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ON SUBGROUPS OF GL_2 OVER A CLASS OF NON-COMMUTATIVE RINGS WHICH ARE NORMALIZED BY ELEMENTARY MATRICES**

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Abstract

Let R be an associative ring with 1 and $Y \neq R$ a quasi-ideal of R. Set $T_2(R,Y) = \{\mathrm{diag}(u,v)a^{1,2}b^{2,1}c^{1,2} : a+c, b \in Y, u, v \in GL_1R$, and $v^{-1}au-a,\ uav^{-1}-a \in Y$ for all $a \in R\}$. It is proved that if R satisfies 2-fold condition, then $[E_2R, T_2(R,Y)] \subset E_2(R,Y) \subset T_2(R,Y)$; and if R satisfies 6-fold condition, then $E_2(R,Y) = [E_2R, E_2(R,Y)] = [E_2R, T_2(R,Y)]$ and the sandwich theorem holds.

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§1. Introduction

Let R be an associative ring with 1, Y a set of R. As usual, GL_2R denotes the group of all invertible 2 by 2 matrices over R, E_2Y denotes the subgroup of GL_2R generated by all elementary matrices $y^{1,2}$ and $y^{2,1}$ where $y \in Y$. We denote by $E_2(R,Y)$ the normal subgroup of E_2R generated by E_2Y.

Vaserstein[7,8] recently introduced a concept of quasi-ideal which is defined as an additive subgroup Y of R such that $yay, aya \in Y$ for all $a \in R$, $y \in Y$, and studied the structure of subgroups of GL_2 over non-commutative local rings which are normalized by elementary matrices[8]. Before Vaserstein's work, many results on the subgroups of GL_2 over some commutative rings were offered by Klingenberg[1], Lacroix[2], Mason[3,4], Mcdonald[5], Zhang-Wang[11] and so on.

For the readers' convenience, we will cite from [7,8] some properties of quasi-ideals and prove a new one.

Lemma 1.1. *Let R be an associative ring with* 1 *and Y a quasi-ideal of R. Then*

(1) $ayb + bya \in Y$, $yab + bay \in Y$ *for all* $y \in Y, a, b \in R$;

(2) $Ry^2 \subseteq Y$ *for all* $y \in Y$;

(3) $R'YR, RYR' \subseteq Y$ *where R' is the ideal of R generated by all additive commutators* $ab - ba$ *with* $a, b \in R$.

The properties (1) and (3) above still hold for an additive group Y satisfying $aya \in Y$ for all $a \in R$ and $y \in Y$.

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Lemma 1.1$'$. *Assume that the condition is the same as in Lemma 1.1. Then $Ry^2R \subseteq Y$ for all $y \in Y$.*

Proof. Let $a, b \in R$ and $y \in Y$. We want to show $ay^2b \in Y$. Since $by(ay) + (ay)yb \in Y$ and $(b(ya) - (ya)b)y \in Y$, but $yaby \in Y$ by the definition of quasi-ideal, we have $byay \in Y$, hence $ayyb = ay^2b \in Y$.

Recall that a ring is called n-fold (or unit n-fold), if for $a_i, b_i \in R(i = 1, \cdots, n)$ with $Ra_i + Rb_i = R((a_i, b_i) = R)$ there is a $c \in R$ (or $c \in GL_1R$) such that $a_i + cb_i(i = 1, \cdots, n) \in GL_1R$.

By [10], we know that n-fold means unit $(n-1)$-fold. See [6,9] for references about n-fold rings, such as semi-local ring R with the fields K_i in $R/J = \prod_{i=1}^{m} M_{n_i}(K_i)$ having sufficient elements, C^*-algebra with unitary stable range 1 and full rings (commutative).

We say that ring R satisfies n-fold condition for a quasi-ideal Y, if $a_i \in R$, $b_i \in Y(i = 1, \cdots, n)$ such that $Ra_i + Rb_i = R$, then there is a $c \in Y$ such that $a_i + cb_i \in GL_1R$.

Lemma 1.2. *That R is n-fold implies that R satisfies n-fold condition for a quasi-ideal Y.*

Proof. We only need to show this for $n = 1$. Assume that $a \in R$, $b \in Y$ with $Ra+Rb = R$. Then $(a, bsb) = R$ for some $s \in R$. Further, $(a, bsbrbsb) = R$ for some $r \in R$. So there is a $t \in R$ such that $a + tbsbrbsb \in GL_1R$. Let us show $tbsbrbs \in Y$. Since $brb \in Y$, we have $(tbs - stb)brbs \in Y$ by Lemma 1.1. But $stbbrbs = stb^2rbs \in Y$ (Lemma 1.1'), so $tbsbrbs \in Y$.

Notice: In fact, $tbsbrbsb$ is also in Y.

Every ideal is quasi-ideal, but a quasi-ideal may not be an ideal. Readers may see the counter example in [8].

For every quasi-ideal $Y \neq R$, we set $T_2(R, Y)$ to denote the set of all elements of the form

$$\mathrm{diag}(u, v)a^{1,2}b^{2,1}c^{1,2},$$

where $a + c$, $b \in Y$, $u, v \in GL_1R$ and $v^{-1}au - a, uav^{-1} - a \in Y$ for all $a \in R$. We set $T_2(R, R) = GL_2R$.

When R satisfies $Sr(R) \leq 1$ (i.e. 1-fold) condition, it is clear that $T_2(R, I) = G_2(R, I)$ for every ideal I of R where $G_2(R, I)$ is the inverse image of the center of GL_2R/I under the homomorphism: $GL_2R \to GL_2R/I$.

In this paper we prove the following theorems.

Theorem 1.1. *Let R be an associative ring with 1 satisfying 2-fold condition. Then $T_2(R, Y)$ is a subgroup of GL_2R and $[E_2R, T_2(R, Y)] \subset E_2(R, Y) \subset T_2(R, Y)$. Therefore $[E_2R, H] \subset H$ for any subgroup H of GL_2R such that $E_2(R, Y) \subset H \subset T_2(R, Y)$. In particular, H is normalized by E_2R.*

Theorem 1.2. *Let R be an associative ring satisfying 6-fold condition. Then*

(a) $E_2(R, Y) = [E_2R, E_2Y] = [E_2R, E_2(R, Y)] = [E_2R, T_2(R, Y)]$ *for any quasi-ideal Y of R (In fact 4-fold condition is enough for* (a)).

(b) *for any subgroup H of GL_2R which is normalized by E_2R, there is a unique quasi-ideal Y of R such that $E_2(R, Y) \subset H \subset T_2(R, Y)$.*

The reason for uniqueness of Y in Theorem 1.2(b) is stated in [8, p.222].

§2. Proof of Theorem 1.1

We point out the following identity

$$a^{2,1}b^{1,2}c^{2,1}d^{1,2} = \begin{pmatrix} p(b,c) & p(b,c,d) \\ p(a,b,c) & p(a,b,c,d) \end{pmatrix},$$

where $p(\ ) = 1$, $p(a) = a$, $p(a,b) = 1 + ab$, $p(a,b,c) = a + c + abc$, $p(a,b,c,d) = 1 + abcd + ab + ad + cd$.

If $p(a,b)$, $p(a,b,c)$ or $p(a,b,c,d) \in GL_1R$, then $\mathrm{diag}(p(a,b), p(b,a)^{-1})$, $\mathrm{diag}(p(a,b,c), p(c,b,a)^{-1})$, $\mathrm{diag}(p(a,b,c,d), p(d,c,b,a)^{-1}) \in E_2R$ (see [6]).

Lemma 2.1. *$(1 + xy - xyxk)a(1 + yx - kxyx) - a \in Y$ for all $y \in Y$ and $x, k, a \in R$.*

Proof.

$$\begin{aligned}&(1 + xy - xyxk)a(1 + yx - kxyx) - a \\ =&a + xya - xyxka + ayx + xyayx - xyxkayx - akxyx - xyakxyx + xyxkakxyx - a \\ =&(xya + ayx) - (xyxka + akxyx) + xyayx - xyxkayx - xyakxyx + xyxkakxyx \in Y.\end{aligned}$$

Proposition 2.1. *The set $T_2(R, Y)$ is invariant under conjugation by E_2R.*

Proof. It is clear that $T_2(R,Y)$ is invariant under conjugation by $x^{1,2}$ for any $x \in R$. We only need to show that $T_2(R,Y)$ is invariant under conjugation by $w(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In fact, it is sufficient to show that $x^{2,1}y^{1,2}z^{2,1} (y \in Y, x + z \in Y)$ can be written as $\mathrm{diag}(u', v')a^{1,2}b^{2,1}c^{1,2}$ where $b, a+c \in Y$ and $u', v' \in GL_1R$ with $u'av'^{-1} - a, v'^{-1}au' - a \in Y$. Since $x + z = y_1 \in Y$, we write z as $-x + y_1$. So

$$x^{2,1}y^{1,2}z^{2,1} = x^{2,1}y^{1,2}(-x)^{2,1}y_1^{2,1} = \begin{pmatrix} 1 - yx & y \\ -xyx & 1 + xy \end{pmatrix} \begin{pmatrix} 1 & \\ y_1 & 1 \end{pmatrix}.$$

Note that $(-xyx, 1 + xy) = R$, $xyx, y_1 \in Y$. We may find a $k \in Y$ such that $1 + xy - xyxk, 1 - ky_1 \in GL_1R$. So

$$\begin{aligned}&\begin{pmatrix} 1 - yx & y \\ -xyx & 1 + xy \end{pmatrix} \begin{pmatrix} 1 & k \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -k \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ y_1 & 1 \end{pmatrix} \\ =&\begin{pmatrix} 1 - yx & k - yxk + y \\ -xyx & 1 + xy - xyxk \end{pmatrix} \begin{pmatrix} 1 - ky_1 & -k \\ y_1 & 1 \end{pmatrix}.\end{aligned}$$

But

$$\begin{pmatrix} 1 - yx & k - yxk + y \\ -xyx & 1 + xy - xyxk \end{pmatrix} = \begin{pmatrix} 1 & (k - yxk + y)(1 + xy - xyxk)^{-1} \\ & 1 \end{pmatrix}$$

$$\cdot \mathrm{diag}((1 + yx - kxyx)^{-1}, 1 + xy - xyxk) \begin{pmatrix} 1 & \\ -(1 + xy - xyxk)^{-1}xyx & 1 \end{pmatrix} \begin{pmatrix} 1 - ky_1 & -k \\ y_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ y_1(1 - ky_1)^{-1} & 1 \end{pmatrix} \mathrm{diag}(1 - ky_1, (1 - y_1k)^{-1}) \begin{pmatrix} 1 & -(1 - ky_1)^{-1}k \\ & 1 \end{pmatrix}.$$

Since $(1 + xy - xyxk)^{-1}xyx = (1 + xy - xyxk)^{-1}xyx(1 + xy - xyxk)(1 + xy - xyxk)^{-1}$ and $xyx(1+xy-xyxk) = xyx + xyxxy - xyxxyxk \in Y$, we have $(1+xy-xyxk)^{-1}xyx \in Y$.

Since $y_1(1 - ky_1)^{-1} = (1 - ky_1)^{-1}(1 - ky_1)y_1(1 - ky_1)^{-1}$ and $(1 - ky_1)y_1 = y_1 - ky_1^2 \in Y$, we have $y_1(1 - ky_1)^{-1} \in Y$.

By Lemma 2.1, we know that $\mathrm{diag}((1 + yx - kxyx)^{-1}, 1 + xy - xyxk)$, $\mathrm{diag}(1 - ky_1, (1 - y_1k)^{-1}) \in T_2(R, Y)$.

Finally,

$$\begin{aligned}&(k-yxk+y)(1+xy-xyxk)^{-1}-(1-ky_1)^{-1}k\\ =&(1+xy-xyxk)^{-1}(1+xy-xyxk)(k-yxk+y)(1+xy-xyxk)^{-1}\\ &-(1-ky_1)^{-1}k\cdot(1-ky_1)(1-ky_1)^{-1}.\end{aligned}$$

We show that

(1) $k-k^2y_1=k(1-ky_1)\in Y(k\in Y)$,

(2) $(1+xy-xyxk)(k-yxk+y)=k-yxk+y-xyyxk+xyk+xy^2-xyxk^2+xyxkyxk-xyxky=k+y+(xy-yx)k-xy^2xk+xy^2-xyxk^2+xyxkyxk-xyxky\in Y$

(Note that $(xyxk-xkxy)yxk\in Y$ and $xkxyyxk\in Y$).

So $(k-yxk+y)(1+xy-xyxk)^{-1}-(1-ky_1)^{-1}k\in Y$.

We finish the proof.

Proposition 2.2. *$T_2(R,Y)$ is a subgroup of GL_2R.*

Proof. Let $\operatorname{diag}(u,v)a^{1,2}b^{2,1}c^{1,2}$, $\operatorname{diag}(u_1,v_1)x^{1,2}y^{2,1}z^{1,2}\in T_2(R,Y)$, where $b,y,a+c,x+z\in Y$ and $u,v;u_1,v_1\in GL_1R$ satisfying the condition in the definition of $T_2(R,Y)$.

Since $T_2(R,Y)$ is invariant under conjugation by E_2R, if we show that

$$z^{1,2}\operatorname{diag}(u,v)a^{1,2}b^{2,1}c^{1,2}\operatorname{diag}(u_1,v_1)x^{1,2}y^{2,1}\in T_2(R,Y)$$

then we finish the proof.

Since

$$\begin{aligned}&z^{1,2}\operatorname{diag}(u,v)a^{1,2}b^{2,1}c^{1,2}\operatorname{diag}(u_1,v_1)x^{1,2}y^{2,1}\\ =&z^{1,2}\operatorname{diag}(u,v)a^{1,2}b^{2,1}c^{1,2}(-z)^{1,2}z^{1,2}\operatorname{diag}(u_1,v_1)x^{1,2}y^{2,1}\end{aligned}$$

and

$$\begin{aligned}z^{1,2}\operatorname{diag}(u_1,v_1)x^{1,2}y^{2,1}&=\operatorname{diag}(u_1,v_1)(u_1^{-1}zv_1-z)^{1,2}z^{1,2}x^{1,2}y^{2,1}\\ &=\operatorname{diag}(u_1,v_1)((u_1^{-1}zv_1-z)+(z+x))^{1,2}y^{2,1}\end{aligned}$$

(Note that $u_1^{-1}zv_1-z+z+x\in Y$), it is sufficient to show that $T_2(R,Y)$ is invariant under right multiplication by $y_1^{1,2}$, $y_1^{2,1}(y_1\in Y)$ and $\operatorname{diag}(u_1,v_1)(u_1,v_1\in GL_1R, u_1av_1^{-1}-a, v_1^{-1}au_1-a\in Y)$. But the proof is the same as that of [8, Lemma (1.3)].

By Propositions 2.1 and 2.2, $T_2(R,Y)$ is a subgroup of GL_2R and it is normalized by E_2R. Since $E_2Y\subset T_2(R,Y)$, it follows that $E_2(R,Y)\subset T_2(R,Y)$. By the definition, $T_2(R,Y)$ normalizes $E_2(R,Y)$. The first inclusion is clear.

§3. Proof of Theorem 1.2

First let us prove Theorem 1.2(a)

Lemma 3.1. *Let Y be a quasi-ideal of R and let $y\in Y$, $1+xy\in GL_1R$. Then* $\operatorname{diag}(1+xy,(1+yx)^{-1})\in E_2(R,Y)$.

Proof. $\operatorname{diag}(1+xy,(1+yx)^{-1})=(x+xyx)^{1,2}(yz)^{2,1}y^{1,2}(-x)^{2,1}(-y)^{1,2}$ where $z=(1+xy)^{-1}-1=-(x+zx)y$. It is clear that $yz=-y(x+zx)y\in Y$.

Now introduce "the lower level" $L(H)$ of a subgroup H of GL_2R. Set

$$L(H)=\{x\in R\mid x^{2,1}\in H\}.$$

It is easy to see that $L(H)$ is an additive subgroup of R and that

$$L(E_2(R,I)) = L(G_2(R,I)) = I$$

for any ideal I of R. By [7,8], we know that $L(E_2(R,Y)) = Y$ and $L(T_2(R,Y)) = Y$ for any quasi-ideal Y of R.

Now we assume that H is normalized by E_2R. Then $[E_2R, H] \subset H$. Then conjugation by $\mathrm{diag}(u, u^{-1})$, or $\mathrm{diag}(1+rs, (1+sr)^{-1})$ $(u, 1+rs \in GL_1R)$, gives the following results:

$$uL(H)u \subset L(H), \quad (1+rs)L(H)(1+sr) \subset L(H).$$

Lemma 3.2. *Assume that R satisfies 4-fold condition and X is an additive group such that $uXu \subset X$, $(1+rs)X(1+sr) \subset X$ for all $u, 1+rs \in GL_1R$. Then $aXa \subset X$ for all $a \in R$.*

Proof. For any $a \in R$, we may write a as $v_1 + v_2$ where $v_1, v_2 \in GL_1R$. Since R satisfies unit 3-fold condition, there is a $u \in GL_1R$ such that $1+ua, 1+uv_1, 1+uv_2 \in GL_1R$.

Since $(1+uv_i)x(1+v_iu) = x+uv_ix+xv_iu+uv_ixv_iu \in X (i=1,2)$, we have $uv_ix+xv_iu \in X (i=1,2)$.

Further, $uax + xau = u(v_1+v_2)x + x(v_1+v_2)u \in X$. From $(1+ua)x(1+au) = x+uax+xau+uaxau \in X$ and $uax+xau \in X$, we have $uaxau \in X \Rightarrow axa \in X$.

Now we can apply the above to the case $H = [E_2R, E_2Y]$ and prove Theorem 1.2(a). By Theorem 1.1, it suffices to prove the inclusion $E_2(R,Y) \subset [E_2R, E_2Y]$. That is, we have to show that $Y \subset X$ where $X = L(H) = L([E_2R, E_2Y])$, assuming the hypotheses of Theorem 1.2(a).

Lemma 3.3. *Let $y \in Y$. Then y can be written as $y_1 + y_2$ with $y_1, y_2 \in Y$ such that $1+y_1, 1+y_2 \in GL_1R$.*

Proof. Since $(1+y, -y) = R$ and $(1, y) = R$, there exists a y' such that $1+y-y'y, 1+y'y \in GL_1R$ and $y'y \in Y$ (see Lemma 1.2). Set $y_1 = y'y$, $y_2 = y - y'y$. It is obvious that $y = y_1 + y_2$, and $1+y_1, 1+y_2 \in GL_1R$.

By Lemma 3.3, it suffices to show that $y \in X$ for any $y \in Y$ such that $1+y \in GL_1R$.

Since $\mathrm{diag}(1+y, (1+y)^{-1}) \in E_2(R,Y)$, we have $[\mathrm{diag}(1+y,(1+y)^{-1}), a^{1,2}] = (ya+ay+yay)^{1,2} \in [E_2R, E_2Y]$, that is,

$$ya + ay + yay \in X \quad \text{for all } a \in R. \tag{3.1}$$

Now we fix a unit $t \in GL_1R$ such that $1+t, 1-yt, 1+y-yty \in GL_1R$ (Note that $(1,1) = R$, $(1,-y) = R$, $(1+y, -y^2) = R$ and $1+y+yty \in GL_1R$ when $1+y-yyt = 1+y(1-yt) \in GL_1R$). We have $[\mathrm{diag}(u,u^{-1}), y^{1,2}] = (uyu-y)^{1,2} \in H$ and hence $uyu - y \in X$, where $u = t, 1+t$. Thus $y+ty+yt \in X$. Using (3.1) with $a = t$, we obtain

$$y - yty \in X. \tag{3.2}$$

Replacing y by $y - yty \in Y$, we obtain

$$y - yty - (y-yty)t(y-yty) \in X. \tag{3.3}$$

Subtracting (3.3) from (3.2), we obtain

$$(y-yty)t(y-yty) = (1-yt)yty(1-ty) \in X.$$

Let $g = \mathrm{diag}(1-yt, (1-ty)^{-1}) \in E_2R$. Then

$$g^{-1}((1-yt)yty(1-ty))^{1,2}g = (yty)^{1,2} \in [E_2R, E_2Y],$$

i.e. $yty \in X$, so $y \in X$.

Next, let us prove Theorem 1.2(b).

Set $K = \{g \in GL_2R \mid [g, E_2R] \subset H\}$.

Then $H \subset K$, K is normalized by E_2R, and $[K, E_2R] \subset H$. If $a, b \in R$ and $a^{1,2}b^{2,1} \in K$, then $a^{1,2}, b^{2,1} \in K$, i.e. $a, b \in L(K)$. This property makes K more convenient than H (see [8]).

If $\mathrm{diag}(u, v)a^{1,2}b^{2,1}c^{1,2} \in K$ where $a + c, b \in K$, then

$$\mathrm{diag}(u, v)(u^{-1}cv + a)^{1,2}b^{2,1} \in K.$$

The following Lemmas are borrowed from [8], readers may find the proof in [8].

Lemma 3.4. *Let $g = x^{1,2}diag(u, v)y^{2,1} \in K$ and u, v be similar in GL_1R. Then $x^{1,2}$, $y^{2,1}$,* $\mathrm{diag}(u, v) \in K$.

Corollary. *If $g = x^{1,2}\mathrm{diag}(u, v)y^{2,1} \in K$, then $(x - y)^{1,2}$, $diag(uv, vu) \in K$.*

Lemma 3.5. *Let $g = x^{1,2}\mathrm{diag}(u, v)x^{2,1} \in K$. Then*

$$(x - uxv^{-1})^{1,2}, [x^{1,2}, x^{2,1}], (2x)^{1,2}\mathrm{diag}(uv^{-1}, vu^{-1})(2x)^{2,1} \in K.$$

Lemma 3.6. *Let $g = y^{1,2}\mathrm{diag}(t, t^{-1})x^{2,1} \in K$. Then*

$$(1 - t^2)^{2,1}(t^2 - 1)^{1,2}\mathrm{diag}(t, t^{-1}), (2x)^{2,1}, \mathrm{diag}(t^2, t^{-2}) \in K.$$

We define

$$\psi(H) = \{b \in R \mid a^{1,2}\mathrm{diag}(u, v)b^{2,1}c^{1,2} \in H \text{ for some } u, v \in GL_1R \text{ and } a, c \in R\},$$
$$\psi'(H) = \{a + c \in R \mid a^{1,2}\mathrm{diag}(u, v)b^{2,1}c^{1,2} \in H \text{ for some } u, v \in GL_1R \text{ and } b \in R\}.$$

Lemma 3.7. *$\psi(H) = \psi'(H)$, $\psi(H)$ is an additive subgroup of R, and $\alpha\psi(H)\alpha' = \psi(H)$ for every $\alpha, \alpha' \in GL_1R$ such that* $\mathrm{diag}(\alpha^{-1}, \alpha') \in E_2R$.

Proof. Let $b \in \psi(H)$. Then $a^{1,2}\mathrm{diag}(u, v)b^{2,1}c^{1,2} \in H$ for some $u, v \in GL_1R$ and $a, c \in R$. We have

$$(a + c)^{1,2}\mathrm{diag}(u, v)b^{2,1} \in H.$$

Taking inverses and conjugating by $w(1)$, we obtain $b \in \psi'(H)$, so $\psi(H) \subset \psi'(H)$. Similarly, $\psi'(H) \subset \psi(H)$.

To prove the second conclusion, let $a + c, a_1 + c_1 \in \psi'(K)$, i.e.

$$g = a^{1,2}\mathrm{diag}(u, v)b^{2,1}c^{1,2} \in H, \quad g_1 = a_1^{1,2}\mathrm{diag}(u_1, v_1)b_1^{2,1}c_1^{1,2} \in H.$$

Then

$$g' = (a + c)^{1,2}\mathrm{diag}(u, v)b^{2,1} \in H, \quad g_1' = (a_1 + c_1)^{1,2}\mathrm{diag}(u_1, v_1)b_1^{2,1} \in H.$$

So

$$\begin{aligned}&(-(a_1 + c_1))^{1,2}g'{g_1'}^{-1}(a_1 + c_1)^{1,2}\\ =&((a + c) - (a_1 + c_1))^{1,2}\mathrm{diag}(uu_1^{-1}, vv_1^{-1})(u_1(b - b_1)v_1^{-1})^{1,2} \in H.\end{aligned}$$

Hence $(a + c) - (a_1 + c_1) \in \psi'(H)$. Thus $\psi(H) = \psi'(H)$ is an additive subgroup of R.

The last conclusion is obvious.

Proposition 3.1. $4\psi(K) \subset L(K)$.

Proof. By Lemmas 3.5 and 3.6.

Set $X = L(K)$, $Y = \psi(K)$.

Lemma 3.8. *$Y = \psi(K)$ is a quasi-ideal of R.*

Proof. By Lemmas 3.7 and 3.2, we know that $aYa \subset Y$ for all $a \in R$.

It remains to show that $yay \in Y$ for all $y \in Y$ and $a \in R$. Since we can write any $a \in R$ as $a_1 + a_2$ such that $1 - ya_i (i = 1, 2) \in GL_1R$ (Note that $(1, -y) = R, (1 - ya, y) = R$), it suffices to prove the result for $a \in R$ with $1 - ya \in GL_1R$. Choose

$$g = (y)^{1,2}\mathrm{diag}(u, v)(*)^{2,1} \in K$$

and let $b = u^{-1}av$. Then

$$[b^{1,2}, g] = \begin{pmatrix} * & * \\ yay & 1 - ya \end{pmatrix} = (*)^{1,2}\mathrm{diag}(*, *)z^{2,1} \in H,$$

where $z = (1 - ya)^{-1}yay$. Hence $z \in Y$ and so $(1 - ya)z(1 - ay) = yay - yayay \in Y$. Since $yayay \in Y$, we have $yay \in Y$.

If 2 is invertible, by Proposition 3.1 we have $\psi(K) = L(K) = Y$. By Theorem 1.2(a), $E_2(R, Y) = [E_2Y, E_2R] \subset [K, E_2R] \subset H$. Hence $L(H) = Y$ and Theorem 1.2(b) follows.

In general, we need the following Lemmas.

Lemma 3.9. *If $\alpha \in GL_1R$ and $\alpha^2 - 1 \in GL_1R$, then $y + ry + yr \in X$, where $r = (\alpha^2 - 1)^{-1}$ and $y \in Y$.*

Proof. Applying the conclusion that if $y \in \psi(K)$ and $u \in GL_1R$ then $u^2yu^2 - y \in L(K)$ to $y' = \alpha^2y\alpha^2 - y$, we get $y' \in X = L(K)$. Then calculate $ry'r$.

Lemma 3.10. *Let $x \in Y$ with $1 - x \in GL_1R$ and $x^2, 2x \in X$. Then $x \in X$ (see [8], p.230).*

Corollary. *If $x \in Y$ with $1 - x, 1 - 2x \in GL_1R$ and $x^2 \in X$, then $x \in X$.*

Proof. By Proposition 3.1, $4Y \subset X$. So $2(2x), (2x)^2 \in X$. By Lemma 3.10, $2x \in X$. Now $2x, x^2 \in X$. Apply Lemma 3.10 again.

Lemma 3.11. *Let $x \in Y$ with $1 - x, 1 - 2x \in GL_1R$. Then $x^4 \in X$.*

Proof. Since $4Y \subset X$, we have $2(2x), (2x)^2 \in X$. By Lemma 3.10, $2x \in X$. Let $h = x^{2,1}\mathrm{diag}(*, *)(*)^{1,2} \in K$. The matrix $(-x)^{2,1}[(-1)^{1,2}, h]x^{2,1}$ has the form

$$h' = (x^2(1 - x)^{-1})^{2,1}\mathrm{diag}(1 - x, (1 - x)^{-1})(*)^{1,2} \in K.$$

By Lemma 3.6, $g = \mathrm{diag}((1 - x)^2, (1 - x)^{-2}) \in K$. So $[g, 1^{1,2}] = (1 - (1 - x)^4)^{1,2} \in K$, i.e. $4x - 6x^2 + 4x^3 - x^4 \in X$, so $x^4 \in X$ (In fact, $2Y \subset X$. See the following proof).

Now we can conclude the proof of the goal that $Y \subset X$. Note that under the hypotheses of Theorem 1.2, every $y \in Y$ may be written as $y_1 + y_2$ with $y_1, y_2 \in Y$ such that $1 - y_i, 1 + y_i, 1 - 2y_i (i = 1, 2) \in GL_1R$ (The proof is similar to Lemma 3.3). So it suffices to prove that for $y \in Y$ with $1 - y, 1 + y, 1 - 2y \in GL_1R$.

Since $2y^2$ may be written as $2x_1 + 2x_2$ with $1 - 2x_1, 1 - 2x_2 \in GL_1R$, and $x_1, x_2 \in Y$, we have $2y^2 \in X$ by Lemma 3.10. By Lemma 3.11, $y^4 \in X$. Because $1 - y^2 = (1 - y)(1 + y) \in GL_1R$, we have $y^2 \in X$. Hence $y \in X$ by Lemma 3.10.

Theorem 1.2(b) is proved.

Actually, we have proved that $K \subset T_2(R, Y)$ also. So $[E_2R, E_2(R, Y)] \subset H \subset T_2(R, Y)$.

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