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THE DERIVATIVES AND INTEGRALS OF FRACTIONAL ORDER ON $\mathfrak{a}$-ADIC GROUPS**

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Abstract

This paper, the author studies fractional derivatives and integrals of fractional order $\alpha > 0$ for functions in $L^r(G_a)$ and Fourier transform for distributions. Under these definitions, the author obtains the formula $\chi_y^{<\alpha>}(x) =| y |^\alpha \chi_y(x)$ for characters χ_y and $(D^{<\alpha>} f)^\wedge =| \cdot |^\alpha f^\wedge$, discusses the existence of the fractional derivatives of test functions, gives relationships between some function spaces, and proves that the fractional derivatives and the fractional integrals are inverse operations one another.

Keywords Derivatives of fractional, Integrals of fractional, Local compact groups.
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§1. Introduction and Preliminaries

The fractional derivatives that we define belowe include the derivative defined by Zheng Weixing[1,p.806], different from the one by C. W. Onneweer[2,3] and the one on G_0 by He Zelin[4]. The fractional integrals on a-adic groups arise as inverse operations to the strong derivatives of fractional order.

Throughout this paper G_a will dentote an a-adic group. We now state some definitions and properties of G_a, (cf [5]). Let a be a fixed but arbitrary doubly infinite sequence of positive integers: $a := \{\cdots, a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots, a_n, \cdots\}$, where each a_n is greater than 1. $G_a := \{x : x = (x_n)_{n\in\mathbb{Z}}\}$, where each x_n is an integer, $0 \le x_n < a_n$, and $x_n = 0$ for $n < n_0$, with n_0 depending on x. Addition operation is defined formally carrying from left to right. For each integer k, let G_k be the set of all $x \in G_a$ such that $x_n = 0$ for all $n < k$. It is well-known that G_a is an abelian topological group which is Hausdorff, locally compact, σ-compact and 0-dimensional. Moreover, the set G_k is a compact subgroup of G_a, and $\{G_k : k \in \mathbb{Z}\}$ defines a topology on G_a.

Next, the character group of G_a is topologically isomorphic with G_{a^*}, where $a^* := (a_n^*)$, $a_n^* = a_{-n}$ for all $n \in \mathbb{Z}$. Thus we associate with every y in Ga^* a continuous character χ_y of G_a in the following way: For a given $y = (y_n) \in Ga^*$ and each $x = (x_n) \in G_a$, if $y_n = 0$ for all $n \le k$ and $x_n = 0$ for all $n < m$, then

$$\chi_y(x) := \exp\Bigg(2\pi i\Bigg(\sum_{n=m}^{-k-1}\Bigg(x_n\Bigg(\sum_{j=n}^{-k-1}\frac{y_{-j}}{a_n a_{n+1}\cdots a_j}\Bigg)\Bigg)\Bigg)\Bigg).$$

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Let $G_k^* := \{y = (y_n) \in Ga^* : y_n = 0 \text{ for all } n \leq k\}$. We have $G_k^* = \{\chi$ in the character group of $G_a : \chi|_{G_{-k}} \equiv 1\}$ and G_k^* are compact subgroups of G_a^*.

We denote the Haar measure on G_a and G_{a^*}, normalized so that the subgroups G_0 and G_0^* have measure 1, by μ and ν, respectively. For each $n \in \mathbb{Z}$ we have $\mu(G_n) = (\nu(G_{-n}^*))^{-1}$ and set $m_n := \mu(G_n)$. Note that $m_0 = 1, m_{-n} = a_{-n}a_{-n+1}\cdots a_{-1}$ and $m_n = (a_0 a_1 \cdots a_{n-1})^{-1}$ for $n \geq 1$. We set $\lambda_{n,j} := a_n a_{n+1} \cdots a_j$ for $n \leq j, \lambda_{n,j} := 1$ for $n > j$.

We denote the ultrametrics on G_a and G_{a^*} by $|\cdot|$, defined by $|x| := m_n$ for $x \in G_n \setminus G_{n+1}$ or $x \in G_{n-1}^* \setminus G_n^*, n \in \mathbb{Z}$. We have the ultrametric inequality $|x_1 + x_2| \leq \max(|x_1|, |x_2|)$, and $G_n = \{x \in G_a : |x| \leq m_n\}$ and $G_n^* = \{y \in G_{a^*} : |y| \leq m_{n+1}\}$. Consequently, the ultrametrics are compatible with the original topologics of G_a and G_{a^*}, respectively.

In what follows, let $e_n := (x_j)$ satisfying $x_n = 1$ and $x_j = 0$ for all $j \neq n$, and set

$$R_n(x) := \begin{cases} 1, & x \in G_n, \\ 0, & x \notin G_n, \end{cases}$$

and

$$R_n^*(y) := \begin{cases} 1, & y \in G_n^*, \\ 0, & y \notin G_n^*. \end{cases}$$

§2. Definitions

In order to consider differentiability and Fourier transform for functions in $L^r(G_a), 1 \leq r \leq \infty$, we introduce the test functions spaces [6, p23 and p37] on G_a and G_{a^*}, denoted respectively by

$$\mathbb{S} := \{h : h(x) = \sum_{j=0}^{n} c_j R_{k_j}(x - v_{k_j}),\ x, v_{k_j} \in G_a\}$$

and

$$\mathbb{S}^* := \{h : h(y) = \sum_{j=0}^{n} c_j R_{k_j}^*(y - w_{k_j}),\ y, w_{k_j} \in G_{a^*}\},$$

where $c_j \in \mathbb{C}$. $\mathbb{S}$(resp. $\mathbb{S}^*$)is provided with a topology as a toplogical vector space as follows: We define a null sequence in $\mathbb{S}$(resp. $\mathbb{S}^*$) as a sequence $\{h_n\}$ of functions in $\mathbb{S}$(resp. $\mathbb{S}^*$) such that there is a fixed pair of integers k and l such that each h_n is constant on each coset of G_k(resp. G_k^*) and is supported on G_l(resp. G_l^*) and the sequence tends (uniformly) to zero. A simple deduction shows that $\mathbb{S}$(resp. $\mathbb{S}^*$) is an algebra of continuous functions with compact support that separates points. Consequently $\mathbb{S}$ is dense in $L^r(G_a)$ and so is $\mathbb{S}^*$ in $L^r(G_{a^*})$ for $1 \leq r \leq \infty$. Let $\hat{h}(y) = \int_{G_a} h(x)\bar{\chi}_y(x)dx$, the Fourier transform of $h \in \mathbb{S}$. We know that the mapping $h \to \hat{h}$ is a homeormorphism between $\mathbb{S}$ and $\mathbb{S}^*$.

The collections $\mathbb{S}'$ and $\mathbb{S}^{*'}$, of continuous linear functionals on $\mathbb{S}$ and $\mathbb{S}^*$, respectively, with their weak* topologies, are called the spaces of distributions. The action of $f \in \mathbb{S}'$(resp. $f \in \mathbb{S}^{*'}$) on $h \in \mathbb{S}$ (resp. $h \in \mathbb{S}^*$) is denoted by (f, h). It is clear that $L^r(G_a) \subset \mathbb{S}'$ and $L^r(G_a^*) \subset \mathbb{S}^{*'}, 1 \leq r \leq \infty$, and for $f \in L^r(G_a)$ (or $L^r(G_a^*)$), we have

$$(f, h) = \int_{G_a} f(x)\bar{h}(x)dx \qquad \left(\text{or} \quad \int_{G_{a^*}} f(y)\bar{h}(y)dy\right)$$

for $h \in \mathbb{S}$ (or $h \in \mathbb{S}^*$).

The Fourier transform of $f \in \mathbb{S}'$ is defined as a distribution $\hat{f} \in \mathbb{S}^{*'}$ by the equality

$$(\hat{f}, \hat{h}) = (f, h), \quad \text{for all } h \in \mathbb{S}.$$

Thus for $1 \le r \le 2$, $f \in L^r(G_a)$, $\hat{f}$ is equal to the original one in $L^r(G_a)$ sense.

For $2 < r \le \infty$, $\hat{f}$ is a distribution but not a function. We define $g\hat{f}$ for a continuous function g. For a continuous function g and an $\hat{h} \in \mathbb{S}^*$, let $\{\hat{h}_k\}$ be a sequence of functions in $\mathbb{S}^*$ such that $\lim_{k\to\infty} \hat{h}_k = \bar{g}\hat{h}$ in $C(G_{a^*})$. Since $\{\hat{h}_k\}$ is a Cauchy sequence in $\mathbb{S}^*$, $(\hat{f}, \hat{h}_k)$ is a Cauchy sequence in $\mathbb{C}$. We write $(\hat{f}, \bar{g}\hat{h}) = \lim_{k\to\infty} (\hat{f}, \hat{h}_k)$ and define product $g\hat{f}$ to be a distribution by $(g\hat{f}, \hat{h}) = (f, \bar{g}\hat{h})$, for all $\hat{h} \in \mathbb{S}^*$.

Note. We claim that the product $g\hat{f}$ is a continuous linear functional on $\mathbb{S}^*$ according to our definition. In fact, the linearity is obvious; as to the continuity, for any given null sequence $\{\hat{h}_n\}$ in $\mathbb{S}^*$, since each $\hat{h}_n$ is supported on a fixed compact subgroup G_l^* it is easy to see that $(g\hat{f}, \hat{h}_n)$ tends to zero.

We now presnt our definitions of derivatives and integrals of fractional order on G_{a^*}.

Definition 2.1. *For $f \in L_{loc}(G_a), \alpha > 0, n \in \mathbb{N}$ and $x \in G_a$, let*

$$E_{n,\alpha} f(x) := \sum_{j=-n}^{n} m_j^{\alpha} \lambda_{-j,n}^{-1} \sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) f(x + l e_{-j}). \tag{2.1}$$

(a) *If $\lim_{n\to+\infty} E_{n,\alpha} f(x)$ exists, the limit is called the pointwise derivative of order α of f at x, denoted by $f^{<\alpha>}(x)$.*

(b) *If $f \in L^r(G_a), 1 \le r \le \infty$, or $f \in C(G_a)$, and if $\lim_{n\to\infty} E_{n,\alpha} f$ exists in the strong sense, the limit is called the strong derivative of order α of f, denoted by $D_r^{<\alpha>} f$ or $D_c^{<\alpha>} f$.*

We denote one of $C(G_a)$ and $L^r(G_a)(1 \le r < \infty)$ by X, and one of $D_c^{<\alpha>} f$ and $D_r^{<\alpha>} f$ $(1 \le r \le \infty)$ by $D^{<\alpha>} f$. We set

$$\mathcal{D}(D^{<\alpha>}) := \{f \in X : D^{<\alpha>} f \text{ exists}\};$$

$$\mathrm{Lip}_X(\alpha) := \{f \in X : \sup_{|t|\le|u|} \|f(\cdot - t) - f(\cdot)\|_X = O(|u|^{\alpha}), |u| \to 0\};$$

$$W_X(|y|^{\alpha}) := \{f \in X : \text{ there is a } g \in L^r(G_a) \text{ such that } \hat{g}(y) = |y|^{\alpha}\hat{f}(y) \text{ a.e. for } 1 \le r \le 2 \text{ and } \hat{g} = |\cdot|^{\alpha}\hat{f} \text{ in the distribution sense otherwise}\}.$$

Defintion 2.2. *For each $n \in \mathbb{Z}, \alpha > 0$, define $V_{n,\alpha}(x)$ by*

$$V_{n,\alpha}(x) := \sum_{l=-n+1}^{\infty} m_{-l+1}^{-\alpha} (m_l^{-1} R_l - m_{l-1}^{-1} R_{l-1})(x). \tag{2.2}$$

If the convolution $V_{n,\alpha} * f(x) = \int_{G_a} V_{n,\alpha}(x-u) f(u) du$ has a pointwise limit or a strong limit in X as $n \to +\infty$, then we call it the pointwise integral of order α of f or the strong integral of order α of f. The strong integral is denoted by $I^{<\alpha>} f$.

§3. Results and Their Proofs

Theorem 3.1. *For each $\alpha > 0$, $y \in G_{a^*}$ and $x \in G_a$, $\chi_y^{<\alpha>}(x)$ exists and $\chi_y^{<\alpha>}(x) = |y|^{\alpha}\chi_y(x)$.*

Proof. For $y \in G_s^* \setminus G_{s+1}^*$, we know that

$$y = (0, \cdots, 0, y_{s+1}, y_{s+2}, \cdots), \ y_{s+1} \neq 0, \text{ and } |y| = m_{s+1}.$$

Now

$$E_{n,\alpha}\chi_y(x) = \sum_{j=-n}^{n} m_j^\alpha \lambda_{-j,n}^{-1} \sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \chi_y(le_{-j}) \chi_y(x).$$

If $j \leq s$ then $e_{-j} \in G_{-s}$ and $\chi_y(le_{-j}) = 1$. So we have

$$\sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \chi_y(le_{-j}) = 0.$$

If $j = s+1$ then $e_{-j} \in G_{-s-1} \setminus G_{-s}$ and we have

$$\chi_y(e_{-s-1}) = \exp(2\pi i y_{s+1}/a_{-s-1}),$$

thus

$$\begin{aligned}
&\sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \chi_y(le_{-j}) \\
&= \sum_{v=1}^{a_{-s-1}-1} \sum_{l=0}^{\lambda_{-s-1,n}-1} \exp(-2\pi i l(v - y_{s+1})/a_{-s-1}) \\
&= \lambda_{-s-1,n} \ (\text{ note that } y_{s+1} \in \{1, \cdots, a_{-s-1} - 1\}).
\end{aligned}$$

If $j > s+1$,

$$\chi_y(e_{-j}) = \exp\left(2\pi i \sum_{k=-j}^{-s-1} \frac{y_{-k}}{a_{-j}a_{-j+1}\cdots a_k}\right) = \exp(2\pi i u a_{-j}^{-1}),$$

where $u = y_j + y_{j-1}/a_{-j+1} + \cdots + y_{s+1}/(a_{-j+1} \cdots a_{-s-1})$ is not an integer, thus

$$\begin{aligned}
&\sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \chi_y(le_{-j}) \\
&= \sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i l(v-u) a_{-j}^{-1}) = 0.
\end{aligned}$$

It follows that for $n > | s |$

$$E_{n,\alpha}\chi_y(x) = m_{s+1}^\alpha \chi_y(x) = | y |^\alpha \chi_y(x).$$

The proof is complete.

Theorem 3.2. *If $h \in \mathbb{S}$, then $h^{<\alpha>}(x)$ and $D^{<\alpha>}h$ exist. Furthermore, if h is constant on each coset of G_s, then*

$$\|E_{n,\alpha}h\|_r \leq \sum_{j=-s+1}^{\infty} m_j^\alpha a_{-j} \|h\|_r, \quad \|D^{<\alpha>}h\|_r \leq \sum_{j=-s+1}^{\infty} m_j^\alpha a_{-j} \|h\|_r$$

and

$$\|h^{<\alpha>}(\cdot)\|_r \leq \sum_{j=-s+1}^{\infty} m_j^\alpha a_{-j} \|h\|_r, \ 1 \leq r \leq \infty.$$

Specially, if $\sup_{j \in \mathbb{Z}}\{a_j\} < \infty$. *Then* $\sum_{j=-s+1}^{\infty} m_j^\alpha a_{-j} = O(m_{-s+1}^\alpha)$.

Proof. $\sum\limits_{j=-s+1}^{\infty} m_j^{\alpha} a_{-j} = O\left(m_{-s+1}^{\alpha}\right)$ provided $\sup\limits_{j\in\mathbb{Z}}\{a_j\} < \infty$ and

$$\|E_{n,\alpha}h\|_r \le \sum_{j=-s+1}^{\infty} m_j^{\alpha} a_{-j}\|h\|_r$$

are obvious.

We first prove that $R_s(x)$ has pointwise derivative for all x in G_a. Let n be sufficiently large.

(i) For $x \in G_s$, we have

$$E_{n,\alpha}R_s(x) = \sum_{j=-s+1}^{n} m_j^{\alpha}\lambda_{-j,n}^{-1} \sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \cdot R_s(le_{-j}),$$

and $le_{-j} \in G_s \Leftrightarrow \lambda_{-j,s-1}^{-1} \cdot l = 0, 1, \cdots, \lambda_{s,n} - 1$, so

$$\begin{aligned} E_{n,\alpha}R_s(x) &= \sum_{j=-s+1}^{n} m_j^{\alpha} \cdot \lambda_{-j,n}^{-1} \cdot (a_{-j}-1) \cdot \lambda_{s,n} \\ &= \sum_{j=-s+1}^{n} m_j^{\alpha} \cdot \lambda_{-j,s-1}^{-1} \cdot (a_{-j}-1). \end{aligned}$$

Thus

$$\begin{aligned} R_s^{<\alpha>}(x) &= \lim_{n\to+\infty} E_{n,\alpha}R_s(x) \\ &= \sum_{j=-s+1}^{\infty} m_j^{\alpha} \cdot \lambda_{-j,s-1}^{-1}(a_{-j}-1) < \infty \text{ for } x \in G_s. \end{aligned} \tag{3.1}$$

(ii) For $x \notin G_s$, if $x = x' + x_{s-1}e_{s-1} + \cdots + x_{s-k}e_{s-k}$, $x' \in G_s$, $x_{s-k} \ne 0$, k is a positive integer, then we have

(a) as $j < -s+k$ (i.e. $-j > s-k$), $R_s(x + le_{-j}) = 0$;

(b) as $j = -s+k$ (i.e. $-j = s-k$), $x + le_{-j} \in G_s$ if and only if

$$\begin{aligned} l =&(a_{s-k} - x_{s-k}) + a_{s-k}(a_{s-k+1} - x_{s-k+1} - 1) + a_{s-k}a_{s-k+1}(a_{s-k+2} - x_{s-k+2} - 1) \\ &+ \cdots + a_{s-k}a_{s-k+1}\cdots a_{s-2}(a_{s-1} - x_{s-1} - 1) + m\lambda_{s-k,s-1}, \ m = 0, 1, \cdots, \lambda_{s,n} - 1. \end{aligned}$$

Thus

$$\sum_{v=1}^{a_{s-k}-1} \sum_{l=0}^{\lambda_{s-k,n}-1} \exp(-2\pi i v l a_{s-k}^{-1}) \cdot R(x + le_{s-k}) = -\lambda_{s,n}.$$

(c) As $j > -s+k$ (i.e. $-j < s-k$), $x + le_{-j} \in G_s$ if and only if

$$\begin{aligned} l = \Big(&(a_{s-k} - x_{s-k}) + a_{s-k}(a_{s-k+1} - x_{s-k+1} - 1) + \cdots + \\ &a_{s-k}a_{s-k+1}\cdots a_{s-2}(a_{s-1} - x_{s-1} - 1)\Big)\lambda_{-j,s-k-1} + m\lambda_{-j,s-1}, \\ &m = 0, 1, \cdots, \lambda_{s,n} - 1. \end{aligned}$$

Thus

$$\sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) \cdot R(x + le_{-j}) = (a_{-j} - 1) \cdot \lambda_{s,n}.$$

From (a), (b) and (c), we have for $x \in G_{s-k} \setminus G_{s-k+1}$

$$\begin{aligned} E_{n,\alpha} R_s(x) &= m^{\alpha}_{-s+k} \lambda^{-1}_{s-k,n}(-\lambda_{s,n}) + \sum_{j=-s+k+1}^{n} m^{\alpha}_j \lambda^{-1}_{-j,n}(a_{-j}-1)\lambda_{s,n} \\ &= -m^{\alpha}_{-s+k} \cdot \lambda^{-1}_{s-k,s-1} + \sum_{j=-s+k+1}^{n} m^{\alpha}_j \cdot \lambda^{-1}_{-j,s-1}(a_{-j}-1), \end{aligned}$$

so that

$$R_s^{<\alpha>}(x) = -m^{\alpha}_{-s+k} \cdot \lambda^{-1}_{s-k,s-1} + \sum_{j=-s+k+1}^{\infty} m^{\alpha}_j \cdot \lambda^{-1}_{-j,s-1}(a_{-j}-1) < \infty$$

$$\text{for } x \in G_{s-k} \setminus G_{s-k+1}. \tag{3.2}$$

(3.1) and (3.2) show that $R_s^{<\alpha>}(x)$ exist everywhere.

Secondly we claim that $R_s^{<\alpha>}(x)$ belongs to $L^r(G_a), 1 \le r \le \infty$, and $C(G_a)$. It is obvious that $R_s^{<\alpha>}(x)$ belongs to $L^{\infty}(G_a)$ and $C(G_a)$ from (3.1) and (3.2). So it is sufficient to prove that $R_s^{<\alpha>}(x)$ is integrable. By (3.1) and (3.2) we have

$$\begin{aligned} \int_{G_a} |R_s^{<\alpha>}(x)| dx &= \int_{G_s} \sum_{j=-s+1}^{\infty} m^{\alpha}_j \lambda^{-1}_{-j,s-1}(a_{-j}-1) dx \\ &+ \sum_{k=1}^{\infty} \int_{G_{s-k}\setminus G_{s-k+1}} \Big| -m^{\alpha}_{-s+k}\lambda^{-1}_{s-k,s-1} + \sum_{j=-s+k+1}^{\infty} m^{\alpha}_j \lambda^{-1}_{-j,s-1}(a_{-j}-1) \Big| dx \\ &\le m_s \sum_{j=-s+1}^{\infty} m^{\alpha}_j \lambda^{-1}_{-j,s-1} a_{-j} + \sum_{k=1}^{\infty} m^{\alpha}_{-s+k} \lambda^{-1}_{s-k,s-1} m_{s-k} \\ &+ \sum_{k=1}^{\infty} m_{s-k} \sum_{j=1}^{\infty} m^{\alpha}_{j+k-s} \lambda^{-1}_{-j-k+s,s-1} a_{-j-k+s} \\ &\le m_s m^{\alpha}_{-s+1} \sum_{j=-s+1}^{\infty} \lambda^{-1}_{-j,s-1} a_{-j} + \sigma_s \sum_{k=1}^{\infty} m^{\alpha}_{-s+k} \\ &+ \sum_{k=1}^{\infty} m^{\alpha}_{1+k-s} \sum_{j=1}^{\infty} \sigma_s \lambda^{-1}_{-j-k+s,s-k-1} a_{-j-k+s}, \end{aligned}$$

where $\sigma_s = \lambda_{s,-1}$ for $s \le -1$ and $\sigma_s = 1$ otherwise. Nothing that $\sum_{j=1}^{\infty} \sigma_s \lambda^{-1}_{-j-k+s,s-k-1} \cdot a_{-j-k+s}$ uniformly converges with respect to k and the other three series converge, we have

$$\int_{G_a} |R_s^{<\alpha>}(x)| dx < \infty.$$

Finally, it is clear that $D_r^{<\alpha>} R_s(x) = R_s^{<\alpha>}(x)$ a.e. for $r = \infty$, and $D_c^{<\alpha>} R_s(x) = R_s^{<\alpha>}(x)$. For $1 \le r < \infty$, using Lebesgue dominant convergence theorem we still have $D_r^{<\alpha>} R_s(x) = R_s^{<\alpha>}(x)$ a.e.. For $h \in \mathbb{S}$, since $h(x)$ is a finite linear combination of $R_{k_j}(x - v_{k_j})$, $D_r^{<\alpha>} h$ and $h^{<\alpha>}(x)$ exist.

By $\|E_{n,\alpha} h\|_r \le \sum_{j=-s+1}^{\infty} m^{\alpha}_j a_{-j} \|h\|_r$, a simple deduction shows that the other two inequalities hold. The proof is complete.

Theorem 3.3. *If f, $D^{<\alpha>}f \in L^r(G_a), \alpha > 0$, then for $1 \le r \le 2$ we have*

$$[D^{<\alpha>}f]^\wedge(y) = |y|^\alpha \hat{f}(y) \text{ a.e.}, \ y \in G_{a^*}, \tag{3.3}$$

and for $2 < r \le +\infty$ we have

$$[D^{<\alpha>}f]^\wedge = |\cdot|^\alpha \hat{f} \quad \text{in the distribution sense.} \tag{3.4}$$

Proof. For $1 \le r \le 2$, since the mapping $f \to \hat{f}$ is a bounded linear transform from $L^r(G_a)$ into $L^{r'}(G_{a^*})$ and $\|\hat{f}\|_{r'} \le \|f\|_r$, where $\frac{1}{r}+\frac{1}{r'}=1, \lim_{n\to+\infty}\|E_{n,\alpha}f - D^{<\alpha>}f\|_r = 0$ implies

$$\lim_{n\to+\infty}\|(E_{n,\alpha}f)^\wedge - (D^{<\alpha>}f)^\wedge\|_{r'} = 0.$$

Thus there exists a subsequence $n_k \to +\infty$ such that

$$\lim_{n_k\to+\infty}[E_{n_k,\alpha}f]^\wedge(y) = [D^{<\alpha>}f]^\wedge(y) \ \text{ a.e..}$$

Now

$$\begin{aligned}[E_{n,\alpha}f]^\wedge(y) &= \int_{G_a} E_{n,\alpha}f(x)\bar{\chi}_y(x)dx \\ &= \hat{f}(y)\sum_{j=-n}^{n} m_j^\alpha \lambda_{-j,n}^{-1}\sum_{v=1}^{a_j-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})\chi_y(le_{-j}),\end{aligned}$$

and we have (in virtue of the proof of Theorem 3.1)

$$\lim_{n\to+\infty}[E_{n,\alpha}f]^\wedge(y) = |y|^\alpha \hat{f}(y).$$

Therefore $[D^{<\alpha>}f]^\wedge(y) = |y|^\alpha\hat{f}(y)$ a.e. for $1 \le r \le 2$.

Secondly, for $2 < r \le \infty$, by definition and the relation between strong convergence and weak convergence, for $h \in \mathbb{S}$, we have

$$\begin{aligned}([D^{<\alpha>}f]^\wedge, \hat{h}) &= (D^{<\alpha>}f, h) = \lim_{n\to+\infty}\int_{G_a} E_{n,\alpha}f(x)\bar{h}(x)dx \\ &= \lim_{n\to+\infty}\sum_{j=-n}^{n} m_j^\alpha\lambda_{-j,n}^{-1}\sum_{v=1}^{a_{-j}-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})\int_{G_a} f(x+le_{-j})\bar{h}(x)dx \\ &= \lim_{n\to+\infty}\sum_{j=-n}^{n} m_j^\alpha\lambda_{-j,n}^{-1}\sum_{v=1}^{a_{-j}-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})\int_{G_a} f(x)\overline{h(x-le_{-j})}dx \\ &= \lim_{n\to+\infty}\sum_{j=-n}^{n} m_j^\alpha\lambda_{-j,n}^{-1}\sum_{v=1}^{a_{-j}-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})(\hat{f}, [h(\cdot - le_{-j})]^\wedge(\cdot)) \\ &= \lim_{n\to+\infty}(\hat{f}, \sum_{j=-n}^{n} m_j^\alpha\lambda_{-j,n}^{-1}\sum_{v=1}^{a_{-j}-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})\bar{\chi}_\cdot(le_{-j})\hat{h}(\cdot)) \\ &= (\hat{f}, |\cdot|^\alpha\hat{h}(\cdot)).\end{aligned}$$

The last equality holds because distribution $\hat{f}$ is continuous and the series converges strongly to $|\cdot|^\alpha$ on the compact support of $\hat{h}(\cdot)$ in virtue of the deduction in Theorem 3.1. By definition we conclude that

$$([D^{<\alpha>}f]^\wedge, \hat{h}) = (|\cdot|^\alpha f^\wedge, \hat{h});$$

this implies $[D^{<\alpha>}f]^\wedge = |\cdot|^\alpha \hat{f}$ in the distribution sense. The proof is complete.

Specially, we have

Theorem 3.4. *For $f \in L^r(G_a)$, $1 \le r \le +\infty$, $\alpha > 0$, $n \in \mathbb{N}$, set*

$$E^{\infty}_{n,\alpha} f(x) = \sum_{j=-n}^{\infty} \{m_j^\alpha \lambda_{-j,n}^{-1} \sum_{v=1}^{a_{-j}-1} \sum_{l=0}^{\lambda_{-j,n}-1} \exp(-2\pi i v l a_{-j}^{-1}) f(x + l e_{-j})\}.$$

We have

$$(E^{\infty}_{n,\alpha} f)^\wedge(y) = \begin{cases} |y|^\alpha \hat{f}(y) \text{ a.e.} & \text{if } y \in G^*_{-n-1}, \\ 0 \text{ a.e.} & \text{if } y \notin G^*_{-n-1}, \end{cases} \quad (1 \le r \le 2) \tag{3.5}$$

and

$$(E^{\infty}_{n,\alpha} f)^\wedge = R^*_{-n-1}(\cdot)|\cdot|^\alpha \hat{f}, \tag{3.6}$$

in the distribution sense for $2 < r \le +\infty$.

Lemma 3.1. *For $V_{n,\alpha}(x)$ in Definition 2.2, we have $V_{n,\alpha} \in L^1(G_a)$ and $\|V_{n,\alpha}\|_1 = O(m_n^{-\alpha})$. Moreover*

$$(V_{n,\alpha})^\wedge(y) = \begin{cases} 0 & \text{if } y \in G^*_n, \\ |y|^{-\alpha} & \text{if } y \notin G^*_n. \end{cases} \tag{3.7}$$

Proof. Firstly,

$$\begin{aligned} \|V_{n,\alpha}\|_1 &= \| \sum_{l=-n+1}^{\infty} m_{-l+1}^{-\alpha} (m_l^{-1} R_l - m_{l-1}^{-1} R_{l-1}) \|_1 \\ &\le \sum_{l=-n+1}^{\infty} m_{-l+1}^{-\alpha} \cdot 2 \le A m_n^{-\alpha} = O(m_n^{-\alpha}). \end{aligned}$$

Secondly,

$$\begin{aligned} (V_{n,\alpha})^\wedge(y) &= \sum_{l=-n+1}^{\infty} m_{-l+1}^{\alpha} (m_l^{-1} R_l^\wedge - m_{l-1}^{-1} R_{l-1}^\wedge)(y) \\ &= \sum_{l=-n+1}^{\infty} m_{-l+1}^{-\alpha} (R^*_{-l}(y) - R^*_{-l+1}(y)) \\ &= \begin{cases} 0 & \text{if } y \in G^*_n, \\ |y|^{-\alpha} & \text{if } y \notin G^*_n. \end{cases} \end{aligned}$$

Lemma 3.2. *If $f \in L^r(G_a), 1 \le r \le \infty$, $g \in L^1(G_a)$, then*

$$(f * g)^\wedge = \hat{f} \cdot \hat{g} \tag{3.8}$$

in the sense stated before the Note in section 2.

Proof. Let $\tilde{g}(x) = \overline{g(-x)}$ and $\hat{h} \in \mathbb{S}^*$. Since $\hat{g}$ is uniformly continuous, we have

$$((f * g)^\wedge, \hat{h}) = (f * g, h) = (f, \tilde{g} * h) = (\hat{f}, \bar{\hat{g}} \hat{h}) = (\hat{f} \cdot \hat{g}, \hat{h}).$$

This means $(f * g)^\wedge = \hat{f} \cdot \hat{g}$.

Lemma 3.3. *If $f \in L^r(G_a)$ $(1 \le r \le \infty)$ and $\hat{f}(y) = 0$ a.e. for $1 \le r \le 2$ and $\hat{f} = 0$ in distribution sense for $2 < r \le \infty$ then $f = 0$ a.e. for $1 \le r \le \infty$ and $f \equiv 0$ if $f \in C(G_a)$.*

The proof is normal.

Lemma 3.4. *If $f \in L_{loc}(G_a)$, then $\lim_{n \to +\infty} m_n^{-1} R_n * f(x) = f(x)$ a.e.. If $f \in X$, then*

$$\lim_{n \to +\infty} \|m_n^{-1} R_n * f - f\|_X = 0. \tag{3.9}$$

The proof is easy (See [6, p.174]). Furthermore, we have

Lemma 3.5. *If $f \in \mathrm{Lip}_X(\alpha)$, then $\|m_n^{-1}R_n * f - f\|_X = O(m_n^\alpha)$.*

Theorem 3.5. *If $\sup_{j\in\mathbb{Z}}\{a_j\} < \infty$ and $a_n = a_{-n-1}$ it for all $n \in \mathbb{Z}$, then for $0 < \alpha < \beta$ we have $\mathrm{Lip}_X(\beta) \subset \mathcal{D}_X(D^{<\alpha>}) \subset \mathrm{Lip}_X(\alpha)$.*

Proof. If $f \in \mathrm{Lip}_X(\beta)$, then by Lemma 3.5 we have

$$\|m_n^{-1}R_n * f - f\|_X \le Am_n^\beta \quad \text{for} \quad n = 1, 2, \cdots, \tag{3.10}$$

where A is a constant which may changes in value from one occurrence to the next. Set $U_1(x) = m_2^{-1}R_2 * f(x)$, $U_n(x) = m_{n+1}^{-1}R_{n+1} * f(x) - m_n^{-1}R_n * f(x)$ for $n > 1$. We see that $U_n(x)$ is constant on each coset of G_n. Consequently, since $a_n = a_{-n-1}, m_n = m_{-n}^{-1}$, we have

$$\|E_{M,\alpha}U_n\|_X \le Am_{-n}^\alpha\|U_n\|_X = Am_n^{-\alpha}\|U_n\|_X \quad \text{for all} \quad M \in \mathbb{N}. \tag{3.11}$$

By (3.10) we have

$$\|U_n\|_X \le \|m_{n+1}^{-1}R_{n+1} * f - f\|_X + \|m_n^{-1}R_n * f - f\|_X \le Am_n^\beta, \ n > 1, \tag{3.12}$$

and

$$f(x) = \sum_{n=1}^{\infty} U_n(x) \quad \text{in} \quad X \quad \text{sense.} \tag{3.13}$$

By (3.11), (3.12) and (3.13), it turns out that

$$\begin{aligned} &\|E_{N,\alpha}f - E_{M,\alpha}f\|_X \\ \le&\|\sum_{n=1}^{L}(E_{N,\alpha}U_n - E_{M,\alpha}U_n)\|_X + \sum_{n=L+1}^{\infty}(\|E_{N,\alpha}U_n\|_X + \|E_{M,\alpha}U_n\|_X) \\ \le&\|E_{N,\alpha}\sum_{n=1}^{L}U_n - E_{M,\alpha}\sum_{n=1}^{L}U_n\|_X + \sum_{n=L+1}^{\infty} Am_n^{\beta-\alpha}. \end{aligned}$$

For $\epsilon > 0$, let L be sufficiently large so that $\sum_{n=L+1}^{\infty} Am_n^{\beta-\alpha} < \epsilon/2$. Then since $m_n^{-1}R_n \in \mathbb{S}(G_a), n = 1, 2, \cdots, L+1$, we can see that $\{E_{M,\alpha}\sum_{n=1}^{L} U_n\}$ is a Cauchy sequence in X sense. Thus

$$\|E_{N,\alpha}\sum_{n=1}^{L}U_n - E_{M,\alpha}\sum_{n=1}^{L}U_n\|_X < \epsilon/2$$

whenever $M, N > G$ for some $G > 0$. Therefore $\{E_{M,\alpha}f\}$ is a Cauchy sequence, and $D^{<\alpha>}f$ exists; that is $f \in \mathcal{D}(D^{<\alpha>})$.

If $f \in \mathcal{D}(D^{<\alpha>})$ and $h \in G_n \setminus G_{n+1}$, by (3.7), (3.8), Theorem 3.3 and Lemma 3.3 we have

$$f(x-h) - f(x) = V_{-n,\alpha} * (D^{<\alpha>}f(x-h) - D^{<\alpha>}f(x)) \quad \text{in} \quad X,$$

so that

$$\|f(\cdot - h) - f(\cdot)\|_X \le 2\|D^{<\alpha>}f\|_X\|V_{-n,\alpha}\|_1 = O(m_{-n}^{-\alpha}) = O(m_n^\alpha) = O(|h|^\alpha).$$

That means that $f \in \mathrm{Lip}_X(\alpha)$. The proof is complete.

Theorem 3.6. *Let $\alpha > 0$. If $\sup_{j\in Z}\{a_j\} < \infty$, then*

$$\mathcal{D}(D^{<\alpha>}) = W_X(|y|^\alpha). \tag{3.14}$$

Proof. For each $f \in \mathcal{D}(D^{<\alpha>})$, from Theorem 3.3 we directly obtain $f \in W_X(|y|^\alpha)$. Conversely, take an $f \in W_X(|y|^\alpha)$ and assume that $g \in X$ satisfies $\hat{g}(y) = |y|^\alpha \hat{f}(y)$ a.e. on G_{a^*} if $f \in L^r(G_a), 1 \le r \le 2$, and $\hat{g} = |\cdot|^\alpha \hat{f}$ in the distribution sense ortherwise. Theorem 3.4 and Lemma 3.3 with the Fourier transforms show that

$$E_{n,\alpha}^{\infty} f(x) = m_{n+1}^{-1} R_{n+1} * g(x) \text{ in } X.$$

Thus

$$\|E_{n,\alpha} f - g\|_X \le \|m_{n+1}^{-1} R_{n+1} * g - g\|_X + \sum_{j=n+1}^{\infty} m_j^\alpha (a_{-j} - 1)\|f\|_X.$$

By (3.9) and $\sup\{a_j\} < \infty$ we have

$$\lim_{n\to\infty} \|E_{n,\alpha} f - g\|_X = 0,$$

that is $f \in \mathcal{D}(D^{<\alpha>})$. The proof is complete.

Lemma 3.6. *Let $f, g \in X$. If $m_{-n}^{-1} R_{-n} * g$ converge in X as $n \to +\infty$, then the following two conditions are equivalent:*

(i) $g = I^{<\alpha>} f$ *in* X.

(ii) $\hat{g}(y) = |y|^{-\alpha} \hat{f}(y)$ *a.e. if* $f, g \in L^r(G_a), 1 \le r \le 2$, *and* $\hat{g} = |\cdot|^{-\alpha} \hat{f}$ *in the distribution sense ortherwise.*

Proof. (i)⇒(ii). If $g = I^{<\alpha>} f$, by continuity of the Fourier transform we have

$$\hat{g} = (\lim_{n\to+\infty} V_{n,\alpha} * f)^\wedge = \lim_{n\to+\infty} (V_{n,\alpha} * f)^\wedge.$$

Then by (3.8) and (3.7) we get (ii).

(ii)⇒(i). With (3.7), (3.8) and Lemma 3.3, a comparison of the Fourier transform shows that

$$V_{n,\alpha} * f(x) - V_{k,\alpha} * f(x) = m_{-k}^{-1} R_{-k} * g(x) - m_{-n}^{-1} R_{-n} * g(x) \text{ in } X.$$

Then that $m_{-n}^{-1} R_{-n} * g$ converge in X as $n \to +\infty$ implies

$$\lim_{n,k\to+\infty} \|V_{n,\alpha} f - V_{k,\alpha} * f\|_X = 0.$$

This means that $I^{<\alpha>} f$ exists. By Theorem 3.3 and Lemma 3.3 we get (i). The Proof is complete.

Theorem 3.7. *If $f, D^{<\alpha>} f \in X$ and $m_{-n}^{-1} R_n * f$ converge in X as $n \to +\infty$, then*

$$f(x) = I^{<\alpha>}(D^{<\alpha>} f)(x) \text{ in } X. \tag{3.15}$$

Proof. By the Fourier transform mathod we have

$$V_{n,\alpha} * D^{<\alpha>} f(x) - V_{k,\alpha} * D^{<\alpha>} f(x) = m_{-k}^{-1} R_{-k} * f(x) - m_{-n}^{-1} R_{-n} * f(x) \text{ in } X.$$

$m_{-n}^{-1} R_{-n} * f$ converge in X as $n \to +\infty$, so do $V_{n,\alpha} * D^{<\alpha>} f$. Thus $I^{<\alpha>}(D^{<\alpha>} f)$ exists in X. By Theorem 3.3 and Lemma 3.6 we have

$$(I^{<\alpha>}(D^{<\alpha>} f))^\wedge = \hat{f} \text{ in themselves sense.}$$

Then by Lemma 3.3 we get (3.15).

Theorem 3.8. *Soppose* $\sup\{a_j\} < \infty$. *if $f, g \in X$ and $g = I^{<\alpha>} f$ in X, then*

$$D^{<\alpha>}(I^{<\alpha>} f)(x) = f(x) \text{ in } X. \tag{3.16}$$

Proof. By Lemma 3.3, Lemma 3.6, (3.7) and (3.8) we have

$$g(x) = (m_n^{-1}R_n * g)(x) + (V_{-n,\alpha} * f)(x) \text{ in } X.$$

Then

$$\|E_{n,\alpha}g - f\|_X \leq \|E_{n,\alpha}(m_n^{-1}R_n * g) - f\|_X + \|E_{n,\alpha}(V_{-n,\alpha} * f)\|_X. \tag{3.17}$$

By Theorem 3.4 we also have

$$E_{n,\alpha}^{\infty}(m_n^{-1}R_n * g)(x) = m_n^{-1}R_n * f(x) \text{ in } X,$$

so that

$$\begin{aligned}&\|E_{n,\alpha}(m_n^{-1}R_n * g) - m_n^{-1}R_n * f\|_X \\ =&\|\sum_{j=n+1}^{\infty} m_j^{\alpha}\lambda_{-j,n}^{-1}\sum_{v=1}^{a_{-j}-1}\sum_{l=0}^{\lambda_{-j,n}-1}\exp(-2\pi i v l a_{-j}^{-1})m_n^{-1}R_n * g(\cdot + le_{-j})\|_X \\ \leq& A\cdot m_{n+1}^{\alpha}\|g\|_X \to 0\ (n\to+\infty).\end{aligned} \tag{3.18}$$

On the orther hand,

$$\begin{aligned}\|E_{n,\alpha}(m_n^{-1}R_n * g) - f\|_X \leq&\|E_{n,\alpha}(m_n^{-1}R_n * g) - m_n^{-1}R_n * f\|_X \\ &+ \|m_n^{-1}R_n * f - f\|_X.\end{aligned}$$

By (3.18) and (3.9) we get

$$\lim_{n\to+\infty}\|E_{n,\alpha}(m_n^{-1}R_n * g) - f\|_X = 0. \tag{3.19}$$

Secondly, $V_{-n,\alpha} \in L^1(G_a)$, $f \in X$, so that $V_{-n,\alpha} * f \in X$, $E_{n,\alpha}(V_{-n,\alpha} * f) \in X$ and $(E_{n,\alpha}(V_{-n,\alpha} * f))^{\wedge} = (f * (E_{n,\alpha}V_{-n,\alpha}))^{\wedge}$. we have

$$E_{n,\alpha}(V_{-n,\alpha} * f)(x) = f * (E_{n,\alpha}V_{-n,\alpha})(x) \text{ in } X.$$

But

$$\begin{aligned}\|f * (E_{n,\alpha}V_{-n,\alpha})\|_X \leq&\|(E_{n,\alpha}V_{-n,\alpha}) * (f - m_k^{-1}R_k * f)\|_X \\ &+ \|E_{n,\alpha}V_{-n,\alpha} * m_k^{-1}R_k * f\|_X.\end{aligned} \tag{3.20}$$

Since $V_{-n,\alpha} * m_k^{-1}R_k(x) \equiv 0$ as $n > k$, the second term in the right side of (3.20) vanishes as $n > k$. Because $\lim\limits_{k\to+\infty}\|f - m_k^{-1}R_k * f\|_X = 0$ and $\|E_{n,\alpha}V_{-n,\alpha}\| \leq A$, we have

$$\lim_{n\to+\infty}\|E_{n,\alpha}(V_{-n,\alpha} * f)\|_X = \lim_{n\to+\infty}\|f * E_{n,\alpha}V_{-n,\alpha}\|_X = 0. \tag{3.21}$$

Applying (3.19) and (3.21) to (3.17), we get

$$\lim_{n\to+\infty}\|E_{n,\alpha}g - f\|_X = 0. \tag{3.22}$$

The proof is complete.

As another application of the derivatives, we study the relationship between functions belonging to $\mathcal{D}(D^{<\alpha>})$ and their Bessel potentials.

For $\alpha > 0$, let

$$B_\alpha = V_{0,\alpha} + R_0 \text{ and } \mu_\alpha := \delta_0 - R_0 + D_1^{<\alpha>}R_0,$$

where δ_0 is the Dirac δ-measure concentrated at $0 \in G_a$. Observe that $B_\alpha \in L^1(G_a)$ and μ_α is a Borel measure on G_a. Furthermore, from (3.7) and (3.3), we have their Fourier transforms,

$$(B_\alpha)^\wedge(y) = \begin{cases} 1 & \text{if } y \in G_0^*, \\ |y|^{-\alpha} & \text{if } \hat{y} \notin G_0^*, \end{cases} \tag{3.23}$$

and

$$(\mu_\alpha)^\wedge(y) = \begin{cases} |y|^\alpha & \text{if } y \in G_0^*, \\ 1 & \text{if } \hat{y} \notin G_0^*. \end{cases} \tag{3.24}$$

For $f \in X$, and $\alpha > 0$, we define $J_\alpha f := B_\alpha * f$. Furthermore, $L(X,\alpha) := \{f \in X : f = J_\alpha g$ for some $g \in X\}$. $J_\alpha f$ is called the Bessel potential of order α of f.

Theorem 3.9. *Let $\alpha > 0$. If $\sup\limits_{j\in\mathbb{Z}}\{a_j\} < \infty$, then $\mathcal{D}(D^{<\alpha>}) = L(X,\alpha)$.*

Proof. Take $f \in L(X,\alpha)$ and let $f = B_\alpha * h$ with $h \in X$. By (3.23), we have

$$\hat{f}(y) = \begin{cases} \hat{h}(y) & \text{if } y \in G_0^*, \\ |y|^{-\alpha}\hat{h}(y) & \text{if } y \notin G_0^*, \end{cases}$$

in the sense of itself. Thus, according to (3.24), we have $|y|^\alpha \hat{f}(y) = (\mu_\alpha)^\wedge(y)\hat{h}(y)$. Consequently, there exists a $g \in X$ with $\hat{g}(y) = |y|^\alpha \hat{f}(y)$ for a.e. $y \in G_{a^*}$ if $1 \le r \le 2$ and in the distribution sense otherwise; that is, $f \in W_X(|y|^\alpha) = \mathcal{D}(D^{<\alpha>})$ by (3.14).

Conversely, take $f \in \mathcal{D}(D^{<\alpha>})$ and let $g = D^{<\alpha>}f$. Set $h = g + (R_0 - D_1^{<\alpha>}R_0) * f$. Clearly, $h \in X$ and we have

$$\hat{h}(y) = \begin{cases} \hat{f}(y) & \text{if } y \in G_0^*, \\ |y|^\alpha \hat{f}(y) & \text{if } y \notin G_0^* \end{cases}$$

in the sense of itself. Thus $f = B_\alpha * h$ by (3.23). Hence, $f \in L(X,\alpha)$. The proof is complete.

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