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ON THE CONTACT COHOMOLOGY OF ISOLATED HYPERSURFACE SINGULARITIES

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Abstract

The author defines, using jets, cohomology $H^p(\Lambda^{\cdot}_{f,k-\cdot})$ for hypersurfaces, which are invariant under contact transformations. For isolated hypersurface singularities, it is proved that

$$H^0(\Lambda^{\cdot}_{f,k-\cdot}) = \mathcal{O}_{U,0}/f^{k+1}\mathcal{O}_{U,0},$$
$$H^p(\Lambda^{\cdot}_{f,k-\cdot}) = 0, \quad 1 \le p \le N-3 \text{ or } p = N,$$
$$\dim H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) - \dim H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot}) = \binom{k}{N}\dim \mathcal{O}_{U,0}/\left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N}\right)\mathcal{O}_{U,0}.$$

The algorithm of computation for H^{N-2} and H^{N-1} is given, and it is proved that $H^{N-1} = 0$ when f is quasi-homogeneous.

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Suppose U is an open set in C^N, $0 \in U$, with coordinates $(x_1, \cdots, x_N)$. $(V, 0) \subset (U, 0)$ is a variety germ at 0 defined by the ideal $I(V) = (f_1, \cdots, f_r)\mathcal{O}_{U,0}$. $U^{(k)}$ (resp. $V^{(k)}$) is the kth infinitesimal neighborhood of U (resp. V) and $\mathcal{O}_U(k)$ (resp. $\mathcal{O}_V(k)$) is the structure sheaf of $U^{(k)}$ (resp. $V^{(k)}$). $f_i(y) = F_i = \sum_{|\alpha|\le k} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(y-x)^\alpha$, $i = 1, \cdots, r$, is the truncated Taylor expansion of $f_i(x)$.

$$\mathcal{O}_V(k) = \mathcal{O}_U(k)/(f_1, \cdots, f_r, F_1, \cdots, F_r)\mathcal{O}_{U^{(k)}}.$$

We investigate the differential form complex of $\mathcal{O}_V(k)$ and their cohomology groups in [1,2,3] for complete intersections and hypersurfaces with isolated singularities. But $\mathcal{O}_U(k)/(F_1, \cdots, F_r)\mathcal{O}_U(k)$ and its differential form complex and cohomology groups still make sense. We will investigate them for hypersurface singularities in this paper.

Let $K = R$ or C, $U \subset K^N$, $0 \in U$ and $W \subset K^M$, $0 \in W$. $U' \subset K^{N'}$, $0 \in U'$ and $W' \subset K^{M'}$, $0 \in W'$. U, W, U' and W' are open sets in $K^N, K^M, K^{N'}$ and $K^{M'}$ respectively.

Definition. *A contact mapping is a C^∞ (resp. real analytic or complex holomorphic) mapping $H : (U \times W, 0 \times 0) \to (U' \times W', 0 \times 0)$ such that there is a C^∞ (resp. real analytic or complex holomorphic) mapping $h : (U, 0) \to (U', 0)$ which makes the following diagram*

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commute

$$\begin{array}{ccccc} (U,0) & \xrightarrow{l} & (U\times W, 0\times 0) & \xrightarrow{\pi} & (U,0) \\ \Big\downarrow h & & \Big\downarrow H & & \Big\downarrow h \\ (U',0) & \xrightarrow{l'} & (U'\times W', 0\times 0) & \xrightarrow{\pi'} & (U',0) \end{array}$$

where $l(x) = (x,0)$, $\pi(x,z) = x$, $x \in V$, $z \in W$ *and* $l'(x') = (x',0)$, $\pi'(x',z') = x'$, $x' \in V'$, $z' \in W'$.

If $U = U'$, $W = W'$, H and h are local diffeomorphisms (resp. local analytic isomorphisms), H is a contact transformation in the sense of Mather[4].

Let $\mathcal{O}_{U,0}$ be the algebra of C^∞ (resp. real analytic or complex holomorphic) function germs and $\mathcal{M}_{U,0}$ be its maximal ideal. If $f:(U,0)\to(W,0)$ is a C^∞ (resp. real analytic or complex holomorphic) mapping germ, $V(f) = f^{-1}(0)$ is the space germ at 0 defined by f.

Proposition. *$f:(U,0)\to(W,0)$ and $f':(U',0)\to(W',0)$ are C^∞ (resp. real analytic or complex holomorphic) mapping germs. The following are equivalent:*

(1) *there exists a contact mapping $H:(U\times W, 0\times 0)\to(U'\times W', 0\times 0)$ such that the following diagram is commutative*

$$\begin{array}{ccc} (U,0) & \xrightarrow{(1,f)} & (U\times W, 0\times 0) \\ \Big\downarrow h & & \Big\downarrow H \\ (U',0) & \xrightarrow{(1,f')} & (U'\times W', 0\times 0) \end{array}$$

where $(1,f)(x) = (x, f(x))$, $x\in U$ *and* $(1,f') = (x', f'(x'))$, $x'\in U'$;

(2) *there exists* $h:(U,0)\to(U',0)$ *such that*

$$h^*(f'^*\mathcal{M}_{W',0}) \subset f^*\mathcal{M}_{W,0};$$

(3) *for analytic cases, there exists* $h:(U,V(f))\to(U',V(f'))$.

For the sake of simplicity we consider the C^∞ case only; all results and proofs are also true for real analytic and complex holomorphic cases.

We take the following complex for the mapping $f:(U,0)\to(W,0)$

$$\frac{\mathcal{O}_{U^{(k)},0}}{\sum\limits_{i=1}^{r} F_i\mathcal{O}_{U^{(k)},0}} \xrightarrow{D} \frac{\Omega_{U,k-1,0}}{\sum\limits_{i=1}^{r} F_i\Omega_{U,k-1,0} + \sum\limits_{i=1}^{r}\mathcal{O}_{U^{(k-1)},0}DF_i} \xrightarrow{D}$$

$$\cdots \xrightarrow{D} \frac{\Omega_{U,k-1,0}}{\sum\limits_{i=1}^{r} F_i\Omega^p_{U,k-p,0} + \sum\limits_{i=1}^{r} DF_i\wedge\Omega^{p-1}_{U,k-p,0}} \xrightarrow{D} \cdots$$

where $\Omega_{U,k-1}$ is the differential module with differentials $Dy_1,\cdots,Dy_N$ (see [1,5]) and $\Omega^p_{U,k-p,0} = \Lambda^p\Omega_{U,k-p,0}$.

If (H,h) (resp. h) satisfies (1) (resp. (2)) of Proposition, it induces a morphism of the complexes

$$h^*: \frac{\Omega^{\cdot}_{U',k-\cdot,0}}{\sum_{i=1}^{r'} F'_i\Omega^{\cdot}_{U',k-\cdot,0} + \sum_{i=1}^{r'} DF_i \wedge \Omega^{\cdot-1}_{U',k-\cdot,0}} \to \frac{\Omega^{\cdot}_{U,k-\cdot,0}}{\sum_{i=1}^{r'} F_i\Omega^{\cdot}_{U,k-\cdot,0} + \sum_{i=1}^{r'} DF_i \wedge \Omega^{\cdot-1}_{U,k-\cdot,0}}$$

and hence the morphisms of their cohomology groups. If f and f' are contact equivalent, h^* is an isomorphism of the complexes and induces isomorphisms of their cohomology groups.

Let $f : (U,0) \to (R,0)$ be a C^∞ function. Denote

$$\Lambda^0_{f,k} = \frac{\mathcal{O}_{U^{(k)},0}}{F\mathcal{O}_{U^{(k)},0}},$$

$$\Lambda^p_{f,k-p} = \frac{\Omega^p_{U,k-p,0}}{F\Omega^p_{U,k-p,0} + DF \wedge \Omega^{p-1}_{U,k-p,0}}, \quad p \geq 1.$$

We get a complex

$$\Lambda^{\cdot}_{f,k-\cdot} : \Lambda^0_{f,k} \xrightarrow{D} \cdots \to \Lambda^p_{f,k-p} \xrightarrow{D} \Lambda^{p+1}_{f,k-p-1} \to \cdots.$$

We have defined in [5]

$$\Omega^p_{f,k-p,0} = \frac{\Omega^p_{U,k-p,0}}{DF \wedge \Omega^{p-1}_{U,k-p,0}}.$$

Hence

$$\Lambda^p_{f,k-p} = \frac{\Omega^p_{f,k-p,0}}{F\Omega^p_{f,k-p,0}}.$$

Theorem 1. *$f : (U,0) \to (R,0)$ is a C^∞ function with* $\dim_R \mathcal{O}_{U,0}/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N})\mathcal{O}_{U,0} < \infty$. *Then*

(1) *if $n \geq 3$,*

$$\begin{aligned} &H^0(\Lambda^{\cdot}_{f,k-\cdot}) = \mathcal{O}_{U,0}/f^{k+1}\mathcal{O}_{U,0}, \\ &H^p(\Lambda^{\cdot}_{f,k-\cdot}) = 0, \quad 1 \leq p \leq N-3 \text{ or } p = N, \\ &\dim_R H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) - \dim_R H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot}) \\ =&\binom{k}{N} \dim_R \mathcal{O}_{U,0}/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N})\mathcal{O}_{U,0}; \end{aligned}$$

(2) *if $N = 2$,*

$$\begin{aligned} &\dim_R H^0(\Lambda^{\cdot}_{f,k-\cdot})\big/(\mathcal{O}_{U,0}/f^{k+1}\mathcal{O}_{U,0}) - \dim_R H^1(\Lambda^{\cdot}_{f,k-\cdot}) \\ =&\binom{k}{N} \dim_R \mathcal{O}_{U,0}/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N})\mathcal{O}_{U,0}, \\ &\dim_R H^2(\Lambda^{\cdot}_{f,k-\cdot}) = 0. \end{aligned}$$

Proof. First we prove the sequences

$$0 \to \Omega^p_{f,k-p,0} \xrightarrow{F} \Omega^p_{f,k-p,0} \to \Lambda^p_{f,k-p} \to 0,$$

$0 \leq p \leq N-1$, are exact.

If $\omega \in \Omega^p_{U,k-p,0}$, $F\omega = DF \wedge \theta$, $\theta \in \Omega^{p-1}_{U,k-p,0}$, multiplying $DF\wedge$, we have $FDF \wedge \omega = 0$, $DF \wedge \omega = 0$. Hence $\omega = DF \wedge \eta$, $\eta \in \Omega^{p-1}_{U,k-p,0}$. Therefore the exactness is proved.

If we take O's to supplant $\Omega^N_{f,k-N,0}$ and $\Lambda^N_{f,k-N}$ and form new complexes

$$\mathcal{O}_{U^{(k)},0} \xrightarrow{D} \Omega_{f,k-1,0} \xrightarrow{D} \cdots \xrightarrow{D} \Omega^{N-1}_{f,k-N+1,0} \to 0$$

and

$$\Lambda^0_{f,k} \xrightarrow{D} \Lambda^1_{f,k-1} \xrightarrow{D} \cdots \xrightarrow{D} \Lambda^{N-1}_{f,k-N+1} \to 0,$$

we have short exact sequence of these new complexes and long exact sequence of cohomology groups of these complexes.

$$0 \to H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \xrightarrow{F} H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \to H^0(\Lambda^{\cdot}_{f,k-\cdot}) \to H^1(\Omega^{\cdot}_{f,k-\cdot,0})$$
$$\to \cdots \to H^p(\Omega^{\cdot}_{f,k-\cdot,0}) \to H^p(\Lambda^{\cdot}_{f,k-\cdot}) \to H^{p+1}(\Omega^{\cdot}_{f,k-\cdot,0}) \to \cdots$$
$$\to H^{N-2}(\Omega^{\cdot}_{f,k-\cdot,0}) \to H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) \to \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} \xrightarrow{F}$$
$$\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} \to \frac{\Omega^{N-1}_{f,k-N+1,0}}{F\Omega^{N-1}_{f,k-N+1,0} + D\Omega^{N-2}_{f,k-N+2,0}} \to 0.$$

For $N = 2$, we get exact sequence

$$0 \to H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \xrightarrow{F} H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \to H^0(\Lambda^{\cdot}_{f,k-\cdot}) \to$$
$$\frac{\Omega_{f,k-1,0}}{D\mathcal{O}_{U^{(k)},0}} \xrightarrow{F} \frac{\Omega_{f,k-1,0}}{D\mathcal{O}_{U^{(k)},0}} \to \frac{\Omega_{f,k-1,0}}{F\Omega_{f,k-1,0} + D\mathcal{O}_{U^{(k)},0}} \to 0. \qquad (\#)$$

For $N \geq 3$, we proved $H^p(\Omega^{\cdot}_{f,k-\cdot,0}) = 0$, $1 \leq p \leq N-2$, in [5]. We have exact sequence

$$0 \to H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \xrightarrow{F} H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \to H^0(\Lambda^{\cdot}_{f,k-\cdot}) \to 0$$

and

$$H^p(\Lambda^{\cdot}_{f,k-\cdot}) = 0, \quad 0 \leq p \leq N-3,$$

and exact sequence too

$$0 \to H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) \to \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} \xrightarrow{F} \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}$$
$$\to \frac{\Omega^{N-1}_{f,k-N+1,0}}{F\Omega^{N-1}_{f,k-N+1,0} + D\Omega^{N-2}_{f,k-N+2,0}} \to 0. \qquad (*)$$

Because $D : \Lambda^{N-1}_{f,k-N+1} \to \Lambda^N_{f,k-N+1}$ is surjective,

$$H^N(\Lambda^{\cdot}_{f,k-\cdot}) = 0.$$

In [5] we also proved

$$\dim_R \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} < \infty, \quad N \geq 2.$$

Hence

$$\dim_R \frac{\Omega^{N-1}_{f,k-N+1,0}}{F\Omega^{N-1}_{f,k-N+1,0} + D\Omega^{N-2}_{f,k-N+2,0}},$$
$$\dim_R H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) < \infty$$

and

$$\dim_R H^{N-2}(\Lambda^{\cdot}_{f,k-.}) = \dim_R \frac{\Omega^{N-1}_{f,k-N+1,0}}{F\Omega^{N-1}_{f,k-N+1,0} + D\Omega^{N-2}_{f,k-N+2,0}}.$$

On the other hand

$$\frac{\Omega^{N-1}_{f,k-N+1}}{F\Omega^{N-1}_{f,k-N+1,0} + D\Omega^{N-2}_{f,k-N+2,0}} = \frac{\Lambda^{N-1}_{f,k-N+1}}{D\Lambda^{N-2}_{f,k-N+2}}$$

and the exact sequence

$$0 \to H^{N-1}(\Lambda^{\cdot}_{f,k-.}) \to \frac{\Lambda^{N-1}_{f,k-N+1}}{D\Lambda^{N-2}_{f,k-N+2}} \to \Lambda^{N}_{f,k-N} \to 0. \qquad (**)$$

Therefore

$$\dim_R H^{N-2}(\Lambda^{\cdot}_{f,k-.}) - \dim_R H^{N-1}(\Lambda^{\cdot}_{f,k-.}) = \dim_R \Lambda^{N}_{f,k-N}.$$

But

$$\begin{aligned} \Lambda^{N}_{f,k-N} &= \frac{\Omega^{N}_{U,k-N,0}}{F\Omega^{N}_{U,k-N,0} + DF \wedge \Omega^{N-1}_{U,k-N,0}} \\ &= \frac{\mathcal{O}_{U^{(k-N)},0}}{\left(F, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_N}\right)\mathcal{O}_{U^{(k-N)},0}} Dy_1 \wedge \cdots \wedge Dy_N. \\ \dim_R \Lambda^{N}_{f,k-N} &= \binom{k}{N} \dim_R \frac{\mathcal{O}_{U,0}}{\left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N}\right)\mathcal{O}_{U,0}}. \end{aligned}$$

In [5], we also proved that for $N \geq 2$

$$H^0(\Omega^{\cdot}_{f,k-\cdot,0}) = \sum_{l=0}^{k} \mathcal{O}_{U,0} F^l$$

and the sum is a direct sum. If

$$\sum_{l=0}^{k} a_l F^l \in H^0(\Omega^{\cdot}_{f,k-\cdot,0}), \quad a_l \in \mathcal{O}_{U,0},$$

$$F \sum_{l=0}^{k} a_l F^l = a_k F^{k+1} + \sum_{l=0}^{k-1} a_l F^{l+1}.$$

Because $H^0(\Omega^{\cdot}_{f,k-\cdot,0}) \subset \mathcal{O}_{U^{(k)},0}$,

$$O = (F-f)^{k+1} = F^{k+1} + \sum_{l=0}^{k}(-1)^{k+1-l}\binom{k+1}{l} f^{k+1-l}F^l,$$

$$F^{k+1} = \sum_{l=0}^{k}(-1)^{k-l}\binom{k+1}{l} f^{k+1-l}F^l,$$

$$F\sum_{l=0}^{k} a_l F^l = (-1)^k a_k f^{k+1} + \sum_{l=1}^{k}\left[(-1)^{k-l}\binom{k+1}{l} f^{k+1-l}a_k + a_{l-1}\right]F^l.$$

$$FH^0(\Omega^{\cdot}_{f,k-\cdot,0}) = f^{k+1}\mathcal{O}_{U,0} + \sum_{l=1}^{k}\mathcal{O}_{U,0}F^l.$$

$$\frac{H^0(\Omega^{\cdot}_{f,k-\cdot,0})}{FH^0(\Omega^{\cdot}_{f,k-\cdot,0})} = \frac{\mathcal{O}_{U,0}}{f^{k+1}\mathcal{O}_{U,0}}.$$

Hence for $n \geq 3$

$$H^0(\Lambda^{\cdot}_{f,k-\cdot}) = \frac{\mathcal{O}_{U,0}}{f^{k+1}\mathcal{O}_{U,0}}.$$

For $N=2$, (#) and the above results induce the conclusions of (2).

The groups $H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot})$ and $H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot})$ are computable. We describe the computational details in the following.

In [5] we proved that the sequence

$$0 \to H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \to \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} \xrightarrow{D} \Omega^{N}_{f,k-N,0} \to 0$$

is exact and the following mappings are $\mathcal{O}_{U,0}$-isomorphisms

$$H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \underset{\approx}{\xrightarrow{\partial}} \frac{DF\wedge\Omega^{N-1}_{U,k-N,0}}{DF\wedge D\Omega^{N-2}_{U,k-N+1,0}} \underset{\approx}{\xleftarrow{DF\wedge}} \frac{\Omega^{N-1}_{f,k-N,0}}{D\Omega^{N-2}_{f,k-N+1,0}}.$$

Denote

$$\delta_{k-N+1} = (DF\wedge)^{-1}\partial : H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \underset{\approx}{\to} \frac{\Omega^{N-1}_{f,k-N,0}}{D\Omega^{N-2}_{f,k-N+1,0}}.$$

We define a filter in $\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}$,

$$F_0\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right) = \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}},$$

$$F_l\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right) = \delta^{-1}_{k-N+1}\cdots\delta^{-1}_{k-N+2-l}\frac{\Omega^{N-1}_{f,k-l-N+1,0}}{D\Omega^{N-2}_{f,k-l-N+2,0}}.$$

$$F_0 \supset F_1 \supset \cdots \supset F_l \supset F_{l+1} \supset \cdots \supset F_{k-N} \supset F_{k-N+1} = 0.$$

The filter has the following simple properties

$$F_1 = H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})$$

and

$$\delta_{k-N+2-t}\cdots\delta_{k-N+1}F_1\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right)\underset{\approx}{\rightarrow}F_{l-t}\left(\frac{\Omega^{N-1}_{f,k-N+1-t,0}}{D\Omega^{N-2}_{f,k-N+2-t,0}}\right),$$

$$\frac{F_l}{F_{l+1}}\xrightarrow[\approx]{\delta_{k-N+2-l}\cdots\delta_{k-N+1}}\frac{\Omega^{N-1}_{f,k-l-N+1,0}}{D\Omega^{N-2}_{f,k-l-N+2,0}}\Big/H^{N-1}(\Omega^{\cdot}_{f,k-l-\cdot,0})$$

$$\underset{\approx}{\overset{D}{\rightarrow}}\Omega^{N}_{f,k-N-l,0},$$

where $t\le l$, $l\ge 1$ and the isomorphisms are $\mathcal{O}_{U,0}$-isomorphisms.

The natural projection

$$\Pi_{k-N+1}:\Omega^{N-1}_{f,k-N+1,0}\rightarrow\Omega^{N-1}_{f,k-N,0}$$

induces the projection

$$\Pi_{k-N+1}:\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\rightarrow\frac{\Omega^{N-1}_{f,k-N,0}}{D\Omega^{N-2}_{f,k-N+1,0}}.$$

Lemma 1. (1) $\Pi_{k-N}\delta_{k-N+1}=\delta_{k-N}\Pi_{k-N+1}$.

(2) $\delta^{-1}_{k-N+1}\Pi_{k-N+1}F_l\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right)=F_{l+1}\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right)$, $l\ge 0$.

We denote

$$\Lambda dy=dy_1\wedge\cdots\wedge dy_N,$$

$$dy_i=dy_1\wedge\cdots\wedge\hat{dy_i}\wedge\cdots\wedge dy_N,\quad i=1,\cdots,N,$$

$$\hat{dy_i}\wedge\hat{dy_j}=dy_1\wedge\cdots\wedge\hat{dy_i}\wedge\cdots\wedge\hat{dy_j}\wedge\cdots\wedge dy_N,\quad 1\le i<j\le N,$$

where "^" means "omit". If $\omega\in\Omega^{N-1}_{U,k-N+1,0}$, $\omega=\sum\limits_{i=1}^{N}(-1)^{i-1}\omega_i\hat{dy_i}$, $\omega_i\in\mathcal{O}_{U(k-N+1),0}$, then $DF\wedge\omega=\sum\limits_{i=1}^{N}\frac{\partial F}{\partial y_i}\omega_i\wedge dy$, and $D\omega=\sum\limits_{i=1}^{N}\frac{\partial\omega_i}{\partial y_i}\wedge dy$.

$\theta\in\Omega^{N}_{U,k-N,0}$, $\theta=\sum\limits_{|\alpha|\le k-N}\theta_\alpha(y-x)^\alpha\wedge dy$, $\theta_\alpha\in\mathcal{O}_{U,0}$, we define

$$\int\theta=\sum_{0\le|\alpha|\le k-N}\frac{\theta_\alpha}{\alpha_1+1}(y_1-x_1)(y-x)^\alpha\hat{dy_1}.$$

It is clear that $D\int\theta=\theta$ and $\Pi_{k-N+1}\int\theta=\int\Pi_{k-N}\theta$.

Under the hypothesis of Theorem 1, let $e_i(y)$, $i=1,\cdots,\mu$, where

$$\mu=\dim_R\mathcal{O}_{U,0}\Big/\left(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_N}\right)\mathcal{O}_{U,0}$$

be the R-basis of $\mathcal{O}_{U,0}(y)\Big/(\frac{\partial f}{\partial y_1},\cdots,\frac{\partial f}{\partial y_N})\mathcal{O}_{U,0}(y)$, then

$$e_i(y)(x-y)^\alpha\wedge Dy\in\Omega^{N}_{U,k-N-l,0},\ i=1,\cdots,\mu,\ 0\le|\alpha|\le k-N-l.$$

Their cosets $[e_i(y)(x-y)^\alpha\wedge Dy]\in\Omega^{N}_{f,k-N-l,0}$, $i=1,\cdots,\mu$, $0\le|\alpha|\le k-N-l$, are the R-basis of $\Omega^{N}_{f,k-N-l,0}$. For $0\le l\le k-N$, $\int e_i(y)(x-y)^\alpha\wedge Dy$, $i=1,\cdots,\mu$, $0\le|\alpha|\le k-N-l$. Their images in $\frac{\Omega^{N-1}_{f,k-N+1-l,0}}{D\Omega^{N-2}_{f,k-N+2-l,0}}$ denoted by $e^{(0)}_{i,\alpha}(k-N+1-l)$, $i=1,\cdots,\mu$, $0\le l\le k-N$,

$0 \le |\alpha| \le k-N-l$, are the R-basis of

$$F_0\left(\frac{\Omega^{N-1}_{f,k-N+1-l,0}}{D\Omega^{N-2}_{f,k-N+2-l,0}}\right) \Big/ F_1\left(\frac{\Omega^{N-1}_{f,k-N+1-l,0}}{D\Omega^{N-2}_{f,k-N+2-l,0}}\right).$$

For $1 \le l \le k-N$, let $e^{(l)}_{i,\alpha}(k-N+1) = \delta^{-1}_{k-N+1}\cdots\delta^{-1}_{k-N+2-l}e^{(0)}_{i,\alpha}(k-N+1-l)$, $i=1,\cdots,\mu$, $0 \le |\alpha| \le k-N-l$. They form an R-basis of

$$F_l\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right) \Big/ F_{l+1}\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right).$$

Clearly for $0 \le l \le k-N$, $e^{(t)}_{i,\alpha}(k-N+1)$, $l \le t \le k-N$, $i=1,\cdots,\mu$, $0 \le |\alpha| \le k-N-t$, are the R-basis of $F_l\left(\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right)$.

Lemma 2. *For* $0 \le l \le k-N$,

(1) $\delta_{k-N+1-l}e^{(t)}_{i,\alpha}(k-N+1-l) = e^{(t-1)}_{i,\alpha}(k-N-l)$, $1 \le t \le k-N-l$, $i=1,\cdots,\mu$, $0 \le |\alpha| \le k-N-l-t$,

(2)

$$\Pi_{k-N+1-l}e^{(t)}_{i,\alpha}(k-N+1-l) = \begin{cases} e^{(t)}_{i,\alpha}(k-N-l), & 0 \le |\alpha| \le k-N-l-t-1, \\ 0, & |\alpha| = k-N-l-t, \end{cases}$$

where $0 \le t \le k-N-l$, $i=1,\cdots,\mu$.

For the sake of simplicity, let $e^{(t)}_{i,\alpha}(k-N+1-l)=0$, if $0 \le t \le k-N-l$, $|\alpha| \ge k-N-l-t$ or $t \ge k-N-l+1$.

Lemma 3. *If* $0 \le l \le k-N-1$, $1 \le t \le k-N-l$, $0 \le |\alpha| \le k-N-l-t$, $0 \le h \le t-1$,

$$\delta_{k-N+1-l-h}\cdots\delta_{k-N+1-l}(Fe^{(t)}_{i,\alpha}(k-N+1-l)) = (h+1)e^{(t-h)}_{i,\alpha}(k-N-l-h) + Fe^{(t-h-1)}_{i,\alpha}(k-N-l-h).$$

Corollary. *If* $i=1,\cdots,\mu$, $0 \le |\alpha| \le k-N-1$, $0 \le l \le k-N$,

$$Fe^{(l)}_{i,\alpha}(k-N+1) = le^{(l+1)}_{i,\alpha}(k-N+1) + \delta^{-1}_{k-N+1}\cdots\delta^{-1}_{k-N+2-l}(Fe^{(0)}_{i,\alpha}(k-N+1-l)).$$

It is sufficient to compute the transformation defined by multiplying $\frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}$ by $f(y)$, in order to compute $H^{N-2}\Lambda^{\cdot}_{f,k-\cdot}$ and $H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot})$ because of (*) and (**).

We can write, in $\mathcal{O}_{U,0}$,

$$f(y) = \sum_{j=1}^{\mu} b_j e_j(y) + \sum_{a=1}^{N} g_a(y)\frac{\partial f(y)}{\partial y_a}, \quad b_j \in R, g_a(y) \in \mathcal{O}_{U,0}(y),$$

and

$$e_i(y)e_j(y) = \sum_{m=1}^{\mu} a^m_{ij}e_m(y) + \sum_{a=1}^{N} u^a_{ij}(y)\frac{\partial f(y)}{\partial y_a}, \quad i,j=1,\cdots,\mu,$$

where $a^m_{ij} \in R$ and $u^a_{ij}(y) \in \mathcal{O}_{U,0}$.

For $0 \le l \le k-N$, $0 \le |\alpha| \le k-N-l$, $i=1,\cdots,\mu$,

$$f(y)e_i(y) = \sum_{j,m=1}^{\mu} b_j a^m_{ij}e_m(y) + \sum_{a=1}^{N}\left(\sum_{j=1}^{\mu} b_j u^a_{ij}(y) + e_i(y)g_a(y)\right)\frac{\partial f(y)}{\partial y_a},$$

$$f(y)[e_i(y)(x-y)^\alpha \wedge Dy] = \sum_{j,m=1}^{\mu} b_j a_{ji}^m [e_m(y)(x-y)^\alpha \wedge Dy],$$

$$f(y)e_{i,\alpha}^{(0)}(k-N+1-l) - \sum_{j,m=1}^{\mu} b_j a_{ji}^m e_{m,\alpha}^{(0)}(k-N+1-l) \in F_1\left(\frac{\Omega_{f,k-N+1-l,0}^{N-1}}{D\Omega_{f,k-N+2-l,0}^{N-2}}\right).$$

$$\delta_{k-N+1-l}\left\{f(y)e_{i,\alpha}^{(0)}(k-N+1-l) - \sum_{j,m=1}^{\mu} b_j a_{ji}^m e_{m,\alpha}^{(0)}(k-N+1-l)\right\}$$

$$= \left[\sum_{a=1}^{N}(-1)^{a-1}\left(\sum_{j=1}^{\mu} b_j u_{ij}^a(y) + e_i(y)g_a(y)\right)(x-y)^\alpha D\hat{y}_a\right]$$

where $[\cdots]$ means the coset of the element in $[\,]$ in $\frac{\Omega_{f,k-N-l,0}^{N-1}}{D\Omega_{f,k-N-l+1,0}^{N-2}}$. If

$$D\left(\sum_{a=1}^{N}(-1)^{a-1}\left(\sum_{j=1}^{\mu} b_j u_{ij}^a(y) + e_i(y)g_a(y)\right)(x-y)^\alpha D\hat{y}_a\right)$$

$$= \sum_{|\beta|\le k-N-l-1}\left(\sum_{r=1}^{\mu} c_{r\beta}e_r(y) + \sum_{b=1}^{N} h_{b\beta}(y)\frac{\partial f(y)}{\partial y_b}\right)(x-y)^\beta \wedge Dy,$$

$$\delta_{k-N-l}\Bigg(\delta_{k-N+1-l}\Bigg(f(y)e_{i,\alpha}^{(0)}(k-N+1-l) - \sum_{j,m=1}^{\mu} b_j a_{ji}^m e_{m,\alpha}^{(0)}(k-N+1-l)$$

$$- e_{i,\alpha}^{(1)}(k-N+1-l)\Bigg) - \sum_{0\le|\beta|\le k-N-l-1}\sum_{r=1}^{\mu} c_{r\beta}e_{r,\beta}^{(0)}(k-N-l)\Bigg)$$

$$= \left[\sum_{b=1}^{N}\sum_{0\le|\beta|\le k-N-l-1}(-1)^{b-1}h_{b\beta}(y)(x-y)^\beta D\hat{y}_b\right].$$

Continuing this process, we can compute $f(y)e_{i,\alpha}^{(0)}(k-N+1-l)$ and hence $f(y)e_{i,\alpha}^{(l)}(k-N+1)$ by the corollary of Lemma 3. For example we have the following results.

Theorem 2. *$f:(U,0)\to(C,0)$ is a quasi-homogeneous function with weight $(w_1,\cdots, w_N)$, $w_i>0$, $i=1,\cdots,N$, and $\dim_R \mathcal{O}_{U,0}/(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_N})\mathcal{O}_{U,0}<\infty$, then*

(1) $\dim_R H^{N-2}(\Lambda^{\cdot}_{f,k-\cdot}) = \binom{k}{N}\dim_R \mathcal{O}_{U,0}/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_N})\mathcal{O}_{U,0}$,

(2) $H^{N-1}(\Lambda^{\cdot}_{f,k-\cdot})=0$.

Proof. We know $f=\sum_{i=1}^{N} c_i x_i\frac{\partial f}{\partial x_i}$, $c_i=\frac{w_i}{d}>0$, $i=1,\cdots,N$ and $d=\deg f$. Let the monomials $x^a = x_1^{a_1}\cdots x_N^{a_N}$, $a\in\Lambda$, a multi-index set, be the R-basis of $\mathcal{O}_{U,0}/(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_N})\mathcal{O}_{U,0}[e_a(y)(x-y)^\alpha\wedge Dy]$, $a\in\Lambda$, $|\alpha|\le k-N-l$, are the R-basis of $\Omega_{f,k-N-l}^{N}$.

$$e_{a,\alpha}^{(0)}(k-N+1-l) = \int e_a(y)(x-y)^\alpha\wedge Dy,\ a\in\Lambda,\ 0\le|\alpha|\le k-N-l,$$

and

$$e_{a,\alpha}^{(l)}(k-N+1) = \delta_{k-N+1}^{-1}\cdots\delta_{k-N+2-l}^{-1}e_{a,\alpha}^{(0)}(k-N+1-l),$$

$$a\in\Lambda,\ 0\le|\alpha|\le k-N-l,\ 0\le l\le k-N.$$

Lemma 4. *If* $0 \le l \le k-N$, $a \in \Lambda$, $0 \le |\alpha| \le k-N-l$,

$$\begin{aligned} f(y)e^{(l)}_{a,\alpha}(k-N+1) = &\left(l+1+\sum_{i=1}^{N} c_i(a_i+1)\right) e^{(l+1)}_{a,\alpha}(k-N+1) \\ &-\sum_{i=1}^{N}\sum_{b\in\Lambda} \alpha_i c_i \mu^a_{ib} e^{(l+1)}_{a,\alpha-1_i}(k-N+1) \\ &+\sum_{t\ge 2}\sum_{b\in\Lambda}\sum_{|\alpha|-t\le|\beta|\le|\alpha|-t+1} \lambda^{(t)}_{b\beta} e^{(1+t)}_{b,\beta}(k-N+1). \end{aligned}$$

Lemma 5. $f(y)\dfrac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}} = F_1\left(\dfrac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}\right)$.

Theorem 2 follows immediately from Lemma 5.

Remark. If V is an analytic variety, $V \subset U \subset C^N$, U is an open set of C^N. $0 \in V$, $I(V) = (f_1, \cdots, f_M)\mathcal{O}_{U,0}$. The complex

$$\Lambda^{\cdot}_{f,k-\cdot} = \Omega^{\cdot}_{U,k-\cdot,0} \Big/ \sum_{i=1}^{M} f_i \Omega^{\cdot}_{U,k-\cdot,0} + \sum_{i=1}^{M} DF_i \wedge \Omega^{\cdot-1}_{U,k-\cdot,0}$$

and its cohomology groups

$$H^n(\Lambda^{\cdot}_{V,k-\cdot}, D), \quad n = 0, 1, \cdots,$$

are the invariances of $(U, V, 0)$, i.e., if V' is another analytic variety, $V' \subset U' \subset C^N$, U' is an open set of C^N, $0 \in V$. If $\phi : (U, V, 0) \to (U', V', 0)$ is a biholomorphic isomorphism, ϕ induces isomorphisms $\Lambda^{\cdot}_{V',k-\cdot} \approx \Lambda^{\cdot}_{V,k-\cdot}$ and $H^n(\Lambda^{\cdot}_{V',k-\cdot}) \approx H^n(\Lambda^{\cdot}_{V,k-\cdot})$.

If V is a hypersurface defined by $f\mathcal{O}_{U,0'}$, and O is an isolated singularity, the above results are still true.

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