64, 124$, there exists a $(4,4,v)$-difference family by Lemma 3.10 and Lemma 3.11. Then by Lemma 3.9, $\{65, 85, 165\} \subset INB(5,4)$.

Theorem 3.2. $\{25, 30, 35\} \subset NGD(5,4,5)$.

Proof. Since there exists a $GD(6,1,5;30)$, constructing an $NB(5,4;6)$ on each block, we get an $NGD(5,4,5;30)$.

For $v = 25, 35$, the following two designs can be found in [1] and here we still use the notation in [1].

$$\begin{aligned} NGD(5,2,5;25): X =& Z(5,2) \times GF(5, f(x) = 0) \\ \mathcal{B} =& \{(\phi,\phi), (0,0), (0,2), (2,1), (2,3)\} \\ & \{(\phi,\phi), (0,1), (0,3), (2,2), (2,4)\}\ (\mathrm{mod}(5,5)). \end{aligned}$$

$$\begin{aligned} NGD(5,2,5;35): X =& Z(5,2) \times GF(7, f(x) = 0) \\ \mathcal{B} =& \{(\phi,\phi), (0,0), (0,3), (2,1), (2,4)\} \\ & \{(\phi,\phi), (0,1), (0,4), (2,2), (2,5)\} \\ & \{(\phi,\phi), (0,2), (0,5), (2,3), (2,6)\}\ (\mathrm{mod}(5,7)). \end{aligned}$$

Let α, γ, φ be three functions defined below:

$$\alpha(a,i) = \begin{cases} (a+1,\ i), & i = 1, \\ (a,\ i), & i \neq 1, \end{cases}$$

$$\gamma(a,i) = \begin{cases} (a+1,\ i), & i = 2, \\ (a,\ i), & i \neq 2, \end{cases}$$

$$\varphi(a,i)=\begin{cases}(a+1,\ i), & i=3,\\ (a,\ i), & i\neq 3.\end{cases}$$

Let $\mathcal{A}:=\{B(\alpha\gamma\varphi):B\in\mathcal{B}\}$, then $(X,\mathcal{G},\mathcal{B}\cup\mathcal{A})$ is an $NGD(5,4,5;v)$ for $v=25,35$.

Corollary 3.1. $\{36,180\}\subset INB(5,4)$.

Proof. Adding an infinite point to each group G of an $NGD(5,4,5;35)$, then form an indecomposable $NB(5,4;6)$ on each $G\cup\{\infty\}$, we get an indecomposable $NB(5,4;36)$. By Lemma 3.4, $180\in INB(5,4)$.

From the above constructions, we have proved the following lemma:

Lemma 3.13. *For $v\in 5M_5$ or $v\in 5M_5+1$, there exists an indecomposable $NB(5,4;v)$.*

Combining Theorem 2.2, Theorem 2.3, Theorem 2.4, Lemma 3.13 and Theorem 3.2 gives our main theorem:

Theorem 3.3. *There exists an indecomposable $NB(5,4;v)$ if and only if $v\equiv 0$ or $1(\text{mod}5)$, $v\geq 6$.*

Appendix

v	X	
11	Z_{11}	$\{0,1,2,3,6\}$, $\{0,1,4,6,8\}$ (mod11)
15	$Z_{14}\cup\{\infty\}$	$\{0,1,3,5,7\}$, $\{0,1,4,6,7\}$, $\{0,1,4,9,\infty\}$ (mod14)
21	Z_{21}	$\{0,1,3,7,15\}$, $\{0,1,3,9,11\}$, $\{0,1,5,10,17\}$, $\{0,1,4,14,19\}$ (mod21)
25	$Z_{24}\cup\{\infty\}$	$\{0,1,4,10,20\}$, $\{0,1,3,9,13\}$, $\{0,1,3,8,17\}$, $\{0,1,3,7,12\}$, $\{0,2,7,13,\infty\}$ (mod26)
26	Z_{26}	$\{0,1,3,9,13\}$, $\{0,1,5,15,24\}$, $\{0,1,6,9,20\}$, $\{0,1,4,9,11\}$, $\{0,2,6,13,18\}$ (mod26)
30	$Z_{29}\cup\{\infty\}$	$\{0,1,3,9,13\}$, $\{0,1,3,8,15\}$, $\{0,1,4,9,19\}$ $\{0,1,6,9,20\}$, $\{0,2,6,13,18\}$, $\{0,2,6,13,\infty\}$ (mod29)
35	$Z_{34}\cup\{\infty\}$	$\{0,1,7,12,16\}$, $\{0,1,12,19,32\}$, $\{0,1,8,13,32\}$, $\{0,1,6,9,16\}$, $\{0,2,12,16,\infty\}$, $\{0,4,8,10,21\}$, $\{0,5,8,14,25\}$ (mod34)
40	$Z_{34}\cup\{\infty_1,\cdots,\infty_6\}$	$\{0,1,12,19,32\}$, $\{0,1,6,9,16\}$, $\{0,4,8,10,21\}$, $\{0,1,10,26,\infty_1\}$, $\{0,1,8,12,\infty_2\}$, $\{0,2,14,17,\infty_3\}$, $\{0,2,5,16,\infty_4\}$, $\{0,4,9,14,\infty_5\}$, $\{0,6,12,27,\infty_6\}$ (mod34) and the block of $NB(5,4;6)$ on $\bigcup_{i=1}^{6}\{\infty_i\}$
41	$Z_{31}\cup\{\infty_1,\cdots,\infty_{10}\}$	$\{0,1,3,17,\infty_1\}$, $\{0,1,5,24,\infty_2\}$, $\{0,1,7,11,\infty_3\}$, $\{0,1,9,22,\infty_4\}$, $\{0,2,6,26,\infty_5\}$, $\{0,2,5,18,\infty_6\}$, $\{0,2,12,16,\infty_7\}$, $\{0,3,11,25,\infty_8\}$, $\{0,3,9,19,\infty_9\}$, $\{0,5,13,24,\infty_{10}\}$ (mod31) and the block of $NB(5,4;10)$ on $\bigcup_{i=1}^{10}\{\infty_i\}$
45	$Z_{34}\cup\{\infty_1,\cdots,\infty_{11}\}$	$\{0,1,7,17,\infty_1\}$, $\{0,1,3,15,\infty_5\}$, $\{0,1,5,11,\infty_9\}$, $\{0,1,8,10,\infty_2\}$, $\{0,2,10,21,\infty_6\}$, $\{0,2,7,16,\infty_{10}\}$, $\{0,3,17,21,\infty_3\}$, $\{0,6,13,25,\infty_7\}$, $\{0,3,11,15,\infty_{11}\}$, $\{0,5,9,21,\infty_4\}$, $\{0,6,11,14,\infty_8\}$ (mod34) and the block of $NB(5,4;11)$ on $\bigcup_{i=1}^{11}\{\infty_i\}$

v	X	
50	$Z_{39}\cup\{\infty_1,\cdots,\infty_{11}\}$	$\{0,1,4,9,19\}$, $\{0,1,7,19,\infty_1\}$, $\{0,1,14,23,\infty_2\}$, $\{0,1,11,19,\infty_3\}$, $\{0,2,7,19,\infty_4\}$, $\{0,2,6,13,\infty_5\}$, $\{0,2,14,24,\infty_6\}$, $\{0,2,7,15,\infty_7\}$, $\{0,3,9,17,\infty_8\}$, $\{0,3,16,31,\infty_9\}$, $\{0,3,12,16,\infty_{10}\}$, $\{0,4,10,15,\infty_{11}\}$ (mod39) and the block of $NB(5,4;11)$ on $\bigcup_{i=1}^{11}\{\infty_i\}$
51	$Z_{41}\cup\{\infty_1,\cdots,\infty_{10}\}$	$\{0,1,5,11,20\}$, $\{0,1,9,28,\infty_2\}$, $\{0,1,14,35,\infty_1\}$, $\{0,1,9,28,\infty_2\}$, $\{0,2,7,18,\infty_3\}$, $\{0,2,12,26,\infty_4\}$, $\{0,2,18,31,\infty_5\}$, $\{0,2,17,37,\infty_6\}$, $\{0,3,7,18,\infty_7\}$, $\{0,3,12,17,\infty_8\}$, $\{0,3,13,35,\infty_9\}$, $\{0,5,8,16,\infty_{10}\}$ (mod41) and the block of $NB(5,4;10)$ on $\bigcup_{i=1}^{10}\{\infty_i\}$
56	$Z_{45}\cup\{\infty_1,\cdots,\infty_{11}\}$	$\{0,9,18,27,36\}$, $\{0,1,5,12,22\}$, $\{0,1,3,9,32\}$, $\{0,1,6,20,\infty_1\}$, $\{0,1,8,23,\infty_2\}$, $\{0,2,12,28,\infty_3,\}$ $\{0,2,13,21,\infty_4\}$, $\{0,2,6,20,\infty_5\}$, $\{0,3,9,21,\infty_6\}$, $\{0,3,10,21,\infty_7\}$, $\{0,3,15,28,\infty_8\}$, $\{0,4,9,38,\infty_9\}$, $\{0,4,14,30,\infty_{10}\}$, $\{0,5,13,30,\infty_{11}\}$ (mod45) and the block of $NB(5,4;11)$ on $\bigcup_{i=1}^{11}\{\infty_i\}$
60	$Z_{49}\cup\{\infty_1,\cdots,\infty_{11}\}$	$\{0,1,3,15,24\}$, $\{0,1,6,13,24\}$, $\{0,1,8,17,21\}$, $\{0,1,10,25,\infty_1\}$, $\{0,2,8,30,\infty_2\}$, $\{0,2,6,31,\infty_3\}$, $\{0,3,13,30,\infty_4\}$, $\{0,3,10,21,\infty_5\}$, $\{0,3,17,23,\infty_6\}$, $\{0,4,16,30,\infty_7\}$, $\{0,5,7,22,\infty_8\}$, $\{0,5,13,43,\infty_9\}$, $\{0,5,9,38,\infty_{10}\}$, $\{0,8,12,39,\infty_{11}\}$ (mod49) and the block of $NB(5,4;11)$ on $\bigcup_{i=1}^{11}\{\infty_i\}$
61	$Z_{46}\cup\{\infty_1,\cdots,\infty_{15}\}$	$\{0,1,3,,24,\infty_1\}$, $\{0,1,5,35,\infty_2\}$, $\{0,1,13,29,\infty_3\}$, $\{0,1,11,38,\infty_4\}$, $\{0,2,7,23,\infty_5\}$, $\{0,2,8,22,\infty_6\}$, $\{0,2,17,35,\infty_7\}$, $\{0,3,7,34,\infty_8\}$, $\{0,3,9,21,\infty_9\}$, $\{0,3,17,27,\infty_{10}\}$, $\{0,4,9,33,\infty_{11}\}$, $\{0,4,10,18,\infty_{12}\}$, $\{0,5,20,31,\infty_{13}\}$, $\{0,6,13,38,\infty_{14}\}$, $\{0,7,16,26,\infty_{15}\}$ (mod46)and the block of $NB(5,4;15)$ on $\bigcup_{i=1}^{15}\{\infty_i\}$
70	$Z_{54}\cup\{\infty_1,\cdots,\infty_{16}\}$	$\{0,1,6,13,27\}$, $\{0,1,9,27,\infty_1\}$, $\{0,1,11,20,\infty_2\}$ $\{0,1,7,32,\infty_3\}$, $\{0,2,15,22,\infty_4\}$, $\{0,2,10,21,\infty_5\}$, $\{0,2,8,25,\infty_6\}$, $\{0,2,14,18,\infty_7\}$, $\{0,3,20,24,\infty_8\}$, $\{0,3,26,32,\infty_9\}$, $\{0,3,8,19,\infty_{10}\}$, $\{0,3,10,24,\infty_{11}\}$, $\{0,4,13,43,\infty_{12}\}$, $\{0,4,16,26,\infty_{13}\}$, $\{0,5,18,25,\infty_{14}\}$, $\{0,6,23,42,\infty_{15}\}$, $\{0,9,14,39,\infty_{16}\}$ (mod54) and the block of $NB(5,4;16)$ on $\bigcup_{i=1}^{16}\{\infty_i\}$
71	$Z_{56}\cup\{\infty_1,\cdots,\infty_{15}\}$	$\{0,1,5,17,28\}$, $\{0,1,7,20,28\}$, $\{0,1,10,26,\infty_1\}$, $\{0,1,15,41,\infty_2\}$, $\{0,2,8,26,\infty_3\}$, $\{0,2,7,25,\infty_4\}$, $\{0,2,11,36,\infty_5\}$, $\{0,2,10,44,\infty_6\}$, $\{0,3,7,24,\infty_7\}$, $\{0,3,19,36,\infty_8\}$, $\{0,3,24,37,\infty_9\}$, $\{0,3,12,18,\infty_{10}\}$, $\{0,4,14,27,\infty_{11}\}$, $\{0,4,11,26,\infty_{12}\}$, $\{0,5,13,25,\infty_{13}\}$, $\{0,5,14,24,\infty_{14}\}$, $\{0,6,27,45,\infty_{15}\}$ (mod56) and the block of $NB(5,4;15)$ on $\bigcup_{i=1}^{15}\{\infty_i\}$

v	X	
81	$Z_{61}\cup\{\infty_1,\cdots,\infty_{20}\}$	$\{0,1,6,31,\infty_1\},\{0,1,10,30,\infty_2\}$, $\{0,1,19,30,\infty_3\}$, $\{0,1,12,44,\infty_4\}$, $\{0,2,5,27,\infty_5\}$, $\{0,2,8,23,\infty_6\}$, $\{0,2,7,28,\infty_7\}$, $\{0,2,16,37,\infty_8\}$, $\{0,3,20,27,\infty_9\}$, $\{0,3,16,39,\infty_{10}\}$, $\{0,3,14,49,\infty_{11}\}$, $\{0,4,20,28,\infty_{12}\}$, $\{0,4,12,27,\infty_{13}\}$, $\{0,4,13,22,\infty_{14}\}$, $\{0,4,14,21,\infty_{15}\}$, $\{0,5,20,28,\infty_{16}\}$, $\{0,6,14,27,\infty_{17}\}$, $\{0,7,19,44,\infty_{18}\}$, $\{0,9,20,48,\infty_{19}\}$, $\{0,10,26,42,\infty_{20}\}$ (mod61) and the block of $NB(5,4;20)$ on $\bigcup_{i=1}^{20}\{\infty_i\}$
86	$Z_{65}\cup\{\infty_1,\cdots,\infty_{21}\}$	$\{0,13,26,39,52\},\{0,1,15,32,\infty_1\}$, $\{0,1,13,32,\infty_2\}$, $\{0,1,9,32,\infty_3\}$, $\{0,1,21,55,\infty_4\}$, $\{0,2,26,30,\infty_5\}$, $\{0,2,18,38,\infty_6\}$, $\{0,5,7,27,\infty_7\}$, $\{0,2,12,30,\infty_8\}$, $\{0,3,11,27,\infty_9\},\{0,3,9,25,\infty_{10}\},\{0,3,17,24,\infty_{11}\}$, $\{0,3,22,42,\infty_{12}\}$, $\{0,4,15,59,\infty_{13}\}$, $\{0,4,17,29,\infty_{14}\}$, $\{0,4,27,57,\infty_{15}\}$, $\{0,5,18,47,\infty_{16}\}$, $\{0,5,15,29,\infty_{17}\}$, $\{0,5,21,30,\infty_{18}\}$, $\{0,6,15,43,\infty_{19}\}$, $\{0,6,17,25,\infty_{20}\}$, $\{0,7,14,33,\infty_{21}\}$ (mod65) and the block of $NB(5,4;21)$ on $\bigcup_{i=1}^{21}\{\infty_i\}$
95	$Z_{74}\cup\{\infty_1,\cdots,\infty_{21}\}$	$\{0,1,11,23,27\},\{0,1,16,30,37\}$, $\{0,1,35,66,\infty_1\}$, $\{0,1,3,18,\infty_2\}$, $\{0,2,21,41,\infty_3\}$, $\{0,2,17,47,\infty_4\}$, $\{0,2,26,66,\infty_5\}$, $\{0,2,20,43,\infty_6\}$, $\{0,3,9,28,\infty_7\}$, $\{0,3,19,32,\infty_8\},\{0,3,20,41,\infty_9\},\{0,3,27,41,\infty_{10}\}$, $\{0,4,16,35,\infty_{11}\}$, $\{0,4,30,53,\infty_{12}\}$, $\{0,5,11,51,\infty_{13}\}$, $\{0,5,32,54,\infty_{14}\}$, $\{0,4,32,64,\infty_{15}\}$, $\{0,4,13,29,\infty_{16}\}$, $\{0,5,12,20,\infty_{17}\}$, $\{0,6,17,24,\infty_{18}\}$, $\{0,6,13,21,\infty_{19}\}$, $\{0,9,31,48,\infty_{20}\}$, $\{0,10,28,50,\infty_{21}\}$ (mod74) and the block of $NB(5,4;21)$ on $\bigcup_{i=1}^{21}\{\infty_i\}$

REFERENCES

[1] Hanani, Balanced incomplete block designs and releted designs, *Discrete Math.*, **11** (1975), 255-369.

[2] Van Buggenhaut, J., On the existence of 2-designs $B(2,3;v)$ without repeated blocks, *Discrete Math.*, **8** (1972), 105-109.

[3] Van Buggenhaut, J., Existence and construction of 2-designs $B(2,3;v)$ without repeated blocks, *J. of Geometry*, **4** (1974), 1-10.

[4] Dehon, M., On the existence of 2-designs $B(2,3;v)$ without repeated blocks, *Discrete Math.*, **43** (1983), 155-171.

[5] Kramer, E., Indecomposable triple systems, *Discrete Math.*, **8** (1974), 173-180.

[6] Guo, H., On the existence of simple $B(4,2;v)$ and simple $B(4,3;v)$, *J. of Shanghai Jiao Tong Univ.*, 2 (1990).

[7] Phelps, K. T. & Rosa, A., Recursive constructions and some properties of two fold designs with block size four, *J. Austral. Math. Soc.*, **44A** (1988), 64-70.

[8] Colbourn, C. J. & Rosa, A., Indecomposable triple systems with $\lambda = 4$, *Studia Sci. Math. Hungarica*, **20** (1985), 139-144.

[9] Guo, H. & Sheng, H., On the existence of simple and indecomposable $B(4,3;v)$ (submitted).
[10] Shen, H., On the existence of simple and indecomposable block designs with block size four, *J. Combinatorial Math. and Combinatorial Computing*, **8**, 31-38.
[11] Wilson, R. M., Cyclotomy and difference families in elementery Abelian groups, *J. Number Th.*, **4**, 17-42.
[12] Jungnikel, D., Composition theorems for difference families and regular plances, *Dicr. Math.*, **23**, 151-158.
[13] Beth, T., Jungnickel, D. & Lenz, H., Design theory, Bibliogmphisches Institute, 1985.
[14] Mulin, R. C., Finite bases for some PBD-closed sets, *Discrete Math.*, **77**, 217-236.