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LOCAL L^p ESTIMATE FOR THE SOLUTION OF $\bar{\partial}$-NEUMANN PROBLEM OVER $D_t = \{(w,z) : \mathrm{Re}\, w \le \frac{|z^m - tw|^2}{m}\}$**

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Abstract

Assume that a distribution u satisfies conditions: $\bar{\partial} u = f$, $u \perp H(D_t)$ on domain D_t, $u \in \mathrm{Dom}(\bar{\partial}_0^*)$, $\bar{\partial} u \in \bar{\partial}_1^*$; $\bar{\partial} f = 0$, $f \perp H^{0,1}$. It is proved that $\varphi_1 u \in L^p_{\beta + \frac{1}{2m} - \varepsilon}$ if $\varphi_2 f \in L^p_\beta$, where L^p_β is the potential space defined in [14]; φ_1, $\varphi_2 \in C_c^\infty(U)$, $\varphi_2 = 1$ on suppt φ_1; U is a neighbourhood of the origin; ε is a small positive number. This result contains a result of D. C. Chang (in [3]) by setting $t = 0$.

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§1. Introduction

$\bar{\partial}$-Neumann Problem is a very important problem in mathematics. C. L. Fefferman and J. J. Kohn have showed the development of $\bar{\partial}$-Neumann Problem in many aspects in [6]. G. B. Folland and J. J. Kohn obtained an important subelliptic estimate for the $\bar{\partial}$-Neumann operator and proved the boundary regularity properties for the solution of the $\bar{\partial}$ equation in case the domain is bounded and strongly pseudoconvex (cf. [7]); In another paper J. J. Kohn studied the same problem on weakly pseudoconvex manifolds of dimension two (cf. [10]). In order to improve the subelliptic estimate in [10], L. P. Rothschild and E. M. Stein introduced the method of Nilpotent Group by which they studied the regularity properties for the solution of equation of Hörmander type (cf. [14]). For the solution of $\bar{\partial}$-equation, detailed results about L^p and Λ_β estimates on strongly pseudoconvex domain can be found in [8]; Λ_β estimate on domains of complex dimension two and three dimensional CR manifolds is obtained by C. L. Fefferman and J. J. Kohn (cf. [6]); L^p estimate on a special weakly pseudo-convex domain in C^2 with boundary being of finite type is obtained by D. C. Chang (cf. [3]). For more information, see the literatures cited in [6]. Our aim is to prove the L^p estimate for the solution of the $\bar{\partial}$ equation in general weakly pseudoconvex domain in C^n with boundary being of finite type. As the first step, we study the $\bar{\partial}$-problem over the domain $D_t = \{(w,z) : \mathrm{Re} w \le \frac{1}{m}|z^m - tw|^2\}$. In forthcoming papers we treat the same problem over general pseudoconvex domain in C^2 of finite type, over domain $\{z \in C^n : \mathrm{Re}\, z_n \le |z_{n-1}|^2 + \sum_{j=1}^{n-2} |z_j^m - z_{j+1} z_n|^2\}$ (example 5.16, p. 633, [1]) (which is typical

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in the sense that: on one hand the order of the ∂D with algebraic curve is $(2m)^{m-1}/2^{n-1}$; on the other hand,its commutator type is $(1,2m,2m,\cdots,2m)$ at the origin and $(1,2,\cdots,2)$ at other boundary points (cf. [4])).

The paper is arranged as follows: We compute at first the $\bar\partial$ equation and then construct parametrices for the associated elliptic equations by calculus of pseudodifferential operators. We compute the boundary Π_b equation (defined below) which is hypoelliptic in some small cone in the phase space (under the Fourier Transform) and elliptic in the rest set. Finally, using the results of [2], [5], [12], [13] and [14], we get the desired L^p estimation for the solution of $\bar\partial$-equation.

§2. Formulation of the $\bar\partial$-Neumann Problem

2.1. Let $R(w,z)=\mathrm{Re}\,w-|z^m-tw|^2/m$. Then the domain D_t (denoted for simplicity by D) can be described as $R(w,z)\le 0$. Let ∂D denote the boundary of D. Associated with the domain D there exist two holomorphic vector fields defined by

$$S=\frac{\overline{R_z}\partial_z+\overline{R_w}\partial_w}{b},\quad T=\frac{R_w\partial_z-R_z\partial_w}{b},\tag{2.1}$$

where $b=\frac{1}{2}[|R_z|^2+|R_w|^2]^{\frac{1}{2}}$; $\partial_z=\frac{\partial}{\partial z}$, $\partial_w=\frac{\partial}{\partial w}$; $R_z=\frac{\partial R}{\partial z}$, $R_w=\frac{\partial R}{\partial w}$.

It is obvious that $S(\frac{R}{4b})|_{\partial D}=S(R/[\sum R_j\bar R_j]^{\frac{1}{2}})|_{\partial D}=1$, $T(R)|_{\partial D}=0$.

Let H(D) denote the space of holomorphic functions defined on D.

Let TD_p denote the tangent vector space spanned by $S,T,\bar S,\bar T$ over ring of C^∞-function germs at point $P\in D$;

$T^{(1,0)}_p$ = subspace of TD_p spanned by S,T at P;

$T^{(0,1)}_p$ = the conjugate space of $T^{(1,0)}_p$;

TD = space of sections of tangent vectors over $C^\infty_c(D)$; similarly one can define $T^{(1,0)}$ and $T^{(0,1)}$.

Let $\Lambda_{(0,1)}$ = space of (0,1)-forms;

$H^{0,1}=\{\varphi:\varphi\in \mathrm{Dom}(\mathrm{clos}\bar\partial)\cap\mathrm{Dom}(\bar\partial^*):\mathrm{clos}\bar\partial(\varphi)$ and $\bar\partial^*(\varphi)=0$ *on* $D\}$.

$\Lambda_{(1,0)}$ = the conjugate space of $\Lambda_{(0,1)}$. Similarly one can define $\Lambda_{(p,q)}$ $(p,q=0,1,2$ in case the complex dimension is two).

On $T_p^{(1,0)}$ there exists a natural metric, i.e., the complex Euclidean Inner Product which satisfies following conditions:

(i) $T\perp S$;

(ii) $\langle\cdot,\cdot\rangle$ can be extended to TD_p such that $T^{(1,0)}_p\perp T^{(0,1)}_p$.

By duality, the complex Euclidean Inner Product can be defined on $\Lambda_{(p,q)}$.

Now we calculate (1,0)-forms w_1,w_2 defined by

$$\langle df,w_1\rangle=S(f),\langle df,w_2\rangle=T(f).\tag{2.2}$$

Lemma 2.1. *The* (1,0)*-forms* w_1,w_2 *satisfy*

$$w_1=\frac{R_zdz+R_wdz}{2b},\quad w_2=\frac{\overline{R_w}dz-\overline{R_z}dw}{2b};$$
$$w_1\perp w_2,\quad \langle w_i,w_j\rangle=2\delta_{ij};\tag{2.3}$$
$$\bar\partial f(=f_{\bar z}d\bar z+f_{\bar w}d\bar w)=\frac{1}{2}(\bar S(f)\bar w_1+\bar T(f)\bar w_2).$$

Proof. It is easy to verify that (2.3) holds.

2.2. $\bar{\partial}$-Neumann Problem

Roughly speaking, the $\bar{\partial}$-Neumann Problem can be stated as follows:

For a given (0,1)-form $f = f_1\bar{w}_1 + f_2\bar{w}_2$ which satisfies conditions $\bar{\partial} f = 0$ and $f \perp H^{0,1}$, what property does the solution of equation $\bar{\partial} u = f$ (u satisfies condition $u \perp H(D)$) has?

Reasoning in the same way as in [7], it is sufficient to solve the following system of differential euqations:

$$\begin{aligned} &\Pi(U) = \bar{\partial}_0\bar{\partial}_0^* + \bar{\partial}_1^*\bar{\partial}_1(U) = f, \\ &U \in \text{Domain of } \bar{\partial}_0^*, \quad \bar{\partial}_1 U \in \text{Domain of } \bar{\partial}_1^*. \end{aligned} \tag{2.4}$$

Here $\bar{\partial}_0, \bar{\partial}_1$ denotes the $\bar{\partial}$-operator acting on function space and $\Lambda_{(0,1)}$ space respectively; $\bar{\partial}^*$ is the formal adjoint operator of $\bar{\partial}$ in L^2 norms.

It is obvious that if U is a solution of (2.4), then for any f satisfying $\bar{\partial} f = 0$, $u = \bar{\partial}_0 U(f)$ is a solution of $\bar{\partial}_0 u = f$ and satisfies condition $u \perp H(D)$.

2.3. Computation of $\bar{\partial}^*$

Throughout the rest of this paper, we always denote $\operatorname{Re} w$, $\operatorname{Im} w$, $\operatorname{Re} z$, $\operatorname{Im} z$ by $x_1, \cdots, x_4$; $\partial_j = \partial/\partial x_j$, $R_j = \partial R/\partial x_j$ for $j = 1,2,3,4$; $x' = (x_2, x_3, x_4)$; $\alpha_j\beta_j = \sum_1^4 \alpha_j\beta_j$ and $\alpha_{j'}\beta_{j'} = \sum_2^4 \alpha_{j'}\beta_{j'}$.

We need the following lemma about the integrals over D and ∂D.

Lemma 2.2. *For any differential vector $A \in TD$, $A = \sum a_j(x)\frac{\partial}{\partial x_j} + c$, it holds that*

$$\int_{R<0} A(f)g dx = \int_{R=0} C_A f g dx' + \int_{R<0} f A^*(g) dx, \forall f, g \in C_c^\infty(R^4). \tag{2.5}$$

where $A^* = -A - \sum \partial_j a_j + c$; $C_A = \frac{\sum a_j R_j}{R_1}$.

Proof. Under coordinate transformation $(x_j) \to (x_1 + r(x'), x')$, the domain D is transformed into $\{x_1 \le 0\}$. Then integrating by parts shows the lemma.

Below we are going to compute $\bar{\partial}_j^* (j = 0, 1)$.

Lemma 2.3. *Let $f = f_1\bar{w}_1 + f_2\bar{w}_2$, g, $h = h\bar{w}_1 \wedge \bar{w}_2$ denote smooth sections in $\Lambda_{(0,1)}$, $\Lambda_{(0,0)}$ and $\Lambda_{(0,2)}$ respectively, where g, h, f_1, f_2 are smooth functions with compact supports. Then*

$$\bar{\partial}_0^* f = S^*(f_1) + T^*(f_2); \quad f \in Dom(\bar{\partial}_0^*) \Leftrightarrow f_1|_{\partial D} = 0.$$

$$S^* = -S - [\partial_z(\frac{\overline{R_z}}{b}) + \partial_w(\frac{\overline{R_w}}{b})] = -S - \bar{c}_2,$$

$$T^* = -T - [\partial_z(\frac{R_w}{b}) - \partial_w(\frac{R_z}{b})] = -T + \bar{c}_1.$$

Proof. From (2.3), applying Lemma 2.2, we get

$$\int_D \langle \bar{\partial}_0^* f, g\rangle dx = \int_D \langle f, \bar{\partial}_0 g\rangle dx = \frac{1}{2}\int_D \langle f_1\bar{w}_1 + f_2\bar{w}_2, \bar{S}(g)\bar{w}_1 + \bar{T}(g)\bar{w}_2\rangle dx$$

$$= \int_D [f_1 S(\bar{g}) + f_2 T(\bar{g})] dx = \int_{\partial D} [C_S f_1 + C_T f_2]\bar{g} dx' + \int_D [S^*(f_1) + T^*(f_2)]\bar{g} dx.$$

Noting that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial Re z} - i\frac{\partial}{\partial Im z})$, we have

$$C_S = \frac{\overline{R_z}R_z + \overline{R_w}R_w}{b\partial_1 R} \ne 0, \quad C_T = \frac{R_w R_z - R_z R_w}{b\partial_1 R} = 0.$$

Since g is arbitrary, it holds that

$$\bar{\partial}_0^* f = S^*(f_1) + T^*(f_2); \qquad f \in \mathrm{Dom}(\bar{\partial}_0^*) \Leftrightarrow f_1|_{\partial D} = 0.$$

In order to compute $\bar{\partial}_1^*$, we should compute the torsion curvature of the changing system in $\Lambda_{(0,1)}$: $\bar{w}_1$ and $\bar{w}_2$.

Lemma 2.4. $\bar{\partial}_1 \bar{w}_j = \frac{1}{2} c_j \bar{w}_1 \wedge \bar{w}_2$, $j = 1, 2$.

Proof. According to Lemma 2.1, we get

$$\begin{aligned} dw \wedge dz &= w_1 \wedge w_2, \\ \partial w_1 &= \Big(\partial_w (\frac{R_z}{2b}) - \partial_z (\frac{R_w}{2b})\Big) w_1 \wedge w_2 = \frac{1}{2} \bar{c}_1 w_1 \wedge w_2, \\ \partial w_2 &= \Big(\partial_w (\frac{\overline{R_w}}{2b}) + \partial_z (\frac{\overline{R_z}}{2b})\Big) w_1 \wedge w_2 = \frac{1}{2} \bar{c}_2 w_1 \wedge w_2. \end{aligned} \tag{2.6}$$

Lemma 2.5. $\bar{\partial}_1^* h = -(T^* - \bar{c}_1)(h)\bar{w}_1 + (S^* + \bar{c}_2)(h)\bar{w}_2$; $h \in \mathrm{Dom}(\bar{\partial}_1^*) \Leftrightarrow h|_{\partial D} = 0$.

Proof. According to Lemma 2.1, Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} &\int_D \langle \bar{\partial}_1^* h, f \rangle dx = \int_D \langle h, \bar{\partial}_1 f \rangle dx \\ &= \frac{1}{2} \int_D \langle h \bar{w}_1 \wedge \bar{w}_2, [\bar{S}(f_2) - \bar{T}(f_1) + c_1 f_1 + c_2 f_2] \bar{w}_1 \wedge \bar{w}_2 \rangle dx \\ &= 2 \int_D h[(S + \bar{c}_2)(\bar{f}_2) - (T - \bar{c}_1)(\bar{f}_1)] dx \\ &= 2 \int_{\partial D} h[C_S \bar{f}_2 - C_T \bar{f}_1] + 2 \int_D (S^* + \bar{c}_2)(h)\bar{f}_2 - (T^* - \bar{c}_1)(h)\bar{f}_1 dx. \end{aligned}$$

Since f is arbitrary, $C_S \neq 0$, $C_T = 0$, we see that $h \in \mathrm{Dom}(\bar{\partial}_1^*) \Leftrightarrow h|_{\partial D} = 0$. We complete the proof of Lemma 2.5.

From Lemma 2.3 and Lemma 2.5, we can prove the following

Theorem 2.1. *For the operator* Π *defined by* (2.4), *for any* $U \in \mathrm{Dom}(\Pi)$,

$$\begin{aligned} \Pi(U) = \frac{1}{2}\{ &(\bar{S}S^* + T^*\bar{T} - \bar{c}_1 \bar{T} - c_1 T^* - T^*(c_1) + c_1 c_2)(U_1)\bar{w}_1 \\ &+ ([\bar{S}, T^*] + \bar{c}_1 \bar{S} - c_2 T^* - T^*(c_2) + \bar{c}_1 c_2)(U_2)\bar{w}_1 \\ &+ (\bar{T}T^* + S^*\bar{S} + \bar{c}_2 \bar{S} + c_2 S^* + S^*(c_2) + \bar{c}_2 c_2)(U_2)\bar{w}_2 \\ &+ ([\bar{T}, S^*] - \bar{c}_2 \bar{T} + c_1 S^* + S^*(c_1) + \bar{c}_2 c_1)(U_1)\bar{w}_2 \}; \end{aligned}$$

$U \in \mathrm{Dom}(\bar{\partial}_0^*) \Leftrightarrow U_1|_{\partial D} = 0$; $\bar{\partial}_1 U \in Dom(\bar{\partial}_1^*) \Leftrightarrow (\bar{S} + c_2)(U_2) - (\bar{T} - c_1)(U_1)|_{\partial D} = 0$.

Proof. From Lemma 2.3 and Lemma 2.5, we get

$$\begin{aligned} \bar{\partial}_0 \bar{\partial}_0^* U &= \bar{\partial}_0 [S^*(U_1) + T^*(U_2)] \\ &= \frac{1}{2}\{\bar{S}[S^*(U_1) + T^*(U_2)]\bar{w}_1 + \bar{T}[S^*(U_1) + T^*(U_2)]\bar{w}_2\}; \\ \bar{\partial}_1^* \bar{\partial}_1 U &= \frac{1}{2} \bar{\partial}_1^* \{[(\bar{S} + c_2)(U_2) - (\bar{T} - c_1)(U_1)]\bar{w}_1 \wedge \bar{w}_2\} \\ &= \frac{1}{2}\{ -(T^* - \bar{c}_1)(\bar{S}U_2 - \bar{T}U_1 + c_1 U_1 + c_2 U_2)\bar{w}_1 \\ &\quad + (S^* + \bar{c}_2)(\bar{S}U_2 - \bar{T}U_1 + c_1 U_1 + c_2 U_2)\bar{w}_2\}. \end{aligned}$$

Summing up, the theorem is proved.

§3. Parametrix and Applications

In this section we give at first a general formula similar to the Green Integral Formula from which two important formulae are deduced, then construct two parametrices by Calculus of Pseudodifferential Operators and compute some kernels.

Theorem 3.1. *Assume that* $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u_i, v_i \in C_c^\infty(U)$ *for i=1,2; U is a neighbourhood of origin. Then*

$$\int_D \langle \Pi(u_1\bar{w}_1 + u_2\bar{w}_2), v_1\bar{w}_1 + v_2\bar{w}_2 \rangle dy$$

$$= \int_{\partial D} -C_{\bar{S}} S(u_1)\bar{v}_1 + C_{S^*}\bar{S}(u_2)\bar{v}_2 + c_2 C_{S^*} u_2\bar{v}_2 + C_{S^*} u_1 \bar{S}^* \bar{v}_1 + C_{\bar{S}} u_2 S \bar{v}_2 dy'$$

$$+ \int_D u_1[S\bar{S}^* + \bar{T}^*T - \bar{c}_1 T^* - \bar{T}^*(\bar{c}_1) - T^*(c_1) + c_1\bar{c}_1](\bar{v}_1)dy$$

$$+ \int_D u_2[\bar{S}^*S + T\bar{T}^* + \bar{c}_2\bar{S}^* + c_2 S + S(c_2) + \bar{S}^*(\bar{c}_2) + \bar{c}_2 c_2](\bar{v}_2)dy$$

$$+ \int_D A_1(\partial)(u_2)\bar{v}_1 + A_1'(\partial)(u_1)\bar{v}_2 dy,$$

where $A_1(\partial), A_1'(\partial)$ *denote differential operators of degree* ≤ 1.

Proof. From Theorem 2.1, the theorem can be proved by Lemma 2.2.

Proposition 3.1. *There exists a neighbourhood U of the origin and a coordinate system* $\{r_j\}_{j=1}^4$ *such that for any* $x \in U \cap D$,

(i) $r_1(x) = R, r_i(x) = x_i$+*higher terms*, $i = 2, 3, 4$;

(ii) $\mathrm{Re}S(r_i) = 0$ for $i = 2, 3, 4$.

Proof. Since $\mathrm{Re}S = \frac{1}{4b}\sum_{j=1}^4 R_j\partial_j$, let $r_j = x_j + h_j$, then it is sufficient to solve equation $4b\mathrm{Re}S(h_j) = -4b\mathrm{Re}S(x_j)$ with condition $h_j(0) = 0$. The existence of h_j is guaranteed by the Cauchy-Kowalevskaya theorem.

Expanding into Taylor series, it holds that

$$r_j(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{i=2}^4 R_j\partial_j\Big)^n (x_j)(-x_1)^n, \forall j = 2, 3, 4.$$

Therefore (i) is satisfied.

3.1. Construction of Parametrix for U_1

Define $\psi = 1/\{|S(\exp\{i\sum r_j\xi_j\})|^2 + |T(\exp\{i\sum r_j\xi_j\})|^2\}(\xi, y)$ and

$$K_1(w, z; w_1, z_1) = K_1(x, y) = \frac{1}{(2\pi)^4}\int_{R^4} \exp\{i\sum_2^4(r_i(x) - r_i(y))\xi_i\}$$

$$\times \big(\exp\{i(r_1(x) - r_1(y))\xi_1\} - \exp\{i(r_1(x) + r_1(y))\xi_1\}\big)\psi d\xi.$$

Lemma 3.1. *The kernel* $K_1(x, y)$ *satisfies following conditions:*

(i) $K_1(x, y)|_{x\in\partial D} = 0, K_1(x, y)|_{y\in\partial D} = 0$;

(ii) $\int K_1(x, y)g(y')\delta(y_1 - r(y'))dy'|_{x\in\partial D} = 0, \forall g \in C_c^\infty(\partial D \cap U)$,*where* δ *is the Dirac distribution* (cf. [12]);

(iii) $\int_D(-\bar{S}S - T\bar{T})_y K_1(x, y)f(y)dy = |J|f(x) + \mathcal{R}_{-1}(f)(x), \forall x : R(x) < 0, \forall f \in C_c^\infty(U)$;

where $\mathcal{R}_{-1}$ represents an operator with kernel

$$\mathcal{R}_{-1}(x,y)=\int \exp\{i\sum_2^4(r_i(x)-r_i(y))\xi_i\}\Big\{\sum_{k=1}^{2}\exp\{i(r_1(x)-(-1)^k r_1(y))\xi_1\}q^k_{-1}\Big\}d\xi,$$

$q^k_{-1}(y,\xi)=\psi(p^k_1+p^k_0)(y,\xi)$*,* p^k_j *is homogeneous in* ξ *of degree* j*,* $k=1,2$*,* $j=0,1$ *respectively;* $|J|$ *is the Jacobian of coordinate transformation* J*:* $x_j\to r_j$*,* $1\le j\le 4$*.*

Proof. From the expreesion of $K_1(x,y)$ we have

$$K_1(x,y)|_{\partial D}=\frac{1}{(2\pi)^4}\int_{R^4}\exp\Big\{i\sum_2^4(r_i(x)-r_i(y))\xi_i\Big\}\Big(-2i\sin(R(y)\xi_1)\psi\Big)d\xi.$$

By Propopsition 3.1, ψ is an even function in ξ_1, so the integral with respect to ξ_1 equals zero.

$K_1(x,y)|_{y\in\partial D}=0$ is obvious. Thus (i) is proved.

(ii) is a special case of (i).

As for (iii), direct computation shows that

$$\begin{aligned}&\int_D(-\bar{S}S-T\bar{T})_y K_1(x,y)f(y)dy\\ &=\int_D\frac{f(y)}{(2\pi)^4}\int_{R^4}\exp\{i\sum_2^4(r_i(x)-r_i(y))\xi_i\}[\exp\{i(r_1(x)-r_1(y))\xi_1\}\\ &\quad-\exp\{ir_1(x)+r_1(y))\xi_1\}]d\xi dy+\mathcal{R}_{-1}(f)(x)\\ &=\int_D[\delta(r_j(x)-r_j(y))-\delta(r_1(x)+r_1(y),r_i(x)-r_i(y))]f(y)dy+\mathcal{R}_{-1}(f)(x).\end{aligned}$$

For any given point $x:R(x)<0$, take a smooth function κ such that

$$\kappa(x)=1,\ \text{suppt}\ \kappa\subset\{y:|x-y|<\delta\}\subset\{R(y)<0\}$$

and a smooth function τ such that

$$\text{suppt}\ \tau\subset\{y:|(x_1+y_1,x'-y')|<\delta\}\subset\{R(y)>0\}.$$

Where δ is a small positive number.

$$\begin{aligned}&\int\frac{1}{(4\pi)^4}\int\exp\{i\sum(r_j(x)-r_j(y))\xi_j\}d\xi f(y)dy\\ &=\int\frac{1}{(4\pi)^4}\int\exp\{i(r_j(x)-r_j(y))\xi_j\}d\xi f\circ J^{-1}(r_j(y))|J|dr_j(y)=|J|f\circ J^{-1}(r_j(x))\\ &=|J|f(x),\forall f\in C^\infty_c(U);\end{aligned}$$

therefore

$$\begin{aligned}&\int_D[\delta(r_j(x)-r_j(y))-\delta(r_1(x)+r_1(y),r_i(x)-r_i(y))]f(y)dy\\ &=\int_{R^4}\delta(r_j(x)-r_j(y))\kappa(y)f(y)-\delta(r_1(x)+r_1(y),r_i(x)-r_i(y))\tau(y)f(y)dy\\ &=[\kappa|J|f](x)-[\tau|J|f](x)=|J|f(x).\end{aligned}$$

Here $\mathcal{R}_{-1}$ is an operator which has the property stated in Lemma 3.1. So (iii) is proved.

In C^∞ choose a sequence of functions which converge to $K_1(x,y)$ with x as parameter; by the Lebesgue dominated convergence theorem, we can set $\bar{v}_1(y)=K_1(x,y)$ and $v_2=0$ in Theorem 3.1. With the help of $u_1|_{\partial D}=0$, we get

Lemma 3.2. *Assume that* $\Pi(u) = f, u = (u_1, u_2)$, $\mathrm{supp}(u)$, $\mathrm{supp}(f) \in U \cap D$. *Then*

$$u_1 = \int_D K_1(x,y)\{f + A_1(\partial)u_2\}(y)dy + \mathcal{R}_{-1}u_1,$$

where $\mathcal{R}_{-1}$ *is defined in Lemma* 3.1.

3.2. Construction of Parametrix for U_2

Since $u_2|_{\partial D}$ is determined by the equation on the boundary of D : $\bar{S}u_2 = \bar{T}u_1 - c_1u_1 - c_2u_2$, we construct a delicate kernel for u_2. Let

$$P_2 = P_2^2 + P_2^1 + c = \bar{S}^*S + T\bar{T}^* + c_2S + \bar{c}_2\bar{S},$$

where $c = \bar{S}^*(\bar{c}_2) + S(c_2) + S^*(c_2) + \bar{c}_2c_2$; P_2^j denotes the part of order j of P_2, $j = 1, 2$ respectively; $P_2^1 = b_j\partial_j$.

Let $R'' = \frac{R}{4b}, R' = dR, d \approx 1$; $\psi_j = 1/\{\sum R'_jR'_j|i\xi_1|^2 + |i\xi'|^2 - 2(-1)^jR'_{j'}i\xi_1i\xi'_{j'}\}$. The kernel for u_2 is defined as follows:

$$\begin{aligned}K_2(x,y) =&\frac{1}{(2\pi)^4}\int \exp\{i(x-y)'\cdot\xi' + i[R''(x) - R'(y)]\xi_1\}\Big\{\psi_1(y,\xi) + i\xi_1[R'_{jj}\psi_1^2\\ &+ 2R'_j\partial'_{y_j}\psi_1(y,\xi)\psi_1 + b_jR'_j\psi_1^2] + 2i\xi_{j'}\partial_{j'}\psi_1(y,\xi)\psi_1 + i\xi_{j'}b_{j'}\psi_1^2\Big\}d\xi\\ &- \frac{1}{(2\pi)^4}\int \exp\{i(x-y)'\cdot\xi' + i[R''(x) + R'(y)]\xi_1\}\Big\{\psi_2(y,\xi) - i\xi_1[R'_{jj}\psi_2^2\\ &+ 2R'_j\partial_j\psi_2(y,\xi)\psi_2 + b_jR'_j\psi_2] + 2i\xi_{j'}\partial_{j'}\psi_2(y,\xi)\psi_2 + i\xi_{j'}b_{j'}\psi_2^2\Big\}d\xi.\end{aligned}$$

Lemma 3.3. *The kernel* K_2 *satisfies following conditions:*

(i) $K_2(x,y)_{x\in\partial D} = 0$,

(ii) $\int_D P_2(\partial_y)K_2(x,y)f(y)dy = f(x) + \mathcal{R}_{-2}(f)(x)$, $\forall x : R(x) < 0; \forall f \in C_c^\infty(U)$,

where $\mathcal{R}_{-2}$ *is an operator with kernel*

$$\begin{aligned}\mathcal{R}_{-2}(x,y) = \int_D &\exp\{i(x-y)'\cdot\xi'\}\{\exp\{i(R''(x) - R'(y))\xi_1\}q_{-2}\\ &- \exp\{i(R''(x) + R'(y))\xi_1\}q'_{-2}\}d\xi,\end{aligned}$$

where $q_{-2}, q'_{-2} \in S^{-2}_{1,0}$.

Proof. For the property (i), we have

$$\begin{aligned}&K_2(x,y)|_{\partial D}\\ =&\frac{1}{(2\pi)^4}\int \exp\{i(x-y)'\cdot\xi'\}\Big(-i\sin[R'(y)\xi_1]\Big)\Big\{\psi_1(y,\xi) + i\xi_1[R'_{jj}\psi_1^2\\ &+ 2R'_j\partial_j\psi_1\psi_1 + b_jR'_j\psi_1^2] + 2i\xi_{j'}\partial_{j'}\psi_1\psi_1 + i\xi_{j'}b_{j'}\psi_1^2\\ &+ \psi_2(y,\xi) - i\xi_1[R'_{jj}\psi_2^2 + 2R'_j\partial_j\psi_2\psi_2 + b_jR'_j\psi_2^2]\\ &+ 2i\xi_{j'}\partial_{j'}\psi_2\psi_2 + i\xi_{j'}b_{j'}\psi_2^2\Big\}d\xi_1d\xi'\\ &+ \cos\Big([R'(y)\xi_1]\Big)\Big\{\psi_1 - \psi_2 + \cdots\Big\}d\xi_1d\xi' = 0,\end{aligned}$$

where we have used the facts that $\psi_1 + \psi_2$ is even in $\xi_1, \psi_1^2 - \psi_2^2$ is odd in ξ_1,etc.

Property (ii) can be read from following facts:

$$P_2^2+P_2^1\Big(\exp\{i(x-y)'\cdot\xi'+i[R''(x)-R'(y)]\xi_1\}\psi_1(y,\xi)\Big)$$
$$=\Big\{[|i\xi'|^2+R'_jR'_j|i\xi_1|^2+2R'_{j'}i\xi_{j'}i\xi_1-i\xi_1R'_{jj}-i\xi_1b_jR'_j-ib_{j'}\xi_{j'}]\psi_1$$
$$-[2i\xi_1R'_j\partial_j\psi_1+2i\xi_{j'}\partial_{j'}\psi_1]\Big\}\exp\{i(x-y)'\cdot\xi'+i[R''(x)-R'(y)]\xi_1\};$$
$$P_2^2+P_2^1\Big(\exp\{i(x-y)'\cdot\xi'+i[R''(x)+R'(y)]\xi_1\}\psi_2(y,\xi)\Big)$$
$$=\Big\{[|i\xi'|^2+R'_jR'_j|i\xi_1|^2-2R'_{j'}i\xi_{j'}i\xi_1+i\xi_1R'_{jj}+i\xi_1b_jR'_j-ib_{j'}\xi_{j'}]\psi_2$$
$$+2[i\xi_1R'_j\partial_j\psi_2-i\xi_{j'}\partial_{j'}\psi_2]\Big\}\exp\{i(x-y)'\cdot\xi'+i[R''(x)+R'(y)]\xi_1\}.$$

The rest is similar to the proof of Lemma 3.1.

Lemma 3.4. *Assume that* $\Pi(u)=f$, $\operatorname{supp}(u)$, $\operatorname{supp}(f)\in U\cap D$. *Then*

$$u_2=\int_D K_2(x,y)\{f_2+A_2(\partial)(u_1)\}(y)dy+\mathcal{R}_{-2}(u_2)(x)-\int_{\partial D}C_{\bar S}(S+\bar c_2)(K_2)u_2dy'.$$

Proof. Letting $\bar v_2=K_2,\bar v_1=0$ and using the boundary condition: $\bar S u_2=\bar T u_1-c_1u_1-c_2u_2, u_1=0$ *on* ∂D, we get by Theorem 2.2

$$u_2=\int_D-\int_{\partial D}C_{S^*}K_2\bar S(u_2)+[C_{\bar S}\bar c_2+C_{S^*}c_2]K_2u_2+C_{\bar S}S(K_2)u_2dy'$$
$$=\int_D-\int_{\partial D}(\bar Tu_1-c_1u_1-c_2u_2)C_{S^*}K_2+[C_{\bar S}\bar c_2+C_{S^*}c_2]K_2u_2+C_{\bar S}S(K_2)u_2dy'$$
$$=\int_D-\int_{\partial D}[u_1\bar T^*(C_{S^*}K_2)+\bar c_2C_{\bar S}K_2u_2+C_{\bar S}S(K_2)u_2]dy'$$
$$=\int_D-\int_{\partial D}C_{\bar S}S(K_2)u_2+\bar c_2C_{\bar S}K_2u_2dy'.$$

Here we have used the properties that $C_T=C_{\bar T}=0$ and $\bar T(f)|_{\partial D}=\bar T|_{\partial D}(f|_{\partial D})$.

3.3. Computation of $S(K_2)$ on the Boundary of D

Lemma 3.5. $S_yK_2(x,y)|_{y\in\partial D}=[1/(2\pi)^3]\int\exp\{i(x-y)'\cdot\xi'\}\{-S(R')\}$

$$\times\Big(\sum_{k=1}^{2}\exp\{-iR''(x)E\}\frac{E}{2i\Delta}+R''(x)q_0+R''(x)^2q_1\Big)d\xi';$$

where $E=(-1)^k(R'_{j'}/\sum R'_jR'_j)\xi_{j'}+i\Delta$; $q_j=q_j(y,\xi')$ *denotes symbol* $\in S^j_{1,0}$, *for j=0,1;*

$$\Delta=[(\xi_{j'}^2/\sum R'_jR'_j)-((R'_{j'}/\sum R'_jR'_j)\xi_{j'})^2]^{\frac12}.$$

Proof. We rewrite K_2 as

$$K_2=\int\exp\{i(x-y)'\cdot\xi'\}\Big(\exp\{i[R''(x)-R'(y)]\xi_1\}(\psi_1+\Xi_1)$$
$$-\exp\{i[R''(x)+R'(y)]\xi_1\}(\psi_2+\Xi_2)\Big)d\xi,$$

where $\Xi_k=i\xi_{j'}\psi_k\partial_{j'}\psi_k+i\xi_{j'}b_{j'}\psi_k^2-(-1)^ki\xi_1\{R'_{jj}\psi_1^2+R'_j\psi_k\partial_j\psi_k+b_jR'_j\psi_k^2\}$. Let S' denote the part of S which is irrelevant to ∂_1. Then

$$\begin{aligned}
&S_y K_2(x,y)|_{y\in\partial D}\\
=&\frac{1}{(2\pi)^4}\int \exp\{i(x-y)'\cdot\xi'+iR''(x)\xi_1\}\Big\{-iS(R')\xi_1\psi_1-iS(R')\xi_1\Xi_1+S\psi_1\\
&-\sigma(S')[\psi_1+\Xi_1]-iS(R')\xi_1\psi_2-iS(R')\xi_1\Xi_2-S\psi_2+\sigma(S')[\psi_2+\Xi_2]\Big\}d\xi\\
=&\frac{1}{(2\pi)^4}\int \exp\{i(x-y)'\cdot\xi'+iR''(x)\xi_1\}\Big\{-iS(R')\xi_1(\psi_1+\psi_2+\Xi_1+\Xi_2)\\
&+S(\psi_1-\psi_2)+\sigma(S')[\Xi_2-\Xi_1+\psi_2-\psi_1]\Big\}d\xi.
\end{aligned}$$

The terms in $\{,\}$ are

$$\begin{aligned}
&\sum_{k=1}^{2}\Big(\frac{-iS(R')\xi_1+q_1}{i^2[R'_jR'_j\xi_1^2+2(-1)^kR'_{j'}\xi_1\xi_{j'}+\xi_{j'}^2]}+\frac{\vartheta\xi_1^2+\vartheta\xi_1q_1+\vartheta q_2}{\{i^2[R'_jR'_j\xi_1^2+2(-1)^kR'_{j'}\xi_1\xi_{j'}+\xi_{j'}^2]\}^2}\\
&+\frac{\vartheta\xi_1^4+\vartheta\xi_1^3q_1+\vartheta\xi_1^2q_2+\vartheta\xi_1q_3+\vartheta q_4}{\{i^2[R'_jR'_j\xi_1^2+2(-1)^kR'_{j'}\xi_1\xi_{j'}+\xi_{j'}^2]\}^3}\Big),
\end{aligned}$$

where $q_j=q_j(y,\xi')\in S^j_{1,0}$, $\vartheta=\vartheta(y)\in C^\infty(U)$ are different from time to time.

We introduce following notations

$$\begin{aligned}
F&=-(R'_{j'}/R'_jR'_j)\xi_{j'}+i\Delta;\\
F'&=(R'_{j'}/R'_jR'_j)\xi_{j'}+i\Delta;\\
E'&=((-1)^kR'_{j'}/\textstyle\sum R'_jR'_j)\xi_{j'}-i\Delta.
\end{aligned}$$

By Contour Integral Formula, the following relation holds

$$\begin{aligned}
&\int\exp\{(iR''(x)\xi_1)\}\xi_1\psi_1(y,\xi)d\xi_1=\frac{2\pi}{i}\exp\{-iR''F\}\frac{F}{2i\Delta\sum R'_jR'_j},\\
&\int\exp\{iR''(x)\xi_1\}\psi_2(y,\xi)\xi_1d\xi_1=\frac{2\pi}{i}\exp\{-iR''F'\}\frac{F'}{2i\Delta\sum R'_jR'_j};\\
&\frac{1}{2\pi i}\int\exp\{iR''\xi_1\}\frac{\xi_1^\lambda}{[\xi_1+E]^3[\xi_1+E']^3}d\xi_1\\
=&(-1)^{\lambda-1}\Big\{\frac{(i\Delta)^\lambda}{(2i\Delta)^5}4(\lambda-3)(\lambda-1)+q_{\lambda-5}\\
&-R''(x)\Big[\frac{(i\Delta)^\lambda}{(2i\Delta)^4}(2\lambda-3)+q_{\lambda-4}\Big]\Big\}\exp\{-iR''E\},\ 0\le\lambda\le5;\\
&\frac{1}{2\pi i}\int\exp\{iR''\xi_1\}\frac{\xi_1^\lambda}{[\xi_1+E]^2[\xi_1+E]^2}d\xi_1\\
=&(-1)^{\lambda-1}\Big\{\frac{(2i\Delta)^\lambda}{(2i\Delta)^3}(\lambda-1)+q_{\lambda-3}\\
&-R''(x)\Big[\frac{(i\Delta)^\lambda}{(2i\Delta)^2}+Q_{\lambda-2}\Big]\Big\}\exp\{-iR''(x)E\},\ 0\le\lambda\le3.
\end{aligned}$$

By above four formulae, we prove the lemma.

3.4. Computation of $\bar{S}_x[S_yK_2(x,y)]|_{y\in\partial D}|_{x\in\partial D}$

From Lemma 3.5, we get

Theorem 3.2.

$$\bar{S}_x[S_y K_2(x,y)]|_{y\in\partial D}|_{x\in\partial D}$$
$$= -\frac{1}{(2\pi)^3}\int \exp\{i(x-y)'\cdot\xi'\}\frac{S(R')}{\sum R_j' R_j'}\Big\{S(R'')\Big[\Delta^2-\Big((R_{j'}'/\sum R_j'R_j')\xi_{j'}\Big)^2\Big]/\Delta$$
$$-\frac{\sigma(\mathrm{Im}S)}{i}+q_0\Big\}d\xi',$$

where $\sigma(\mathrm{Im}S)$ is the symbol of $\mathrm{Im}S$.

§4. Various Estimates

In this section we give various estimates for the previous operators in L^p_β and $\Lambda^{p,p}_\alpha$ norms (for definitions, cf. [14]). We will use extensively the following propositions from several papers.

Proposition 4.1 (Theorem 6.3 in p.536 of [10]).

(D) *N is pseudo-local in the sense that if U is a neighbourhood in $\bar{M}$, and $\alpha\in L_2^{0,1}(M)$ such that $\alpha|_U\in C^\infty(U)$, then $N\alpha|_U\in C^\infty(U)$. Furthermore if $\alpha|_U\in H_t(U)$, for $t\ge 0$, then $N\alpha|_U\in H_{t+2s}(U)$.*

(E) *Let H denote the space of holomorphic functions in $L_2(M)$. If $\alpha\in$ Dom (clos$\bar\partial$), $\bar\partial\alpha\perp H^{0,1}$. Then there exists a unique $u\perp H$ such that $\bar\partial u=\alpha$. Thus α may be expressed by $u=\bar\partial^* N\alpha$. Furthermore, by (D), $\bar\partial^*N$ is pseudo-local, and if $\alpha|_U\in H_t(U)$, then $u|_U\in H_{t+s}(U)$.*

Proposition 4.2. *The restriction operator $\varphi\to\varphi|_{\partial D}$ maps $L^p_\beta(\bar D)$ to $\Lambda^{p,p}_{\beta-\frac{1}{p}}(\partial D)$ boundedly.*

The proof of Lemma 2.4 can be found in [1] with the help of interpolation theorem in Banach Space (cf. [25]) and the mapping properties of the Bessel Potential Operator in L^p_β and $\Lambda^{p,q}_\alpha$ spaces (cf. [24]).

Proposition 4.3[5]. *Assume that $\sigma: R^n\times R^n\to C$ is a continuous function such that $\forall\alpha=(\alpha_1,\cdots,\alpha_n),\beta=(\beta_1,\cdots,\beta_n),\alpha_i,\beta_i=0,1,2,3$, derivatives $\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)\in L^\infty(R^n\times R^n)$. Then $\sigma(x,\xi)$ defines an operator $\sigma(x,\partial)$ which is bounded on $L^2(R^n)$.*

Proposition 4.4 (implicitly contained in [5]). *If $\sigma(x,\xi)\in S^m_{1,0}$, then $\sigma(x,\partial)$ maps L^p_α boundedly into $L^p_{\alpha-m}$.*

Proposition 4.5 (a variant of Theorem 4 in [11]). *Assume that A,B are given by $Au(x)=(a(x,\xi)\hat u(\xi))^\vee$, $Bu(x)=(b(x,\xi)\hat u(\xi))^\vee$, where $a\in S^\lambda_{1,0}, b\in S^\mu_{1,0}$. Then the symbol of $A\circ B$ is*

$$\sigma(A\circ B)=\sum\frac{1}{\alpha!}(\frac{1}{i}\partial_\xi)^\alpha a(x,\xi)(\frac{\partial}{\partial x})^\alpha b(x,\xi).$$

Furthermore, $\sigma(A\circ B)-\sum\limits_{|\alpha|\le N}=r_N\in S^{\lambda+\mu-N-1}_{1,0}$.

Proposition 4.6 (Theorem 5 in [11]). *Assume that A is given by*

$$Au(x)=(a(x,\xi)\hat u(\xi))^\vee,\ \max\{|\alpha(x,\xi)|:x\in U,|\xi|=1\}=K,$$

where $a(x,\xi)\in S^\sigma_{1,0}$. Then for s real and any $\epsilon>0$ there is a constant C such that

$$\|Au\|_{s-\sigma}\le(K+\epsilon)\|u\|_s+C\|u\|_{s-1/2},\forall u\in C_c^\infty(U).$$

($\|\cdot\|_s$ *is the Sobolev norm*).

Proposition 4.7 (Theorem 18 in [13]). *Suppose* $L = \sum_{j=1}^{n} X_j^2 + X_0$, *where all commutators of weight* $\le r$ *span the tangent space at each point, and* $L(f) = g, f \in L^p(M), 1 < p < \infty$.

(a) *If* $g \in L^p_\alpha(M)$, *then* $f \in L^p_{\alpha+\frac{2}{r}}, \alpha > 0$.

(b) *If* $g \in \Lambda_\alpha(M)$, *then* $f \in \Lambda_{\alpha+\frac{2}{r}}, \alpha > 0$.

(c) *If* $g \in L^\infty(M)$, *then* $f \in \Lambda_{\frac{2}{r}}(M)$.

Proposition 4.8[2]. *Assume that* $\mathcal{A}(v) = f$ *on* $U \cap \partial D$, *and the commutators of length* $\le 2m$ *of* $\bar{T}_b, T_b$ *span the tangent space. Then* $v = \wp(f) + E(v)$, *where* $\wp, E$ *satisfy conditions:*

(1) E *is bounded from* $L^p(U)$ *to* $L^p_\epsilon(U)$ *for some* $\epsilon > 0$;

(2) $\wp$ *maps* $L^p(U \cap \partial D)$ *boundedly into* $L^p_{\frac{1}{m}}(U \cap \partial D)$;

(3) $\wp \circ T_b$ *(or* $\bar{T}_b$*)*, T_b *(or* $\bar{T}_b$*)* $\circ \wp$ *map* $L^p(U \cap \partial D)$ *boundedly into* $L^p_{\frac{1}{2m}}(U \cap \partial D)$;

(4) T_b*(or* $\bar{T}_b$*)* $\circ \wp \circ \bar{T}_b$*(or* T_b*)*, $\bar{T}_b$*(or* T_b*)* $\circ T_b$*(or* $\bar{T}_b$*)* $\circ \wp$ *and* $\wp \circ T_b$*(or* $\bar{T}_b$*)* $\circ T_b$*(or* $\bar{T}_b$*) map* $L^p(U \cap \partial D)$ *boundedly into itself.*

4.1. Estimate of $\|u_1\|_{L^p_\alpha}$

Lemma 4.1. $\|u_1\|_{L^p_{\beta+2}} \le C\Big(\|u_1\|_{L^p_{\beta+1}} + \|u_2\|_{L^p_{\beta+1}} + \|f\|_{L^p_\beta}\Big)$.

Proof. From Lemma 3.2, we have

$$u_1 = \int_D K_1(x,y)\{f_1 + A_1(\partial)u_2\}(y)dy + \mathcal{R}_{-1}(u_1)(x).$$

Since

$$K_1(x,y) = \frac{1}{(2\pi)^4}\int_{R^4} \exp\{i\sum_2^4 (r_i(x) - r_i(y))\xi_i\} \times \big(\exp\{i(r_1(x) - r_1(y))\xi_1\}$$
$$- \exp\{i(r_1(x) + r_1(y))\xi_1\}\big)\frac{1}{|\sigma(S)|^2 + |\sigma(T)|^2}(\xi, y)d\xi,$$

we have

$$\partial_{x_j} K_1(x,y) = \int e^{\cdots} i\sigma(\partial_{x_j})/\{|\sigma(S)|^2 + |\sigma(T)|^2\}d\xi.$$

Generally we have

$$\partial_j\partial_k K_1(x,y) = \int e^{\cdots} \sum_{j,k=1}^{4} \omega_{jk}(x)\xi_j\xi_k/[|\sigma(S)|^2 + |\sigma(T)|^2]d\xi. \tag{4.1}$$

Since the operator defined by the kernel $K_1(x,y)$ is an oscillatory integral operator which is approximately a pseudodifferential operator, we treat them by doing coordinate transformation $J : x \to (r_j)_1^4$. By Proposition 3.1, J is regular; therefore function space $L^p_\alpha(D)$ remains the same under the transformation J. Now the theory of Singular Integral Operator (Theorem 4.7 in [15]) can be applied and we get

$$\|\partial_j\partial_k \int K_1(x,y)f(y)dy\|_{L^p_\beta} \le C\|f\|_{L^p_\beta}. \tag{4.2}$$

Applying (4.2), we get

$$\|\int K_1(x,y)A_1(y,\partial)u_2(y)dy\|_{L^p_{\beta+2}} \le C\|u_2\|_{L^p_{\beta+1}}. \tag{4.3}$$

The term $\mathcal{R}_{-1}(u_1)$ is treated in the same way since $\mathcal{R}_{-1}$ has the expression given in Lemma 3.2.

4.2. Estimate for u_2: Part I

From Lemma 3.4, we have

$$\begin{aligned} u_2 = &\int_D K_2(x,y)\{f_2 + A_2(\partial)(u_1)\}(y)dy + \mathcal{R}_{-1}(u_2)(x) \\ &- \int_{\partial D} \Big\{C_{\bar{S}}(y)S_y[K_2(x,y)]|_{y\in\partial D} + C_{\bar{S}}\bar{c}_2 K_2(x,y)|_{y\in\partial D}\Big\}(u_{2b})dy'. \end{aligned} \tag{4.4.1}$$

Therefore

$$\begin{aligned} \bar{S}_x u_2(x) = &\bar{S}\int_D K_2(x,y)\{f_2 + A_2(\partial)(u_1)\}(y)dy + \bar{S}\mathcal{R}_{-2}u_2 \\ &- \bar{S}\int_{\partial D} \Big\{C_{\bar{S}}(y)S_y[K_2(x,y)]|_{y\in\partial D} + C_{\bar{S}}\bar{c}_2 K_2(x,y)|_{y\in\partial D}\Big\}u_{2b}(y')dy'. \end{aligned} \tag{4.4.2}$$

4.3. Estimate of Terms of $\bar{S}_x u_2|_{\partial D}$ Arising from $\int_D$

Denote $\Lambda^{p,p}_\beta$ by $B(\beta)$.

Lemma 4.2. *For any $f, u \in L^p(U)$, $\mathrm{supp}(f)$, $\mathrm{supp}(u) \in U$, following estimates hold:*

$$\|\int \bar{S}K_2(x,y)f_2(y)dy|_{x\in\partial D}\|_{B(\beta+1-\frac{1}{p})} \le C\|f_2\|_{L^p_\beta}; \tag{4.5}$$

$$\|\bar{S}\mathcal{R}_{-2}u_2|_{x\in\partial D}\|_{B(\beta+1-\frac{1}{p})} \le C\|u_2\|_{L^p_\beta}; \tag{4.6}$$

$$\|\int S\circ A_2(u_1)K_2 dy|_{x\in\partial D}\|_{B(\beta+1-\frac{1}{p})} = 0; \tag{4.7}$$

$$\|\int A_2(\partial)(u_1)\bar{S}K_2(x,y)dy|_{x\in\partial D}\|_{B(\beta+1-\frac{1}{p})} \le C\|u_1\|_{L^p_{\beta+1}}. \tag{4.8}$$

Proof. Similar to the proof of Lemma 4.1, we can prove Lemma 4.2 with the help of Proposition 4.2 and Lemma 3.3.

4.4. Estimates of $u_2|_{\partial D}$ arising from $\int_{\partial D} C_{\bar{S}}\bar{c}_2 K_2(x,y)u_{2b}(y)_{\partial D}dy'$

Lemma 4.3. (i) $K_2(x,y)|_{x,y\in\partial D} = \int \exp\{i(x-y)'\cdot\xi'\}\sum_k \theta_k(y)q^k_{-2}(\xi)d\xi$; where $q^k_{-2}(\xi)$ are homogeneous of degree -2 in ξ';

(ii) $$\|\int_{\partial D} C_{\bar{S}}\bar{c}_2(y)K_2(x',y')u_{2b}(y')dy'|_{\partial D}\|_{B(\beta+2-\epsilon)} \le C\|u_{2b}\|_{B(\beta)};$$

(iii) $$\bar{S}_x\int_{\partial D} C_{\bar{S}}\bar{c}_2(y)K_2(x,y)u_{2b}(y')dy'|_{\partial D} = \Big(\bar{S}(R")\frac{\sum R'_{j'}\xi_{j'}}{i^2(\sum R'_j R'_j)^2\Delta} + B_{-1}(\xi')\Big)^\wedge,$$

$$\|\int B_{-1}(\partial')u_{2b}(y')dy'\|_{B(\beta+1-\epsilon)} \le C\|u_{2b}\|_{B(\beta)}.$$

Proof. By following Lemmas 4.6 and 4.7, using Proposition 4.4, (i) is verified from the expression of K_2. (ii) and (iii) are treated similarly.

4.5. Estimate for u_2: Part II, the Boundary Equation

Since

$$\bar{S}u_2|_{\partial D} = \Big(\bar{T}u_1 - c_1u_1 - c_2u_2\Big)|_{\partial D} = 0,$$
$$\bar{T}u_1|_{\partial D} = \bar{T}|_{\partial D}(u_1|_{\partial D}) = 0,$$

from (4.4.2), we get

$$-\int_{\partial D} C_{\bar{S}}\bar{S}_x[S_y K_2(x,y)|_{y\in\partial D}]|_{\partial D}u_{2b}(y')dy' = -c_2u_{2b} + g, \tag{4.9}$$

where

$$g = -\bar S \int K_2\{f - A_2(\partial)u_1\} - \bar S \mathcal{R}_{-2}u_2 + \bar S \int_{\partial D} C_{\bar S}\bar c_2 K_2 u_{2b}.$$

From Theorem 3.2, we get

$$\bar S^x \int C_{\bar S}(y')S_y K_2(x,y')u_{2b}dy' = -\frac{1}{(2\pi)^3}\int C_{\bar S}\frac{S(R')}{\sum R'_j R'_j}u_{2b}(y)dy'$$
$$\times \int e^{i(x-y)'\cdot\xi'}\Big\{S(R'')(x)\frac{\Delta^2 - ((R'_{j'}/\sum R'_j R'_j)\xi_{j'})^2}{\Delta} - \frac{\sigma(\operatorname{Im} S)}{i}\Big\}d\xi',$$

where $\operatorname{Im} S = \frac{1}{4b}R_1\partial_2 + o(1)\partial_3 + o(1)\partial_4$; $\frac{\sigma(\operatorname{Im} S)}{i} = \frac{1}{4b}\xi_2 + o(1)\xi_3 + o(1)\xi_4$.

Lemma 4.4. *The boundary value of u_2 satisfies the following equation:*

(i) $\Pi_b(u_{2b})(x) = g - c_2 u_{2b} = g_1$;

(ii) $$\Pi(v_{2b})(x) = \frac{1}{(2\pi)^3}\int e^{i(x-y)'\cdot\xi'}\frac{S(R')}{\sum R'_j R'_j}(y)$$
$$\times \Big\{S(R'')(x)\frac{\Delta^2 - ((R'_{j'}/\sum R'_j R'_j)\xi_{j'})^2}{\Delta} - \frac{\sigma(\operatorname{Im} S)}{i}\Big\}u_{2b}(y')d\xi' dy';$$

(iii) $\|g_1\|_{B(\beta+1-\frac{1}{p}-\epsilon)} \le C\Big(\|f\|_{L^p_\beta} + \|u_1\|_{L^p_{\beta+1}} + \|u_2\|_{L^p_\beta} + \|u_{2b}\|_{B(\beta+1-\frac{1}{p})}\Big)$.

In order to estimate u_{2b},we must look for a parametrix for the pseudodifferential operator Π_b (whose symbol is $S(R')(x)\frac{\Delta^2-((R'_{j'}/\sum R'_j R'_j)\xi_{j'})^2}{\Delta} - \frac{\sigma(\operatorname{Im} S)}{i}$) which is elliptic in region: $\{\xi' \in R^3 : |\xi_3,\xi_4| \ge \delta|\xi_2|, |\xi'| \ge M(\delta)\}$ for any $\delta > 0$; $M(\delta)$ is a constant depending on δ.

Since $|\xi'|$ is involved, we modify it at first.

Lemma 4.5. $f \mapsto (|\xi'|^\lambda \hat f(\xi'))^\vee$ *is bounded from L^p_β to $L^p_{\beta-\lambda}$ and from $\Lambda^{p,p}_\beta$ to $\Lambda^{p,p}_{\beta-\lambda}$, $\forall\lambda \ge 0$.*

Proof. We give the proof of the lemma only in case $0 \le \lambda \le 2$. If $\lambda = 2$, the lemma holds by the definition of function spaces L^p_β and $\Lambda^{p,p}_\beta$. If $\lambda = 0$, the map is identity, the lemma is obviously valid. For the rest λ, making use of theorem of interpolation, we are led to verify that the following operator

$$T_z f = (|\xi'|^{2z}\hat f(\xi'))^\vee, 0 \le z \le 1,$$

satisfies all conditions in Theorem 4.1 in [15](p.205). 6.12 in [14] (p.51) gives the details of the proof.

We analysis the Π_b operator defined in Lemma 4.5 in two cases. Fix a small $\delta > 0$. Decompose the phase space (ξ') into three parts:

$$\text{Region I} : |\xi'| \le 2;$$
$$\text{Region II} : |\xi'| \ge 2, |\xi'| + \xi_2 \le \delta|\xi'|; \text{ and}$$
$$\text{Region III} : |\xi'| \ge 2, |\xi'| + \xi_2 \ge \delta|\xi'|.$$

Choose three smooth functions κ, ρ and χ such that

$\kappa(\xi') = 1$ over $|\xi'| \le 1$, $\operatorname{supp}(\kappa) \subset \{|\xi'| \le 2\}$;

$\rho = 0$ over $|\xi'| \le 1$, $\rho + \chi = 1 - \kappa$;

$\operatorname{supp}(\rho) \subset \{|\xi'| + \xi_2 \le 2\delta|\xi'|\}$, $\rho = 1$ over the region II;

$\operatorname{supp}(\chi) \subset \{|\xi'| + \xi_2 \ge \delta|\xi'|\}$, $\chi = 1$ over $\{|\xi'| + \xi_2 \ge 2\delta|\xi'|\}$.

4.6. Elliptic Case

We solve the following equation

$$\Pi_b(\partial)\chi(\partial)u_{2b} = \chi(\partial)g_1 + [\Pi_b, \chi(\partial)]u_{2b} = g_2. \tag{4.10}$$

Lemma 4.6. (i) *There exists a kernel* k_1 *of type* 1 (p. 208, [13]) *such that*

$$k_1\Pi_b(\partial) = I + \mathcal{R}_{-1}, \Pi_b(\partial)k_1 = I + \mathcal{R}'_{-1};$$

(ii) *For the solution of* (4.10), *we have*

$$\|u_{2b}\|_{B(\beta+2-\frac{1}{p}-\epsilon)} \le C\|g_2\|_{B(\beta+1-\frac{1}{p})},$$

therefore

$$\|u_{2b}\|_{B(\beta+2-\frac{1}{p}-\epsilon)} \le C\|g_1\|_{B(\beta+1-\frac{1}{p})} + \|u_{2b}\|_{B(\beta+1-\frac{1}{p})}.$$

Proof. (i) Consider the part of Π_b of degree one which is elliptic over suppt $\chi(\xi')$, we can construct the parametrix k_1 by Proposition 4.5. As for (ii), it can be proved by general theory of elliptic equation with constant coefficients. We sketch it as follows: one obtains the L^p_β estimate by Proposition 4.4 and the $\Lambda^{p,p}_\beta$ estimate by the following Lemma 4.6 and thus prove the first inequality; by the following Lemmas 4.6 and 4.7, we get the Sobolev inequality in $\Lambda^{p,p}_\beta$ norms. The second inequality thus follows.

4.7. Hypoelliptic Case

We are going to estimate the term $\rho(\partial')u_{2b}$. We derive boundary equation directly from the definition of Π operator. For equation $\Pi u = f$, the condition $\bar\partial f = 0$ gurantees that $\bar\partial\bar\partial^*\bar\partial u = 0$; therefore

$$\langle\bar\partial^*\bar\partial u, \bar\partial^*\bar\partial u\rangle = \langle\bar\partial\bar\partial^*\bar\partial u, \bar\partial u\rangle = 0.$$

Thus $\Pi u = f$ is reduced to $\bar\partial_0\bar\partial_0^* u = f$. From the computation of the Π, we get

$$\overline{T}T^*u_2 = f_2 - \overline{T}S^*u_1.$$

Making use of the property $T(R)|_{\partial D} = 0$,we have $(Th)_b = T_b h_b$ for any C^∞ function h with compact support. Therefore the boundary value u_{2b} satisfies

$$\overline{T}_bT_bu_{2b} = g_3 = f_2 - \overline{T}(cu_{2b}) - \overline{T}_b[S^*(u_1)]|_{\partial D}, \tag{4.11}$$

so

$$\begin{aligned}\overline{T}_bT_b\rho(\partial')u_{2b} =&\rho(\partial')g'' + [\overline{T}_bT_b, \rho(\partial')]u_{2b}\\ =&\rho(\partial')g'' + [[\overline{T}_b, \rho(\partial')], T_b]u_{2b} + \overline{T}_b[T_b, \rho(\partial')]u_{2b} + T_b[\overline{T}, \rho(\partial')]u_{2b}\\ =&g_3.\end{aligned} \tag{4.12}$$

In order to treat equation (4.12), we make use of the theory developed by M. Christ[2] and J. J. Kohn and C. L. Fefferman[6]. Let us introduce a new pseudodifferential operator $\mathcal{A}$ which is defined by

$$\mathcal{A}(f) = \Big\{\frac{1}{2}(T_b\overline{T}_b + \overline{T}_bT_b) + \frac{1}{2}|[T_b, \overline{T}_b]|\Big\}f.$$

Here $[T_b, \overline{T}_b] = \lambda\partial_2 \mathrm{mod}\{T_b, \overline{T}_b\}$, $\lambda \ge 0$ since ∂D is convex; $|\partial_2|$ is defined to be the map

$$f \mapsto (|\xi_2|\hat{f})^\vee.$$

M. Christ showed the Proposition 8 in [2]. For more information, see [2].

In order to obtain the $\Lambda_\beta^{p,p}$ estimate for the $\rho(\partial')u_{2b}$,we need the following two lemmas

Lemma 4.7. *For any* $0 \le \alpha < \beta < \gamma$, $L^p_\gamma(\partial D \cap U) \subset \Lambda_\beta^{p,p}(\partial D \cap U) \subset L^p_\alpha(\partial D \cap U)$.

Lemma 4.8. *For any* $0 \le \alpha < \beta < \gamma$, *for any* $\epsilon > 0$, *there exists a constant* $C > 0$ *such that*

$$\|v\|_{L^p_\beta} \le \epsilon \|v\|_{L^p_\gamma} + C\|v\|_{L^p_\alpha}, \forall v \in C_c^\infty(\partial D \cap U).$$

Proof. By the mapping properties of the Bessel Potential Operator (cf. (41) in p.135 and Theorem 4′ in p.153 of [14]), Lemma 4.8 can be deduced to the case $0 \le \alpha < \beta < \gamma < 1$. Then the conclusions are consequences of the definition of $\Lambda_\alpha^{p,p}$ and L^p_β (cf. (60) in p.151 and 6.12 in p.162 of [14]).

Making use of the characterization of L^p_β (cf. 6.12 in p.162 of [14]), since for any $\epsilon > 0$, there exists a constant $C > 0$ such that $\frac{1}{t^\beta} \le \epsilon \frac{1}{t^\gamma} + C\frac{1}{t^\alpha}$, we prove the lemma.

Lemma 4.9. *For the equation* (4.11), *we have*

$$\|\rho(\partial')u_{2b}\|_{B(\gamma+\frac{1}{m}-\epsilon)} \le C\big(\|f_2\|_{L^p_\beta} + \|u_{2b}\|_{B(\gamma+\frac{1}{2m}-\frac{1}{p})} + \|u_1\|_{L^p_{\beta+\frac{1}{2m}+1}}\big).$$

Proof. From (4.22) and the definition of $\mathcal{A}$, we get

$$\mathcal{A}(\rho(\partial')u_{2b}) = g_3.$$

Define Λ^γ by $f \to (\hat f(1+|\xi'|^2)^{\frac{\gamma}{2}})^\wedge$; then $\Lambda^\gamma \in S^\gamma_{1,0}$ and

$$\begin{aligned}
[\mathcal{A},\Lambda^\gamma] =& \frac{1}{2}[T_b\overline{T}_b + \overline{T}_bT_b],\Lambda^\gamma] + [\|[T_b,\overline{T}_b]\|,\Lambda^\gamma];\\
[T_b\overline{T}_b,\Lambda^\gamma] =& T_b[\overline{T}_b,\Lambda^\gamma] + [T_b,\Lambda^\gamma]\overline{T}_b\\
=& T_b[\overline{T}_b,\Lambda^\gamma] + \overline{T}_b[T_b,\Lambda^\gamma] + [[T_b,\Lambda^\gamma],\overline{T}_b]\\
=& T_bQ_\gamma + \overline{T}_bQ'_\gamma + Q''_\gamma.
\end{aligned} \tag{4.13}$$

Here Q_γ, Q'_γ and $Q"_\gamma \in S^\gamma_{1,0}$.

From (4.11) and (4.13), we get

$$\mathcal{A}\circ\Lambda^\gamma\rho(\partial')u_{2b} = \Lambda^\gamma\rho f_2 + (T_bQ_\gamma + \overline{T}_bQ'_\gamma + Q''_\gamma)(u_{2b}) + (\overline{T}_bQ'_{\gamma+1} + Q''_{\gamma+1})(u_1).$$

By Proposition 4.8,

$$\begin{aligned}
\Lambda^\gamma\rho(\partial')u_{2b} =& \wp\Big(\Lambda^\gamma\rho f_2 + (T_bQ_\gamma + \overline{T}_bQ'_\gamma + Q''_\gamma)(u_{2b})\\
&+ (\overline{T}_bQ'_{\gamma+1} + Q''_{\gamma+1})(u_1)\Big) + E(\Lambda^\gamma\rho(\partial')u_{2b}).
\end{aligned} \tag{4.14}$$

By Proposition 4.8, together with Lemmas 4.7 and 4.8, we prove the lemma.

Theorem 4.1. $\|u_{2b}\|_{B(\beta+\frac{1}{m}-\frac{1}{p}-\epsilon)}$

$$\le C\Big(\sum_1^2 \|g_j\|_{B(\beta+1-\frac{1}{p})} + \|f_2\|_{L^p_\beta} + \|u_{2b}\|_{B(\beta+\frac{1}{2m}-\frac{1}{p})} + \|u_1\|_{L^p_{\beta+1+\frac{1}{2m}}}\Big).$$

Proof. Summing up the estimates in elliptic and hypoelliptic cases, we prove the theorem.

4.8. Conclusions

From (4.4.1), noticing that the term

$$\int S_yK_2(x,y)|_{y\in\partial D}u_{2b}(y')dy'$$

is indeed the Poisson Integral with boundary value u_{2b} (which can be read off from Lemma 3.5), we get by Lemmas 4.7 and 4.8 the following

Theorem 4.2. $\|u_2\|_{L^p_{\beta+\frac{1}{m}-\epsilon}} \le C\Big(\|f_2\|_{L^p_\beta} + \|f_1\|_{L^p_\beta} + \|u_2\|_{L^p_{\beta-N}} + \|u_1\|_{L^p_{\beta-N}}\Big).$

Finally we give

Theorem 4.3. *For* $u = \bar{\partial}_0^*(u_1\bar{w}_1 + u_2\bar{w}_2) = S^*(u_1) + T^*(u_2)$, *we have*

$$\|u\|_{L^p_{\beta+\frac{1}{2m}-\epsilon}} \le C\Big\{\|f\|_{L^p_\beta} + \|u_1\|_{L^p_{\beta-N}} + \|u_2\|_{L^p_{\beta-N}}\Big\}.$$

Proof. Apply Lemma 4.1, we get the estimate for $S^*(u_1)$. Noticing that T^* is of degree 1, we get the estimate for $T^*(u_2)$ by Theorem 4.2.

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