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NONLINEAR DIFFUSIVE PHENOMENA OF NONLINEAR HYPERBOLIC SYSTEMS***

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Abstract

The authors study the nonlinear hyperbolic system which describes the motion of isentropic gas flow with external friction acting on it, such as a flow through porous media, and show the nonlinear diffusive phenomena for the large time behavior of solutions for this system by proving that the solutions tend to those of a nonlinear diffusion equation time-asymptotically.

Keywords Nonlinear hyperbolic systems, Nonlinear diffusive phenomena, Isentropic gas flow.

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§1. Introduction

We study the large time behavior of solutions for the following system

$$\begin{cases} \dfrac{\partial \rho}{\delta t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \\ \dfrac{\partial(\rho u)}{\partial t} + \dfrac{\partial(\rho u^2 + p(\rho))}{\partial x} = -\alpha \rho u, \end{cases} \tag{1.1}$$

which describes the motion of isentropic gas flow with external friction acting on it, such as flow through porous media. Here the friction coefficient α is a positive constant and the function $p(\rho)$ satisfies the condition

$$p'(\rho) > 0 \quad \text{for} \quad 0 < \rho < \infty \tag{1.2}$$

under which the system (1.1) is hyperbolic.

Ignoring the convection term in $(1.1)_2$, one obtains

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ p(\rho)_x = -\alpha \rho u, \end{cases} \tag{1.3}$$

which can be rewritten as a nonlinear diffusion equation, the porous media equation for ρ plus a decoupled equation for u, namely,

$$\begin{cases} \rho_t = \dfrac{1}{\alpha} \dfrac{\partial^2 p(\rho)}{\partial x^2}, \\ u = -\dfrac{1}{\alpha \rho} \dfrac{\partial p(\rho)}{\partial x}. \end{cases} \tag{1.4}$$

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One would expect that the system (1.1) is accurately approximated by (1.4) time-asymptotically since the convection terms in $(1.1)_2$ are small time-asymptotically compared to the terms in $(1.3)_2$ for similarity solution of $(1.4)_1$ with u defined by $(1.4)_2$.

It would be interesting to prove the expectation since it shows certain relations between the theory of nonlinear hyperbolic equations and nonlinear diffusive phenomena. Moreover, the simplified nonlinear diffusion equation has been understood much better than the original nonlinear hyperbolic system.

This expectation was proved rigorously in [1] with the systems in Lagrange coordinate which are equivalent to (1.1) and (1.3) respectively for $\rho > 0$. However, for dealing with more general situation when vacuum may occur one has to consider the system (1.1) and (1.3) directly in Euler coordinate.

For the first step, we prove the above expectation with Euler coordinate for the case when the initial data $(\rho_0(x), u_0(x))$ satisfy the following condition in this paper.

$$\rho_0(x) > 0 \text{ with } \lim_{x\to\mp\infty} \rho_0(x) = \rho_\mp, \quad \lim_{x\to\mp\infty} u_0(x) = u_\mp. \tag{1.5}$$

For comparing the solution of (1.1) and (1.3) we use variables (ρ, m) instead of (ρ, u) where $m = \rho u$, in which the system (1.1) and (1.3) becomes

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\dfrac{m^2}{\rho} + p(\rho))_x + \alpha m = 0 \end{cases} \tag{1.6}$$

and

$$\begin{cases} \rho_t + m_x = 0, \\ p(\rho)_x + \alpha m = 0 \end{cases} \tag{1.7}$$

or

$$\begin{cases} \rho_t = \dfrac{1}{\alpha}\dfrac{\partial^2 p(\rho)}{\partial x^2}, \\ m = \dfrac{-1}{\alpha}\dfrac{\partial p(\rho)}{\partial x}. \end{cases} \tag{1.8}$$

It is known that there exists a similarity solution $\rho^*(\eta), \eta = \frac{x}{\sqrt{t+1}}$ for $(1.8)_1$, satisfying the boundary condition $\rho^*(\eta) = \rho_\mp$ as $\eta \to \mp\infty$ under certain condition on $p(\rho)$ (see [2]). Furthermore, we can define a constant x_0 by the equation

$$\int_{-\infty}^{\infty} (\rho_0(x) - \rho^*(x + x_0))dx = \frac{(\rho_+ u_+ - \rho_- u_-)}{\alpha}, \tag{1.9}$$

and then obtain the similarity solution $\rho^*(\frac{x+x_0}{\sqrt{t+1}})$ for $(1.8)_1$, namely $\rho^*(\frac{x+x_0}{\sqrt{t+1}})$ satisfies the equation as follows

$$\begin{cases} \rho^*_t = \dfrac{1}{\alpha}\dfrac{\partial^2 p(\rho^*)}{\partial x^2}, & (1.10) \\ \rho^*|_{t=0} = \rho^*(x + x_0), & (1.11) \end{cases}$$

where $\rho^*(x + x_0)$ satisfies the condition (1.9).

Define

$$m^* = \frac{-1}{\alpha} p(\rho^*)_x. \tag{1.12}$$

We will compare the solution (ρ, m) for (1.6) with the initial condition

$$\rho(x, 0) = \rho_0(x), \quad m(x, 0) = m_0(x) = \rho_0(x)u_0(x) \tag{1.13}$$

to the functions $\rho^* + \widehat{\rho}(x,t), m^* + \widehat{m}(x,t)$ where the functions $\widehat{\rho}$ and $\widehat{m}$ are defined as follows

$$\widehat{\rho}(x,t) = \widetilde{\rho}(x) \cdot \frac{(\rho_+ u_+ - \rho_- u_-)}{\alpha} e^{-\alpha t}, \tag{1.14}$$

$\widetilde{\rho}(x) \geq 0$ with compact support $|x| \leq k$ such that

$$\int_{-\infty}^{\infty} \widetilde{\rho}(x) dx = 1 \tag{1.15}$$

and

$$\widehat{m}(x,t) = \rho_- u_- e^{-\alpha t} + (\rho_+ u_+ - \rho_- u_-) e^{-\alpha t} \int_{-\infty}^{x} \widetilde{\rho}(\xi) d\xi. \tag{1.16}$$

Let $w = \rho - \rho^* - \widehat{\rho}, z = m - m^* - \widehat{m}$. It follows by (1.6) (1.10) (1.12) (1.14) (1.16) that

$$\begin{cases} w_t + z_x = 0, \\ z_t + [\frac{(z + m^* + \widehat{m})^2}{w + \rho^* + \widehat{\rho}} + p(w + \rho^* + \widehat{\rho}) - p(\rho^*)]_x + \alpha z - \frac{1}{\alpha} p(\rho^*)_{xt} = 0. \end{cases} \tag{1.17}$$

Introducing $y(t,x) \equiv \int_{-\infty}^{x} -w(t,s) ds$, it is obvious that $y_x = -w$. Due to $(1.17)_1$ (1.16) and the facts that $m(-\infty,t) = \rho_- u_- e^{-\alpha t}$ and $m^*(-\infty,t) = 0$, it is known that $y_t = z$. Therefore, (1.17) becomes

$$y_{tt} + [\frac{(y_t + m^* + \widehat{m})^2}{\rho^* + \widehat{\rho} - y_x} + p(\rho^* + \widehat{\rho} - y_x) - p(\rho^*)]_x + \alpha y_t - \frac{1}{\alpha} p(\rho^*)_{xt} = 0. \tag{1.18}$$

It is clear that $y(-\infty,t) = 0, y(+\infty,t) = 0$ in view of (1.9), $(1.17)_1$ (1.15), (1.16) and the facts that $m(\mp\infty,t) = m_\mp e^{-\alpha t}, m^*(\mp\infty,t) = 0$.

We study the Cauchy problem of (1.18) with the following initial condition

$$y(x,0) = y_0(x) = \int_{-\infty}^{x} \{\rho^*(s + x_0) + \widetilde{\rho}(s) \frac{(\rho_+ u_+ - \rho_- u_-)}{\alpha} - \rho_0(s)\} ds, \tag{1.19}$$

$$\begin{aligned} y_t(x,0) =& y_1(x) \\ =& m_0(x) + \frac{1}{\alpha} p(\rho^*(x + x_0))_x - [\rho_- u_- + (\rho_+ u_+ - \rho_- u_-) \int_{-\infty}^{x} \widetilde{\rho}(\xi) d\xi], \end{aligned} \tag{1.20}$$

where x_0 is defined by (1.9), $\widetilde{\rho}(x)$ is defined in (1.15) and $m_0(x) = \rho_0(x) u_0(x)$.

For any given initial data $(\rho_0(x), u_0(x))$ such that $y_0(x) \in H^3(\mathbb{R}), y_1(x) \in H^2(\mathbb{R})$, we are going to prove that (1.18)–(1.20) has a unique smooth solution in the large in time provided the initial data are small (the precise description for the smallness will be given later). Furthermore, the solution y and its derivatives y_t, y_x decay to zero in the L_∞-norm as $t \to \infty$ with a rate $(t+1)^{-\frac{1}{4}}$, which implies that the system (1.1) is accurately approximated by (1.3) time-asymptotically since the functions $\widehat{\rho}(x,t)$ and $\widehat{m}(x,t)$ decay to zero exponentially fast.

For convenience, we only give the proof for the case when $u_- = u_+ = 0$ in which $\widehat{m} \equiv 0, \widehat{\rho} \equiv 0$ and (1.9) becomes

$$\int_{-\infty}^{\infty} (\rho_0(x) - \rho^*(x + x_0)) dx = 0. \tag{1.21}$$

The general case can be treated in a similar way by using the properties of $\widehat{m}(x,t)$ and $\widehat{\rho}(x,t)$. Moreover, we assume $\alpha = 1$ for simplicity.

§2. Priliminary Remarks

Consider the Cauchy problem

$$\begin{cases} y_{tt}+[\dfrac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}+p(\rho^*-y_x)-p(\rho^*)]_x+y_t-p(\rho^*)_{xt}=0, & (2.1)\\ y(x,0)=y_0(x)=\displaystyle\int_{-\infty}^{x}\{\rho^*(s+x_0)-\rho_0(s)\}ds, & (2.2)\\ y_t(x,0)=y_1(x)=m_0(x)+p(\rho^*(x+x_0))_x, & (2.3)\end{cases}$$

where $\rho^*(\frac{x+x_0}{\sqrt{t+1}})$ satisfies

$$\begin{cases} \rho_t^*=\dfrac{\partial^2 p(\rho^*)}{\partial x^2}, & (2.4)\\ \rho^*|_{t=0}=\rho^*(x+x_0) & (2.5)\end{cases}$$

while $\rho^*(x+x_0)$ satisfies the condition (1.21).

For the global existence of the solution to (2.1)–(2.3) we need the a priori estimate in a suitable norm which will be established with the help of the a priori estimate for $\rho^*(\frac{x+x_0}{\sqrt{t+1}})$.

Let $\eta=\frac{x+x_0}{\sqrt{t+1}}$. It is easy to know that the function $\rho^*(\eta)=\rho^*(\frac{x+x_0}{\sqrt{t+1}})$ satisfies

$$\begin{cases} -\dfrac{\eta}{2}\rho_\eta^*=p(\rho^*)_{\eta\eta}, & (2.6)\\ \rho^*(\mp\infty)=\rho_\mp. & (2.7)\end{cases}$$

Hypothesis 2.1. $p(\rho)$ is a smooth function of ρ in Ω such that the derivatives $p^{(i)}(\rho)$ up to $i=5$ are bounded in Ω and $p'(\rho)>0$ in Ω, where Ω is defined as $\Omega:\{\rho:\rho_0\le\rho\le\rho_1, 0<\rho_0<\rho_1<\infty\}$.

Assume the initial data $\rho_0(x)$ is given such that $\rho_-,\rho_+\in[\rho_0,\rho_1]$ at this moment. By a similar argument as in [2], it can be shown that the solution $\rho^*(\eta)$ of (2.6), (2.7) exists which is a monotone function, increasing if $\rho_+>\rho_-$ and decreasing if $\rho_+<\rho_-$. Moreover, $\rho_\eta^*(0)$ is small if $|\rho_--\rho_+|$ is small and $\rho_\eta^*(0)$ depends only on $\rho_\mp$ and $p(\rho)$. For definiteness, let us assume $\rho_+>\rho_-$ from now on, the case when $\rho_+<\rho_-$ can be dealt with in the same way. Therefore, $\rho_-\le\rho^*(\eta)\le\rho_+$ for $-\infty\le\rho\le\infty$ and $\rho_\eta^*(\mp\infty)=0$. Furthermore, we can establish the L^2-estimates on the derivatives of ρ^*.

Lemma 2.1 *Under the Hypothesis* 2.1, *the following estimates hold*

$$\int_{-\infty}^{\infty}(\rho_t^*)^2(x,t)dx\le C\cdot\frac{(\rho_\eta^*(0))^2}{(t+1)^{3/2}}, \tag{2.8}$$

$$\int_{-\infty}^{\infty}(\rho_{tx}^*)^2(x,t)dx\le C\cdot\frac{(\rho_\eta^*(0))^2}{(t+1)^{5/2}}, \tag{2.9}$$

$$\int_{-\infty}^{\infty}(\rho_{tt}^*)^2(x,t)dx\le C\cdot\frac{(\rho_\eta^*(0))^2}{(t+1)^{7/2}}, \tag{2.10}$$

$$\int_{-\infty}^{\infty}(\rho_{ttx}^*)^2(x,t)dx\le C\cdot\frac{(\rho_\eta^*(0))^2}{(t+1)^{9/2}}, \tag{2.11}$$

$$\int_{-\infty}^{\infty}(\rho_{ttxx}^*)^2(x,t)dx\le C\cdot\frac{(\rho_\eta^*(0))^2}{(t+1)^{11/2}}, \tag{2.12}$$

where C only depends on Ω and the function $p(\rho)$. Moreover, the L^2-estimates on ρ_x^{*2} and ρ_{xx}^* are the same as in (2.8); the L^2-estimate on ρ_{xxx}^* is the same as in (2.9); the L^2-estimate on ρ_{xxt}^* is the same as in (2.10); the L^2-estimate on ρ_{xxxt}^* is the same as in (2.11).

Proof. The equation (2.6) can be rewritten as

$$\rho_{\eta\eta}^* + \frac{\frac{\eta}{2} + p''(\rho^*(\eta)) \cdot \rho_\eta^*(\eta)}{p'(\rho^*(\eta))} \cdot \rho_\eta^* = 0. \tag{2.13}$$

It follows from (2.13) that

$$\rho_\eta^*(\eta) = \rho_\eta^*(0) e^{-\int_0^\eta A(s)ds}, \tag{2.14}$$

where

$$A(\eta) = \frac{\frac{\eta}{2} + p''(\rho^*(\eta)) \cdot \rho_\eta^*(\eta)}{p'(\rho^*(\eta))}. \tag{2.15}$$

Due to Hypothesis 2.1, we denote the bounds of $p'(\rho)$ by α_i, the bound of $|p''(\rho)|$ by β respectively. Namely, $0 < \alpha_1 \le p' \le \alpha_2, |p''| \le \beta$ in Ω, which, together with the fact that $0 \le \rho_\eta^* \le \gamma_0$, implies the following estimate about $e^{-\int_0^\eta A(s)ds}$

$$e^{-\int_0^\eta A(s)ds} \le \begin{cases} e^{-\frac{\eta(\eta+4\beta\gamma_0)}{4\alpha_2}} & \text{for } -\infty < \eta \le -2\beta\gamma_0, \\ e^{-\frac{\eta(\eta+4\beta\gamma_0)}{4\alpha_1}} & \text{for } -2\beta\gamma_0 \le \eta \le 0, \\ e^{-\frac{\eta(\eta-4\beta\gamma_0)}{4\alpha_1}} & \text{for } 0 \le \eta \le 2\beta\gamma_0, \\ e^{-\frac{\eta(\eta-4\beta\gamma_0)}{4\alpha_2}} & \text{for } 2\beta\gamma_0 \le \eta < \infty. \end{cases} \tag{2.16}$$

Since $\rho_t^* = \rho_\eta^* \cdot \frac{-\eta}{2} \cdot \frac{1}{t+1}$, it follows that

$$\int_{-\infty}^{\infty} \rho_t^{*^2}(x,t)dx = \int_{-\infty}^{\infty} \frac{\rho_\eta^{*^2}(0)}{(t+1)^{3/2}} \cdot \frac{\eta^2}{4} e^{-\int_0^\eta 2A(s)ds} d\eta,$$

by using (2,14).

With the help of (2.16), one obtains

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\eta^2}{4} e^{-\int_0^\eta 2A(s)ds} d\eta \le & \int_{-\infty}^{-2\beta\gamma_0} \frac{\eta^2}{4} e^{-\frac{\eta(\eta+4\beta\gamma_0)}{2\alpha_2}} d\eta + \int_{-2\beta\gamma_0}^{0} \frac{\eta^2}{4} e^{-\frac{\eta(\eta+4\beta\gamma_0)}{2\alpha_1}} d\eta \\ & + \int_0^{2\beta\gamma_0} \frac{\eta^2}{4} e^{-\frac{\eta(\eta-4\beta\gamma_0)}{2\alpha_1}} d\eta + \int_{2\beta\gamma_0}^{+\infty} \frac{\eta^2}{4} e^{-\frac{\eta(\eta-4\beta\gamma_0)}{2\alpha_2}} d\eta \\ \le & C(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_0). \end{aligned} \tag{2.17}$$

Therefore (2.8) follows, namely

$$\int_{-\infty}^{\infty} \rho_t^{*^2}(x,t)dx \le C \cdot \frac{[\rho_\eta^*(0)]^2}{(t+1)^{3/2}},$$

where C only depends on Ω and the function $p(\rho)$.

Since

$$\rho_x^* = \rho_\eta^* \cdot \frac{1}{\sqrt{t+1}} \tag{2.18}$$

and $\rho_\eta^*(\eta) \le C\rho_\eta^*(0)$ by (2.14) and (2.16), where C only depends on Ω and p, it is clear that ρ_x^* is small uniformly in t if $(\rho_+ - \rho_-)$ is small.

In order to estimate (2.9), we consider $\rho_{\eta\eta}^*$ first.

Differentiate (2.13) once more and denote $\rho^*_{\eta\eta}$ by Y. One obtains the following equation for Y with the help of (2.6)

$$Y_\eta + B(\eta)Y = E(\eta), \tag{2.19}$$

where

$$\begin{aligned} B(\eta) &= \frac{\frac{\eta}{2} + 3p''(\rho^*(\eta)) \cdot \rho^*_\eta(\eta)}{p'(\rho^*(\eta))}, \quad E(\eta) = e(\eta)\rho^*_\eta(\eta), \\ e(\eta) &= -\frac{\frac{1}{2} + p'''(\rho^*(\eta))(\rho^*_\eta(\eta))^2}{p'(\rho^*(\eta))}. \end{aligned} \tag{2.20}$$

It is known that there exists a finite number η_0 such that $Y(\eta_0) = 0$. Thus, the solution of (2.19) with $Y(\eta_0) = 0$ can be expressed by

$$Y(\eta) = \int_{\eta_0}^{\eta} e(\xi)\rho^*_\eta(\xi) e^{-\int_\xi^\eta B(s)ds} d\xi. \tag{2.21}$$

Substituting (2.14) into (2.21), we get

$$Y(\eta) = \rho^*_\eta(0) e^{-\int_0^\eta A(s)ds} \int_{\eta_0}^{\eta} e(\xi) e^{\int_\xi^\eta [A(s)-B(s)]ds} d\xi, \tag{2.22}$$

where

$$A(s) - B(s) = \frac{-2p''(\rho^*(s))\rho^*_\eta(s)}{p'(\rho^*(s))}.$$

It can be shown that

$$e^{\int_\xi^\eta [A(s)-B(s)]ds} \le \begin{cases} e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta-\xi)} & \text{for } \eta > \xi, \\ e^{\frac{2\beta\gamma_0}{\alpha_1}(\xi-\eta)} & \text{for } \eta < \xi; \end{cases}$$

therefore

$$|Y(\eta)| \le \begin{cases} \rho^*_\eta(0)e_0 \cdot e^{-\int_0^\eta A(s)ds} \int_{\eta_0}^{\eta} e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta-\xi)} d\xi & \text{for } \eta > \eta_0, \\ \rho^*_\eta(0)e_0 \cdot e^{-\int_0^\eta A(s)ds} \int_{\eta}^{\eta_0} e^{\frac{2\beta\gamma_0}{\alpha_1}(\xi-\eta)} d\xi & \text{for } \eta < \eta_0, \end{cases}$$

where e_0 is defined by $e_0 = \sup\limits_{\eta \in \mathbb{R}} |e(\eta)|$ which depends only on $\rho_\mp, \Omega$ and $p(\rho)$. It follows then that

$$|\rho^*_{\eta\eta}| \le \begin{cases} \rho^*_\eta(0)e_0 \cdot e^{-\int_0^\eta A(s)ds} \cdot \dfrac{\left[e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta-\eta_0)} - 1\right]}{\frac{2\beta\gamma_0}{\alpha_1}} & \text{for } \eta > \eta_0, \\ \rho^*_\eta(0) \cdot e_0 \cdot e^{-\int_0^\eta A(s)ds} \cdot \dfrac{\left[e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta_0-\eta)} - 1\right]}{\frac{2\beta\gamma_0}{\alpha_1}} & \text{for } \eta < \eta_0. \end{cases} \tag{2.23}$$

Since $\rho^*_{tx} = \frac{(-1)[\rho^*_{\eta\eta}\cdot\eta + \rho^*_\eta]}{2(t+1)^{3/2}}$, it reads off that

$$\int_{-\infty}^{\infty} \rho^{*2}_{tx}(x,t)dx \le \int_{-\infty}^{\infty} \frac{(\rho^{*2}_{\eta\eta} \cdot \eta^2 + \rho^{*2}_\eta)}{(t+1)^{5/2}} d\eta.$$

Due to (2.14), (2.16) and (2.23), a similar calculation as we made for getting (2.8) implies (2.9) then. In view of $\rho^*_{tt} = \frac{\rho^*_{\eta\eta}\cdot\eta^2 + 3\rho^*_\eta\cdot\eta}{4(t+1)^2}$, a similar treatment implies (2.10) as well.

For establishing the estimate for ρ^*_{ttx}, we consider $\rho^*_{\eta\eta\eta}$ next. Differentiate (2.13) twice and denote $\rho^*_{\eta\eta\eta}$ by Z, one obtains that

$$Z_\eta + D(\eta)Z = F(\eta), \tag{2.24}$$

where

$$
\begin{aligned}
D(\eta) &= \frac{\frac{\eta}{2} + 4p''(\rho^*(\eta)) \cdot \rho^*_\eta(\eta)}{p'(\rho^*(\eta))}, \\
F(\eta) &= f_1(\eta) \cdot \rho^*_{\eta\eta}(\eta) + f_2(\eta) \cdot \rho^*_\eta(\eta), \\
f_1(\eta) &= \frac{1 + 3p''(\rho^*(\eta)) \cdot \rho^*_{\eta\eta}(\eta) + 6p^{(3)}(\rho^*(\eta)) \cdot \rho^{*^2}_\eta(\eta)}{p'(\rho^*(\eta))}, \\
f_2(\eta) &= \frac{p^{(4)}(\rho^*(\eta)) \cdot \rho^{*^3}_\eta(\eta)}{p'(\rho^*(\eta))},
\end{aligned}
$$

$|f_i| \le f_i^0 < \infty$, the constants f_i^0 only depend on $\Omega, \rho_\mp$ and $p(\rho)$. It is clear that there exists a finite value η_1 such that $\rho^*_{\eta\eta\eta}(\eta_1) = 0$. Therefore, the solution of (2.24) with $Z(\eta_1) = 0$ can be expressed by

$$
Z(\eta) = \int_{\eta_1}^{\eta} [f_1(\xi)\rho^*_{\eta\eta}(\xi) + f_2(\xi)\rho^*_\eta(\xi)] e^{-\int_\xi^\eta D(s)ds} d\xi. \tag{2.25}
$$

Substitute (2.14), (2.22) into (2.25), it turns out that

$$
\begin{aligned}
Z(\eta) =& \rho^*_\eta(0) e^{-\int_0^\eta A(s)ds} \Big[\int_{\eta_1}^{\eta} e^{\int_\xi^\eta (A(s)-D(s))ds} \Big\{ f_2(\xi) \\
&+ f_1(\xi) \int_{\eta_0}^{\xi} e(\lambda) e^{\int_\lambda^\xi (A(s)-B(s))ds} d\lambda \Big\} d\xi \Big].
\end{aligned}
$$

For definiteness, let us assume that $\eta_1 < \eta_0$. It can be shown then that

$$
Z(\eta) \le \begin{cases}
\rho^*_\eta(0)e^{-\int_0^\eta A(s)ds} \left[\int_\eta^{\eta_1} e^{\frac{3\gamma_0\beta}{\alpha_1}(\xi-\eta)} \{ f_2^0 + f_1^0 e_0 \frac{(e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta_0-\xi)} - 1)}{\frac{2\beta\gamma_0}{\alpha_1}} \} d\xi \right] & \text{for } \eta < \eta_1, \\
\rho^*_\eta(0)e^{-\int_0^\eta A(s)ds} \left[\int_{\eta_1}^{\eta} e^{\frac{3\gamma_0\beta}{\alpha_1}(\eta-\xi)} \{ f_2^0 + f_1^0 e_0 \frac{(e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta_0-\xi)} - 1)}{\frac{2\beta\gamma_0}{\alpha_1}} \} d\xi \right] & \text{for } \eta_1 < \eta < \eta_0, \\
\rho^*_\eta(0)e^{-\int_0^\eta A(s)ds} \left[\int_{\eta_1}^{\eta_0} e^{\frac{3\gamma_0\beta}{\alpha_1}(\eta-\xi)} \{ f_2^0 + f_1^0 e_0 \frac{(e^{\frac{2\beta\gamma_0}{\alpha_1}(\eta_0-\xi)} - 1)}{\frac{2\beta\gamma_0}{\alpha_1}} \} d\xi \right. & \\
\left. + \int_{\eta_0}^{\eta} e^{\frac{3\gamma_0\beta}{\alpha_1}(\eta-\xi)} \{ f_2^0 + f_1^0 e_0 \frac{[e^{\frac{2\beta\gamma_0}{\alpha_1}(\xi-\eta_0)} - 1]}{\frac{2\beta\gamma_0}{\alpha_1}} \} d\xi \right] & \text{for } \eta > \eta_0.
\end{cases} \tag{2.26}
$$

Since $\rho^*_{ttx} = \frac{\rho^*_{\eta\eta\eta} \cdot \rho^*_\eta \cdot \eta^2 + 5\rho^*_{\eta\eta} \cdot \eta + 3\rho^*_\eta}{4(t+1)^{5/2}}$, by using (2.14), (2.16), (2.23), (2.26) and the similar argument as before, one obtains (2.11).

The other estimates in Lemma 2.1 can be established by a similar argument which is omitted.

We seek the smooth solution $y(t,x) \in C^2(t \ge 0, x \in R)$ and

$$
|y(t)|_{C^2} \equiv |y(t,\cdot)|_{c^2} + |y_t(t,\cdot)|_{c^1} + |y_{tt}(t,\cdot)|_{c^0}, \tag{2.27}
$$

where

$$
|f(\cdot)|_{c^k} \equiv \sum_{0 \le j \le k} \sup |d^j f(x)/dx^j|.
$$

By Sobolev's lemma, we have

$$
|f(\cdot)|_{c^k} \le \widehat{C} \|f(\cdot)\|_{H^{k+1}}. \tag{2.28}
$$

Thus using the L^2-energy method we will solve the Cauchy problem (2.1)–(2.3) in the Banach space X_3 defined by

$$X_m = \{y(t) \in L^\infty(t; H^m), y_t \in L^\infty(t; H^{m-1}),\ y_{tt} \in L^\infty(t; H^{m-2}), 0 \le t \le T, \forall T\}\ m \ge 3.$$

Hypothesis 2.2. $y_0(x) \in H^3(\mathbb{R})$, $y_1(x) \in H^2(\mathbb{R})$, $\rho^*(x + x_0) - y'_0(x) \in \Omega$, $\rho_0 + r \le \rho_-$, $\rho_+ \le \rho_1 - r$ for a positive constant r such that $0 < r < \frac{\rho_1 - \rho_0}{4}$. We assume $0 < \rho_0 \le 1$ for convenience.

It is known that the classical local existence theorem gives the solution for the Cauchy problem (2.1)–(2.3) in the space X_3 locally in time. For the global smooth solution in $t > 0$ we only need the a priori estimate in the norm (2.27) for which the a priori estimate in the norm of X_3 is sufficient by (2.28), namely,

$$|||y(t)|||_3^2 \equiv \|y(t)\|_{H^3}^2 + \|y_t(t)\|_{H^2}^2 + \|y_{tt}(t)\|_{H^1}^2 < \infty \text{ for } t \ge 0. \tag{2.29}$$

In order to obtain the a priori estimate in the norm (2.29), it suffices to obtain the a priori estimate of

$$E(t) = \sum_{j=1}^{3} E_j(t) \tag{2.30}$$

for the solution y with $\rho^* - y_x \in \Omega$ and $\left(\frac{y_t - p(\rho^*)_x}{\rho^* - y_x}\right)^2 \le \frac{\alpha_1}{2}$ in each (t, x), where

$$E_1(t) = \frac{4}{\alpha_1} \int_{-\infty}^{\infty} \{y \cdot y_t + \frac{y^2}{2} + y_t^2 + \sigma(y_x, \rho^*)\}(x, t)dx, \tag{2.31}$$

$$\begin{aligned} E_2(t) = \int_{-\infty}^{\infty} \{&y_{tt}^2 + [1 + p'(\rho^* - y_x) - (\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2]y_{xt}^2 \\ &+ [p'(\rho^* - y_x) - (\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2]y_{xx}^2\}(x, t)dx, \end{aligned} \tag{2.32}$$

$$\begin{aligned} E_3(t) = \frac{1}{2} \int_{-\infty}^{\infty} \{&y_{ttx}^2 + [p'(\rho^* - y_x) + 1 - (\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2]y_{xxt}^2 \\ &+ [p'(\rho^* - y_x) - (\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2]y_{xxx}^2\}(x, t)dx, \end{aligned} \tag{2.33}$$

where $\sigma(\xi, \rho^*) = \int_0^\xi [p(\rho^*) - p(\rho^* - \lambda)]d\lambda$.

It follows by (2.28) then that

$$|y(t)|_{C^2}^2 \le \widehat{C}|||y(t)|||_3^2 \le \widehat{C}^2 E(t) \tag{2.34}$$

for $y(t, x)$ with $\rho^* - y_x \in \Omega$ and $(\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2 \le \frac{\alpha_1}{2}$.

Lemma 2.2. *Under the Hypotheses* 2.1 *and* 2.2 *there exists an*

$$\varepsilon = \varepsilon(\Omega, p) \le \min\{r, \frac{\rho_0}{2}\sqrt{\frac{\alpha_1}{2}}\}$$

such that if the solution $y(t) \in X_3$ *to the Cauchy problem* (2.1)–(2.3) *is small as*

$$|y(t)|_{C^2 < \varepsilon} \tag{2.35}$$

and $\rho^*(\frac{x + x_0}{\sqrt{t+1}})$ *satisfies the condition*

$$|\rho_x^*|_{C^1} < \varepsilon, \tag{2.36}$$

then one has the a priori estimate

$$E(t) \le E(0) + R\rho_\eta^{*^2}(0) \quad in \quad 0 \le t \le T, \tag{2.37}$$

where R depends only on p, Ω.

First we assume that the solution $y(t)$ belongs to the space X_4 with $y_0(x) \in H^4, y_1(x) \in H^3$. In establishing the following energy estimates, a lot of troubles concerning the term $(\frac{[y_t - p(\rho^*)_x]^2}{\rho^* - y_x})_x$ in (2.1) occur for which we have to make very careful treatment.

Multiply the equation (2.1) by y and y_t respectively and integrate then over $[s,t] \times (-\infty, \infty)$. After the integration by part we have two equalities

$$\begin{aligned}
&\int_{-\infty}^{\infty} (y \cdot y_t + \frac{y^2}{2})(x,t)dx + \int_s^t \int_{-\infty}^{\infty} p'(\rho^* + \theta y_x) y_x^2(x,\xi)dxd\xi \\
&- \int_s^t \int_{-\infty}^{\infty} y_x \{\frac{[y_t - p'(\rho^*)\rho_x^*]^2}{\rho^* - y_x} - p'(\rho^*)\rho_t^*\}(x,\xi)dxd\xi \\
&= \int_{-\infty}^{\infty} (y \cdot y_t + \frac{y^2}{2})(x,s)dx + \int_s^t \int_{-\infty}^{\infty} y_t^2(x,\xi)dxd\xi,
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} [\frac{y_t^2}{2} + \sigma(y_x, \rho^*)](x,t)dx + \int_s^t \int_{-\infty}^{\infty} y_t^2(x,\xi)dxd\xi \\
&- \int_s^t \int_{-\infty}^{\infty} \frac{\partial \sigma}{\partial \rho^*} \cdot \rho_t^*(x,\xi)dxd\xi \\
&+ \int_s^t \int_{-\infty}^{\infty} y_t \{(\frac{1}{\rho^* - y_x})_x \cdot [y_t - p'(\rho^*)\rho_x^*]^2 - \frac{2[y_t - p(\rho^*)_x]}{\rho^* - y_x}\rho_t^* \\
&+ \frac{2y_t \cdot y_{tx}}{\rho^* - y_x} + y_t \cdot (\frac{2p_x}{\rho^* - y_x})_x - p(\rho^*)_{xt}\}(x,\xi)dxd\xi \\
&= \int_{-\infty}^{\infty} [\frac{y_t^2}{2} + \sigma(y_x, \rho^*)](x,s)dx,
\end{aligned} \tag{2.39}$$

where $\sigma(\xi, \rho^*) = \int_0^\xi [p(\rho^*) - p(\rho^* - \lambda)]d\lambda, \quad 0 < \theta < 1.$

By using Cauchy inequality with (2.38) and (2.39) respectively, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} \{y \cdot y_t + \frac{y^2}{2} + y_t^2 + \sigma(y_x, \rho^*)\}(x,t)dx \\
&+ (\frac{1}{4} - \frac{\delta}{\rho_0^2}) \int_s^t \int_{-\infty}^{\infty} y_t^2(x,\xi)dxd\xi \\
&+ (\frac{\alpha_1}{4} - \frac{\delta}{\rho_0}) \int_s^t \int_{-\infty}^{\infty} y_x^2(x,\xi)dxd\xi \\
\le &\frac{1}{2}\int_{-\infty}^{\infty} \{y \cdot y_t + \frac{y^2}{2} + y_t^2 + \sigma(y_x, \rho^*)\}(x,s)dx \\
&+ (\frac{\alpha_2^4}{2\rho_0^2\alpha_1} + \delta) \int_s^t \int_{-\infty}^{\infty} \rho_x^{*4}(x,\xi)dxd\xi \\
&+ (\frac{\alpha_2^2}{2\alpha_1} + \delta) \int_s^t \int_{-\infty}^{\infty} \rho_t^{*2}(x,\xi)dxd\xi + \alpha_2^2 \int_s^t \int_{-\infty}^{\infty} \rho_{xt}^{*2}(x,\xi)dxd\xi,
\end{aligned}$$

where ρ_0 is assumed to be less than or equal to 1 as before and $\delta = \delta(|y(t)|_{C^2} + |\rho_x^*|_{C^1})$.

It is clear that there exists an $\varepsilon > 0$ such that if (2.35), (2.36) are true, then

$$\delta \leq \min(\frac{1}{8}\rho_0^2, \frac{\alpha_1}{8}\rho_0). \tag{2.40}$$

Let us assume $\alpha_1 \leq 1$ for convenience. Therefore

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{y\cdot y_t + \frac{y^2}{2} + y_t^2 + \sigma(y_x,\rho^*)\}(x,t)dx \\
&+\frac{\alpha_1}{8}\int_s^t\int_{-\infty}^{\infty}(y_x^2+y_t^2)(x,\xi)dxd\xi \\
\leq&\frac{1}{2}\int_{-\infty}^{\infty}\{y\cdot y_t + \frac{y^2}{2} + y_t^2 + \sigma(y_x,\rho^*)\}(x,s)dx \\
&+(\frac{\alpha_2^4}{2\rho_0^2\alpha_1}+\frac{\rho_0^2}{8})\int_s^t\int_{-\infty}^{\infty}\rho_x^{*4}(x,\xi)dxd\xi \\
&+(\frac{\alpha_2^2}{2\alpha_1}+\frac{\rho_0^2}{8})\int_s^t\int_{-\infty}^{\infty}\rho_t^{*2}(x,\xi)dxd\xi + \alpha_2^2\int_s^t\int_{-\infty}^{\infty}\rho_{xt}^{*2}(x,\xi)dxd\xi.
\end{aligned} \tag{2.41}$$

Differentiate (2.1) with respect to t and multiply by y_{tt}, integrate over $[s,t]\times(-\infty,\infty)$ then with doing the integration by part whenever we need and the careful treatment on the term $y_{tt}\cdot[\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}]_{xt}$, it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{\frac{y_{tt}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot\frac{y_{xt}^2}{2}\}(x,t)dx \\
&+\int_s^t\int_{-\infty}^{\infty}y_{tt}^2(x,\xi)dxd\xi - \int_s^t\int_{-\infty}^{\infty}y_{tt}^2[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]_x(x,\xi)dxd\xi \\
&-\int_s^t\int_{-\infty}^{\infty}\frac{y_{xt}^2}{2}[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_t(x,\xi)dxd\xi \\
&+\int_s^t\int_{-\infty}^{\infty}y_{tt}\{[\rho_t^*(p'(\rho^*-y_x)-p'(\rho^*))]_x - y_{tx}[(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_x \\
&+2(y_t-p(\rho^*)_x)[(\frac{1}{\rho^*-y_x})_x(y_t-p(\rho^*)_x)_t+(\frac{1}{\rho^*-y_x})_t(y_t-p(\rho^*)_x)_x] \\
&+\frac{2}{\rho^*-y_x}[(y_{tt}-p(\rho^*)_{xt})(y_{tx}-\rho_t^*)-(y_t-p(\rho^*)_x)\rho_{tt}^*] \\
&+\frac{(y_t-p(\rho^*)_x)^2}{(\rho^*-y_x)^3}[2(\rho_t^*-y_{xt})(\rho_x^*-y_{xx})-(\rho^*-y_x)\rho_{xt}^*]\}(x,\xi)dxd\xi \\
&-\int_s^t\int_{-\infty}^{\infty}y_{tt}p(\rho^*)_{xtt}(x,\xi)dxd\xi \\
=&\int_{-\infty}^{\infty}\{\frac{y_{tt}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot\frac{y_{xt}^2}{2}\}(x,s)dx.
\end{aligned} \tag{2.42}$$

Differentiate (2.1) with respect to x and multiply by y_{xt}, integrate over $[s,t]\times(-\infty,\infty)$ then with doing the integration by part whenever it needs and the careful treatment on the term $y_{xt}\cdot[\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}]_{xx}$, it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\{\frac{y_{xt}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot\frac{y_{xx}^2}{2}\}(x,t)dx+\int_s^t\int_{-\infty}^{\infty}y_{xt}^2(x,\xi)dxd\xi\\&-\int_s^t\int_{-\infty}^{\infty}y_{xt}\rho_{tt}^*(x,\xi)dxd\xi-\int_s^t\int_{-\infty}^{\infty}y_{xt}^2[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]_x(x,\xi)dxd\xi\\&+\int_s^t\int_{-\infty}^{\infty}\frac{y_{xx}^2}{2}[(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2-p'(\rho^*-y_x)]_t(x,\xi)dxd\xi+\int_s^t\int_{-\infty}^{\infty}y_{tx}\{[(p'(\rho^*-y_x)\\&-p'(\rho^*))\rho_x^*]_x-y_{xx}[(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_x+4(y_t-p(\rho^*)_x)(\frac{1}{\rho^*-y_x})_x\cdot(y_t-p(\rho^*)_x)_x\\&+\frac{2}{\rho^*-y_x}[(y_{xt}-\rho_t^*)^2-(y_t-p(\rho^*)_x)\cdot\rho_{xt}^*]\\&+\frac{(y_t-p(\rho^*)_x)^2}{(\rho^*-y_x)^3}[2(\rho_x^*-y_{xx})^2-(\rho^*-y_x)\rho_{xx}^*]\}(x,\xi)dxd\xi\\=&\int_{-\infty}^{\infty}\{\frac{y_{xt}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot\frac{y_{xx}^2}{2}\}(x,s)dx.\end{aligned}\tag{2.43}$$

Express y_{xx} in terms of y_{tt}, y_t, etc by (2.1) and multiply the equation by y_{xx}, integrate it over $[s,t]\times(-\infty,\infty)$, one obtains

$$\begin{aligned}&\int_s^t\int_{-\infty}^{\infty}p'(\rho^*-y_x)y_{xx}^2(x,\xi)dxd\xi\\=&\int_s^t\int_{-\infty}^{\infty}y_{xx}\{y_{tt}+y_t-p(\rho^*)_{xt}+[p'(\rho^*-y_x)-p'(\rho^*)]\rho_x^*\\&+[\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}]_x\}(x,\xi)dxd\xi.\end{aligned}\tag{2.44}$$

By using Cauchy inequalities with the above three equations respectively, it turns out that

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xt}^2+[p'(\rho^*-y_x)\\&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xx}^2\}(x,t)dx+(\frac{1}{2}-\frac{\delta}{\rho_0^3})\int_s^t\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)(x,\xi)dxd\xi\\\le&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot y_{xt}^2\\&+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xx}^2\}(x,s)dx\\&+\frac{\delta}{\rho_0^3}\int_s^t\int_{-\infty}^{\infty}\{y_t^2+y_x^2+\rho_t^{*2}+\rho_x^{*4}+\rho_{xt}^{*2}\}(x,\xi)dxd\xi\\&+(\beta^2+\frac{1}{2})\int_s^t\int_{-\infty}^{\infty}\rho_{tt}^{*2}(x,\xi)dxd\xi+\alpha_2^2\int_s^t\int_{-\infty}^{\infty}\rho_{ttx}^{*2}(x,\xi)dxd\xi.\end{aligned}$$

It is clear that there exists an $\varepsilon>0$ such that if (2.35) (2.36) are true, then

$$\delta\le\min\{\frac{\alpha_1}{8}\rho_0^2,\frac{1}{4}\rho_0^3\}.\tag{2.45}$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xt}^2+[p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xx}^2\}(x,t)dx+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2)(x,\xi)dxd\xi\\
\le&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xt}^2+[p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xx}^2\}(x,s)dx\\
&+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}(y_t^2+y_x^2)(x,\xi)dxd\xi+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}\{\rho_t^{*2}+\rho_x^{*4}+\rho_{xt}^{*2}\}(x,\xi)dxd\xi\\
&+(\beta^2+\frac{1}{2})\int_s^t\int_{-\infty}^{\infty}\rho_{tt}^{*2}(x,\xi)dxd\xi+\alpha_2^2\int_s^t\int_{-\infty}^{\infty}\rho_{ttx}^{*2}(x,\xi)dxd\xi. \qquad (2.46)
\end{aligned}
$$

Differentiate (2.1) with respect to x and t successively and multiply the resulting equation by y_{ttx}, integrate it then over $[s,t]\times(-\infty,\infty)$ with doing the integration by part wherever one needs and doing the careful treatment on the term $y_{ttx}\cdot[\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}]_{xxt}$, it follows that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\{\frac{y_{ttx}^2}{2}+[p'(\rho^*-y_x)-[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]^2]\cdot\frac{y_{xxt}^2}{2}\}(x,t)dx+\int_s^t\int_{-\infty}^{\infty}y_{ttx}^2(x,\xi)dxd\xi\\
&-\int_s^t\int_{-\infty}^{\infty}\frac{y_{xxt}^2}{2}[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_t(x,\xi)dxd\xi\\
&-\int_s^t\int_{-\infty}^{\infty}y_{ttx}^2[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]_x(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}y_{ttx}\{[(p'(\rho^*-y_x)-p'(\rho^*))\rho_x^*]_t-[p'(\rho^*-y_x)]_t\cdot y_{xx}\}_x(x,\xi)dxd\xi\\
&-\int_s^t\int_{-\infty}^{\infty}y_{ttx}\{[(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_x\cdot y_{xxt}+\frac{2(y_t-p(\rho^*)_x)}{\rho^*-y_x}\cdot p(\rho^*)_{xxxt}\}(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}y_{ttx}\frac{(y_t-p(\rho^*)_x)^2}{(\rho^*-y_x)^4}\{(\rho_x^*-y_{xx})[4(\rho^*-y_x)(\rho_{xt}^*-y_{xxt})-(\rho_t^*-y_{xt})(\rho_x^*-y_{xx})]\\
&+(\rho^*-y_x)[2(\rho_t^*-y_{xt})(\rho_{xx}^*-y_{xxx})-(\rho^*-y_x)\rho_{xxt}^*]\}(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}y_{ttx}\{(\frac{1}{\rho^*-y_x})_{xx}\cdot((y_t-p(\rho^*)_x)^2)_t+2(\frac{1}{\rho^*-y_x})_{xt}\cdot((y_t-p(\rho^*)_x)^2)_x\\
&+2(\frac{1}{\rho^*-y_x})_x((y_t-p(\rho^*)_x)^2)_{xt}+(\frac{1}{\rho^*-y_x})_t((y_t-p(\rho^*)_x)^2)_{xx}\\
&+\frac{2}{\rho^*-y_x}[2(y_{tx}-p(\rho^*)_{xx})(y_{ttx}-p(\rho^*)_{xxt})+(y_{tt}-p(\rho^*)_{xt})(y_{txx}-p(\rho^*)_{xxx})]\}(x,\xi)dxd\xi\\
&-\int_s^t\int_{-\infty}^{\infty}y_{ttx}p(\rho^*)_{xxtt}(x,\xi)dxd\xi\\
=&\int_{-\infty}^{\infty}\{\frac{y_{ttx}^2}{2}+[p'(\rho^*-y_x)-[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]^2]\cdot\frac{y_{xxt}^2}{2}\}(x,s)dx. \qquad (2.47)
\end{aligned}
$$

Differentiate (2.1) with respect to x twice and multiply it by y_{txx}, integrate then over $[s,t]\times(-\infty,\infty)$. With the help of integration by part and the careful treatment on the term

$y_{txx}[\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x}]_{xxx}$, one obtains that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{\frac{y_{txx}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\frac{y_{xxx}^2}{2}\}(x,t)dx+\int_s^t\int_{-\infty}^{\infty}y_{txx}^2(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}\frac{y_{xxx}^2}{2}\{[(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]_t-[p'(\rho^*-y_x)]_t\}(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}y_{xxt}^2[\frac{y_t-p(\rho^*)_x}{\rho^*-y_x}]_x(x,\xi)dxd\xi\\
&+\int_s^t\int_{-\infty}^{\infty}y_{txx}\{[(p'(\rho^*-y_x)-p'(\rho^*))\rho_x^*]_x-[p'(\rho^*-y_x)]_x\cdot y_{xx}+(y_t-p(\rho^*)_x)^2\\
&[\frac{2(\rho_x^*-y_{xx})^2-\rho_{xx}^*(\rho^*-y_x)}{(\rho^*-y_x)^3}]+2(\frac{1}{\rho^*-y_x})_x[(y_t-p(\rho^*)_x)^2]_x\\
&+\frac{2(y_t-p(\rho^*)_x)_x(y_{tx}-\rho_t^*)-2(y_t-p(\rho^*)_x)\rho_{xt}^*}{\rho^*-y_x}\}_x(x,\xi)dxd\xi\\
&-\int_s^t\int_{-\infty}^{\infty}y_{txx}\rho_{ttx}^*(x,\xi)dxd\xi\\
&=\int_{-\infty}^{\infty}\{\frac{y_{txx}^2}{2}+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot\frac{y_{xxx}^2}{2}\}(x,s)dx.
\end{aligned}\tag{2.48}$$

Differentiate (2.1) with respect to x and multiply then by y_{xxx}, integrate it over $[s,t]\times(-\infty,\infty)$, we obtain that

$$\begin{aligned}
&\int_s^t\int_{-\infty}^{\infty}p'(\rho^*-y_x)y_{xxx}^2(x,\xi)dxd\xi\\
=&\int_s^t\int_{-\infty}^{\infty}y_{xxx}\{y_{ttx}+y_{tx}-\rho_{tt}^*+(\frac{(y_t-p(\rho^*)_x)^2}{\rho^*-y_x})_{xx}\\
&+[(p'(\rho^*-y_x)-p'(\rho^*))\rho_x^*]_x-p''(\rho^*-y_x)(\rho_x^*-y_{xx})y_{xx}\}(x,\xi)dxd\xi.
\end{aligned}\tag{2.49}$$

Using Cauchy inequalities with the above three equations respectively, it can be obtained, similarly as before, that

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{ttx}^2+[p'(\rho^*-y_x)+1-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot y_{xxt}^2+[p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot y_{xxx}^2\}(x,t)dx+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}(y_{txx}^2+y_{ttx}^2)(x,\xi)dxd\xi\\
\le&\frac{1}{2}\int_{-\infty}^{\infty}\{y_{ttx}^2+[p'(\rho^*-y_x)+1-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot y_{xxt}^2+[p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xxx}^2\}(x,s)dx+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}(y_{tt}^2+y_{tx}^2+y_t^2)(x,\xi)dxd\xi\\
&+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}(\rho_t^{*2}+\rho_x^{*4}+\rho_{xt}^{*2}+\rho_{tt}^{*2}+\rho_{xx}^{*2}+\rho_{xxt}^{*2}+\rho_{xxx}^{*2})(x,\xi)dxd\xi\\
&+\frac{3}{4}\int_s^t\int_{-\infty}^{\infty}\rho_{ttx}^{*2}(x,\xi)dxd\xi+\frac{\alpha_2^2}{2}\int_s^t\int_{-\infty}^{\infty}\rho_{xxtt}^{*2}(x,\xi)dxd\xi.
\end{aligned}\tag{2.50}$$

Due to (2.41), (2.46) and (2.50), it reads off that

$$\begin{aligned}
&\frac{4}{\alpha_1}\int_{-\infty}^{\infty}\{y\cdot y_t+\frac{y^2}{2}+y_t^2+\sigma(y_x,\rho^*)\}(x,t)dx+\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xt}^2+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]\cdot y_{xx}^2\}(x,t)dx\\
&+\frac{1}{2}\int_{-\infty}^{\infty}\{y_{ttx}^2+[p'(\rho^*-y_x)+1-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xxt}^2+[p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xxx}^2\}(x,t)dx\\
&+\frac{1}{4}\int_s^t\int_{-\infty}^{\infty}\{y_x^2+y_t^2+y_{tt}^2+y_{tx}^2+y_{ttx}^2+y_{xxt}^2\}(x,\xi)dxd\xi\\
\le&\frac{4}{\alpha_1}\int_{-\infty}^{\infty}\{y\cdot y_t+\frac{y^2}{2}+y_t^2+\sigma(y_x,\rho^*)\}(x,s)dx+\int_{-\infty}^{\infty}\{y_{tt}^2+[1+p'(\rho^*-y_x)\\
&-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xt}^2+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xx}^2\}(x,s)dx\\
&+\frac{1}{2}\int_{-\infty}^{\infty}\{y_{ttx}^2+[p'(\rho^*-y_x)+1-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xxt}^2\\
&+[p'(\rho^*-y_x)-(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2]y_{xxx}^2\}(x,s)dx+C\cdot\rho_\eta^*(0)^2[(s+1)^{-\frac{1}{2}}-(t+1)^{-\frac{1}{2}}],
\end{aligned}\tag{2.51}$$

where C depends only on Ω and p.

This gives (2.37) in view of the definition of $E(t)$. Therefore, we arrive at the a priori estimate (2.37) under the assumption (2.35) (2.36). This a priori estimate is also valid for the solution $y(t)$ in X_3 by use of the Friedrich's mollifier under the same assumption (2.35) (2.36). Lemma 2.2 follows then.

§3. The Main Theorems

Theorem 3.1 *Under the Hypotheses* 2.1 *and* 2.2, *there exists a constant* $0<\varepsilon\le\min\{r,\frac{\rho_0}{2}\sqrt{\frac{\alpha_1}{2}}\}$ *such that if the initial data are small as* $E(0)<\varepsilon$, *then the Cauchy problem* (2.1)–(2.3) *has a unique smooth solution in the large in time provided that* $|\rho_+-\rho_-|$ *is sufficiently small so that* $|\rho_x^*|_{C^1}<\varepsilon$ *holds. Moreover, the solutions* y *and* y_t, y_x *decay to zero in the* L_∞*-norm as* $t\to\infty$, *with a rate* $(t+1)^{-\frac{1}{4}}$.

Proof. We choose the initial data so small that

$$E(0)<\frac{\varepsilon^2}{4\widehat{C}^2},\tag{3.1}$$

where ε is the same as in Lemma 2.2, $\widehat{C}$ is the same as in (2.34).

By the local existence theorem there exists $t_0>0$ such that the solution $y(t)\in X_3$ exists in $0\le t\le t_0$ and satisfies

$$E(t)\le 2E(0) \text{ and } \rho^*-y_x\in\Omega,$$
$$(\frac{y_t-p(\rho^*)_x}{\rho^*-y_x})^2\le\frac{\alpha_1}{2} \text{ in } 0\le t\le t_0.$$

It follows by (2.34) then that

$$|y(t)|_2^2 \le \widehat{C}^2 E(t) \le 2\widehat{C}^2 E(0) \le \frac{\varepsilon^2}{2} < \varepsilon^2 \text{ in } 0 \le t \le t_0. \tag{3.2}$$

Therefore, Lemma 2.2 implies

$$E(t) \le E(0) + R\rho_\eta^*(0)^2 \text{ in } 0 \le t \le t_0. \tag{3.3}$$

Due to (2.51), it is always possible to choose $|\rho_+ - \rho_-|$ so small that

$$R\rho_\eta^*(0)^2 \le \frac{\varepsilon^2}{4\widehat{C}^2}. \tag{3.4}$$

Next, by the local existence theorem for $t \ge t_0$, there exists $\widehat{t} = \widehat{t}(E(0) + R\rho_\eta^*(0)^2)$ such that the solution $y(t)$ exists in $0 \le t \le t_0 + \widehat{t}$ and satisfies

$$\begin{aligned} &E(t) \le 2E(t_0) \text{ and } \rho^* - y_x \in \Omega, \\ &(\frac{y_t - p(\rho^*)_x}{\rho^* - y_x})^2 \le \frac{\alpha_1}{2} \text{ in } t_0 \le t \le t_0 + \widehat{t}. \end{aligned} \tag{3.5}$$

Combine (2.34) (3.1) (3.3) (3.4) and (3.5), one obtains in $t_0 \le t \le t_0 + \widehat{t}$,

$$\begin{aligned} |y(t)|_{C^2}^2 &\le \widehat{C}^2 E(t) \le 2\widehat{C}^2 E(t_0) \\ &\le 2\widehat{C}^2 [E(0) + R\rho_\eta^*(0)^2] < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \end{aligned} \tag{3.6}$$

Therefore, (3.2), (3.6) and Lemma 2.2 imply

$$E(t) \le E(0) + R\rho_\eta^*(0)^2 \text{ in } 0 \le t \le t_0 + \widehat{t}.$$

Repeat the same procedure with the same time interval $\widehat{t} > 0$, we complete the proof of the global existence of the solution.

The decay of y, y_t, y_x can be obtained easily with the help of (2.51).

The decay rate can be obtained then by using the following Lemma 3.2 which can be shown by a similar argument as used in [1] and the detail is omitted.

Lemma 3.2. *The solution y in Theorem 3.1 satisfies the estimate*

$$\|y_t(t,\cdot)\|_{H^1} + \|y_2(t,\cdot)\|_{H^1} = CE(0)(t+1)^{-\frac{1}{4}}. \tag{3.7}$$

Since $y_t = z = m - m^* - \widehat{m}$ and $y_t = -w = -(\rho - \rho^* - \widehat{\rho})$, it turns out that

$$m(x,t) + \frac{1}{\alpha} p(\rho^*(\frac{x+x_0}{\sqrt{t+1}}))_x - \widehat{m}(x,t) \to 0$$

as $t \to \infty$ uniformly in x, in a rate of $(t+1)^{-\frac{1}{4}}$;

$$\rho(x,t) - \rho^*(\frac{x+x_0}{\sqrt{t+1}}) - \widehat{\rho}(x,t) \to 0$$

as $t \to \infty$ uniformly in x, in a rate of $(t+1)^{-\frac{1}{4}}$. Therefore, the following theorem holds

Theorem 3.3 *Under the same assumption of Theorem* 3.1, *the solution* $(\rho(x,t), m(x,t))$ *of* (1.6) *with* (1.13) *approaches to the solution of* (1.8),

$$(\rho^*(\frac{x+x_0}{\sqrt{t+1}}), \frac{-1}{\alpha}\frac{\partial p(\rho^*(\frac{x+x_0}{\sqrt{t+1}}))}{\partial x}),$$

as $t \to \infty$ uniformly in x, in a rate of $(t+1)^{-\frac{1}{4}}$.

On the other hand, it can be shown that[3] the solution $\rho(x,t)$ of (1.8) with $\rho(x,0) = \rho_0(x)$ approaches to $\rho^*(\frac{x}{\sqrt{t+1}})$ as $t \to \infty$ uniformly in x with a rate $(t+1)^{-\frac{1}{4}}$ if $\rho_0(x) \to \rho_\mp$ with a rate

$$\rho_0(x) - \rho_- = O((-x)^{-k_1}) \text{ as } x \to -\infty,$$
$$\rho_0(x) - \rho_+ = O((x^{-k_2})) \text{ as } x \to +\infty,$$

where $k_i > 1$ $(i = 1, 2)$. Therefore, we have the last theorem.

Theorem 3.4. *Under the Hypothesis* 2.1, *assume* $\rho_0(x) \to \rho_\mp$ *as* $x \to \mp\infty$ *with a rate* $|x|^{-k}$, $k > \frac{3}{2}$,

$$u_0(x) \to u_\mp \text{ as } x \to \mp\infty, \ \rho_-, \ \rho_+ \in [\rho_0 + r, \ \rho_1 - r],$$
$$\rho_0(x) - \widetilde{\rho}(x) \cdot \frac{\rho_+ u_+ - \rho_- u_-}{\alpha} \in \Omega$$

and the initial data satisfies $E(0) < \varepsilon$. *Then the solution* $(\rho(x,t), m(x,t))$ *of* (1.6) (1.13) *approaches to the solution* $(\rho(x + x_0, t), m(x + x_0, t))$ *of* (1.8) $(1.13)_1$ *as* $t \to \infty$, *uniformly in* x, *with a rate* $t^{-\frac{1}{4}}$.

This shows that the system (1.1) is accurately approximated by (1.4) time-asymptotically.

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