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HOMOLOGICAL PROPERTIES OF TORSION CLASSES UNDER CHANGE OF RINGS

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Abstract

Let R be a ring with identity, x be a central element of R which is neither a unit nor a zero divisor. $S = R/xR$ is the quotient ring of R and $\varphi : R \to R/xR$ is the natural map. R-Mod (resp. S-Mod) denotes the category of unital left R-modules(resp. S-modules). In this paper, relationships betwee torsion theories on R-Mod and torsion theories on S-Mod are investigated. Properties of the functor $\mathrm{Ext}^n_R(N,-)$ are given. Properties of the localization functor $Q_\sigma$are also investigated.

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Let R be a ring with identity, x be a central element of R which is neither a unit nor a zero divisor. $S = R/xR$ is the quotient ring of R and $\varphi : R \to R/xR$ is the natural map. R-Mod(resp. S-Mod) denotes the category of unital left R-modules (resp.S-modules). There is a canonical way to define each left S-module M to be a left R-module:

$$r \cdot a = \varphi(r)a \qquad \text{for any} \quad a \in M,\ r \in R.$$

The family of all hereditary torsion theories defined on R-Mod (resp. S-Mod) will be denoted by R-tors (resp. S-tors). Let T_τ (resp. F_τ) be the torsion class (resp. torsionfree class) of R-Mod determined by $\tau \in R-$tors. According to [1], define a map φ_* from R-tors to S-tors: for each $\tau \in R$-tors, $\varphi_*(\tau) = \sigma$ is defined by the condition that a left S-module M is σ torsion if and only if M is τ-torsion as a left R-module. If τ is perfect, then σ is perfect. In general, the converse is not true. A counterexample will be given in section 2 which is a negative answer to a problem in [1]. Nevertheless, the perfectness of σ does provide information of τ. For instance, we will show in section 1 that the functor $\mathrm{Ext}^2_R(N,-)$ with $N\tau$-torsion kills all σ-closed left S-modules. Other properties of $\mathrm{Ext}_R(N,-)$ will be given in section 1.

In section 2, we will investigate the properties of the localization functor Q_σ and give some interesting results.

Throughout this paper, R is a ring with identity, $S = R/xR$. The maps φ and φ_* are defined as above. We always assume that $\tau \in R$-tors, $\sigma = \varphi_*(\tau)$. We say τ is compatible with φ if it happens that any S-module M is τ-torsionfree iff M is σ torsionfree. By [1], every $\tau \in R$-tors is compatible with the surjective homomorphism φ. As for the notations

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and terminologies of torsion theory we refer to [1]. Notations of spectral sequence are the same as [3].

§1.

Lemma 1.1. *Let M be an R-module which is τ-closed, then $(0:x)_M = \{m \in M \mid xm = 0\}$ is also a τ-closed module.*

Proof. Evidently $M/(0:x) \cong xM$. But xM is τ-torsionfree. Therefore $(0:x)$ is a closed submodule of M. Since M is τ-closed, $(0:x)$ is τ-closed.

In the following we will use the trivial fact: $\mathrm{Hom}_R(N, M) = \mathrm{Hom}_S(N, M)$ for S-modules N and M.

Lemma 1.2. *Let $\sigma = \varphi_*(\tau)$ and M' be a σ-torsionfree S-module. Then M' is τ-closed (regarding M' as an R-module canonically) if and only if M' is σ-closed as an S-module.*

Proof. It sufficies to show that M' is τ-injective iff M' is σ-injective. Let K' be any dense left ideal of S. Then $K' = K/xR$, where K is a dense left ideal of R which contains xR. Let f be any S-homomorphism from $K' \to M'$, μ be the canonical homomorphism $K \to K'$. We have a diagram as follows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{j} & R & \xrightarrow{g} & M' \\
 & & \mu\downarrow & & \downarrow\varphi & & \| \\
0 & \longrightarrow & K' & \xrightarrow{j'} & S & \xrightarrow{\bar{g}} & M' \\
 & & f\downarrow & & & & \\
 & & M' & & & &
\end{array}$$

where j and j' are injections. Since M' is τ-injective, there is a homomorphism $g: R \to M'$ such that $f\mu = gj$. Then g induces an R-homomorphism $\bar{g}: S \to M'$ which is also an S-homomorphism. Since μ is onto, we get $f = \bar{g}j'$, which shows that M' is σ-injective.

Conversely, assume that M' is σ-injective. We have to show that $\mathrm{Ext}_R^1(N, M') = 0$ for every τ-torsion R-module N (see [1]). There is a spectral sequence

$$\mathrm{Ext}_S^p(\mathrm{Tor}_q^R(S, N)) \Longrightarrow_p \mathrm{Ext}_R^n(N, M'), \qquad p + q = n. \tag{1.1}$$

Now $n = 1$.

$$E_2^{1,0} = \mathrm{Ext}_S^1(S \otimes_R N, M'),\ S \otimes_R N = R/xR \otimes_R N \cong N/xN.$$

Since N is τ-torsion, $S \otimes_R N$ is σ-torsion. Thus $E_2^{1,0} = 0$. On the other hand,

$$E_2^{0,1} = \mathrm{Hom}_S(\mathrm{Tor}_1^R(S, N), M').$$

From the short exact sequence

$$0 \longrightarrow xR \longrightarrow R \longrightarrow S \longrightarrow 0, \tag{1.2}$$

we get an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(S, N) \longrightarrow xR \otimes N \longrightarrow R \otimes N \longrightarrow S \otimes N \longrightarrow 0.$$

Since N is τ-torsion, $xR \otimes N$ is also τ-torsion. Then $\mathrm{Tor}_1^R(S, N)$ is τ-torsion, and hence is σ-torsion. Therefore $\mathrm{Hom}_S(\mathrm{Tor}_1^R(S, N), M') = 0$ for M' is σ-torsionfree. We have already shown that $E_2^{1,0}=0$, $E_2^{0,1}=0$. By Grothendieck cohomology 5-exact sequence, $H'=\mathrm{Ext}_R^1(N,M')=0$. This is the desired result.

Theorem 1.1.. *Let $\tau\in R$-tors, N be any τ-torsion R- module. If $\sigma=\varphi_*(\tau)$ is a perfect torsion theory in S-tors, then $\mathrm{Ext}_R^2(N,M')=0$ for every σ-closed module M'.*

Proof. We use (1.1) again. Let $n=2$.

$$E_2^{2,0}=\mathrm{Ext}_S^2(S\otimes N,M').$$

Evidently $S\otimes N$ is σ-torsion. Then $E_2^{2,0}=\mathrm{Ext}_S^2(S\otimes N,M')=0$ for σ is perfect. Now,

$$E_2^{1,1}=\mathrm{Ext}_S^1(\mathrm{Tor}_1^R(S,N),M').$$

By the proof of Lemma 1.2, $\mathrm{Tor}_1^R(S,N)$ is σ-torsion. Therefore $E_2^{1,1}=0$ for M' is σ-injective.

Since x is not a zero divisor, xR is a projective R-module. By the long exact sequence induced by (1.2), we get $\mathrm{Tor}_2^R(S,N)=0$. This implies $E_2^{0,2}=0$. An easy calculation shows that $\mathrm{Ext}_R^2(N,M')=0$.

Corollary 1.1. *Let K be a dense left ideal of R. Then for any σ-closed module M', $\mathrm{Ext}_R^1(K,M')=0$.*

Proof. We have a short exact sequence:

$$0\longrightarrow K\longrightarrow R\longrightarrow R/K\longrightarrow 0, \tag{1.3}$$

where R/K is τ-torsion. By Theorem 1.1, $\mathrm{Ext}_R^2(R/K,M')=0$. Then the long exact sequence induced by (1.3) gives the desired result.

Corollary 1.2. *Let M be any τ-closed R-module. Then xM is also a τ-closed module. Moreover, M/xM is σ-torsionfree as an S-module.*

Proof. Evidently $(0:x)_M$ can be regarded as an S-module. By Lemma 1.1, $(0:x)$ is τ-closed, so it is σ-closed as an S-module by Lemma 1.2. Clearly, the following sequence is exact:

$$0\longrightarrow(0:x)\longrightarrow M\longrightarrow xM\longrightarrow 0. \tag{1.4}$$

Then for any τ-torsion R-module N, we have the following exact sequence:

$$\cdots\longrightarrow\mathrm{Ext}_R^1(N,M)\longrightarrow\mathrm{Ext}_R^1(N,xM)\longrightarrow\mathrm{Ext}_R^2(N,(0:x))\longrightarrow\cdots.$$

But M is τ-injective. Then $\mathrm{Ext}_R^1(N,M)=0$. Therefore $\mathrm{Ext}_R^1(N,xM)=0$, which implies that xM is τ-injective. Now xM is a τ-injective submodule of a τ-closed module M. Then xM is τ-closed, i.e., M/xM is τ-torsionfree. Since φ is a surjective map, every $\tau\in R$-tors is compatible with φ (see [1]). This implies that M/xM is σ-torsionfree.

Lemma 1.3. *Let $\tau\in R$-tors, $\sigma=\varphi_*(\tau)$ which is perfect. Assume that N is a τ-torsion R-module, M' is any σ-closed S-module. Then $\mathrm{Ext}_R^3(N,M')=0$.*

Proof. We have a spectral sequence:

$$\mathrm{Ext}_S^p(\mathrm{Tor}_q^R(S,N),M')\Longrightarrow_p\mathrm{Ext}_R^n(N,M'),\qquad p+q=n=3.$$

First we want to show

$$E_2^{3,0}=\mathrm{Ext}_S^3(S\otimes_R N,M')=0.$$

Let $E_S(M')$ be the injective hull of M' as an S-module. There is an exact sequence

$$0\longrightarrow M'\longrightarrow E_S(M')\longrightarrow E_S(M')/M'\longrightarrow 0. \tag{1.5}$$

For any σ-torsion module M', we have an exact sequence

$$0 = \mathrm{Ext}_S^2(N', E(M')) \longrightarrow \mathrm{Ext}_S^2(N', E(M')/M') \longrightarrow \mathrm{Ext}_S^3(N', M') \longrightarrow 0.$$

Since σ is perfect, $E_S(M')/M'$ is σ-closed. Then $\mathrm{Ext}_S^2(N', E_S(M')/M') = 0$. This implies that $\mathrm{Ext}_S^3(N', M') = 0$. On the other hand, $S \otimes N$ is σ- torsion. Therefore

$$E_2^{3,0} = \mathrm{Ext}_S^3(S \otimes N, M') = 0.$$

Moreover, since σ is perfect and $\mathrm{Tor}_1^R(S, N)$ is σ-torsion, we have

$$E_2^{2,1} = \mathrm{Ext}_S^2(\mathrm{Tor}_1^R(S, N), M') = 0.$$

In the proof of Theorem 1.1, $\mathrm{Tor}_2^R(S, N) = 0$. Similarly one can easily see that $\mathrm{Tor}_3^R(S, N) = 0$. These facts imply

$$E_2^{1,2} = \mathrm{Ext}_S^1(\mathrm{Tor}_2^R(S, N), M') = 0$$

and

$$E_2^{0,3} = \mathrm{Hom}_S(\mathrm{Tor}_3^R(S, N), M') = 0.$$

Thus we have shown that

$$E_2^{3,0} = E_2^{2,1} = E_2^{1,2} = E_2^{0,3} = 0.$$

A routine verification shows that $\mathrm{Ext}_R^3(N, M') = 0$.

Theorem 1.2. *Let $\tau \in R$-tors, $\sigma = \varphi_*(\tau)$ which is perfect. Assume that N is any τ-torsion R-module, M is any τ-closed R-module. Then*

$$\mathrm{Ext}_R^2(N, M) \cong \mathrm{Ext}_R^2(N, xM).$$

Moreover, the isomorphism is induced by the multiplication of x.

Proof. For any τ-torsion R-module N, the exact sequence

$$0 \longrightarrow (0 : x)_M \longrightarrow M \longrightarrow xM \longrightarrow 0 \qquad (1.6)$$

induces a long exact sequence

$$\longrightarrow \mathrm{Ext}_R^2(N, (0 : x)) \longrightarrow \mathrm{Ext}_R^2(N, M) \longrightarrow \mathrm{Ext}_R^2(N, xM) \longrightarrow \mathrm{Ext}_R^3(N, (0 : x)).$$

Since $(0 : x)$ is τ-closed, we have

$$\mathrm{Ext}_R^2(N, (0 : x)) = 0$$

by Theorem 1.1. By Lemma 1.3, $\mathrm{Ext}_R^3(N, (0 : x)) = 0$, therefore

$$\mathrm{Ext}_R^2(N, M) \longrightarrow \mathrm{Ext}_R^2(N, xM)$$

is an isomorphism and it is not difficult to verify that the isomorphism is multiplication by x.

Corollary 1.3. *Let E be τ-torsionfree injective R-module. Then $\mathrm{Ext}_R^2(N, xE) = 0$ for every τ-torsion R-module N. Moreover, E/xE is a σ-closed S-module.*

Proof. $\mathrm{Ext}_R^2(N, xE) = 0$ is trivial. The short exact sequence

$$0 \longrightarrow xE \longrightarrow E \longrightarrow E/xE \longrightarrow 0 \qquad (1.7)$$

induces an exact sequence for any τ-torsion R-module N:

$$0 \longrightarrow \mathrm{Ext}_R^1(N, E/xE) \longrightarrow \mathrm{Ext}_R^2(N, xE) \longrightarrow 0.$$

Then

$$\mathrm{Ext}^1_R(N, E/xE) \cong \mathrm{Ext}^2_R(N, xE) = 0.$$

This means that E/xE is τ-injective. By Corollary 1.2, E/xE is τ-torsionfree. Therefore E/xE is τ-closed, which implies that E/xE is σ-closed (Lemma 1.2).

Corollory 1.4. *Let N and M be as in Theorem* 1.2. *Then for any natural number k,*

$$\mathrm{Ext}^2_R(N, M) \cong \mathrm{Ext}^2_R(N, x^k M).$$

So far, we study the properties of the functor $\mathrm{Ext}^2_R(N, -)$ with $N\tau$-torsion. Now we turn to the investigation of the functor $\mathrm{Ext}^2_R(N', -)$ with $N'\sigma$-torsion (i.e., N' is a σ torsion S'-module, but it is regarded as an R-module here). We have the following

Theorem 1.3. *Assume τ, σ as in Theorem* 1.1. *Let N' be a σ-torsion S-module. Then for any τ-closed R-module M,*

$$\mathrm{Ext}^2_R(N', M) \cong \mathrm{Ext}^1_S(N', M/xM).$$

Proof. There is a spectral sequence of change of rings:

$$\mathrm{Ext}^p_S(N', \mathrm{Ext}^q_R(S, M)) \Longrightarrow_p \mathrm{Ext}^n_R(N', M), \qquad p + q = n.$$

For $n = 2$,

$$E_2^{0,2} = \mathrm{Hom}_S(N', \mathrm{Ext}^2_R(S, M)).$$

Note that xR is projective and the following sequence is exact

$$0 \longrightarrow xR \longrightarrow R \longrightarrow S \longrightarrow 0. \tag{1.8}$$

We have $E_2^{0,2} = \mathrm{Ext}^2_R(S, M) = 0$. Furthermore,

$$E_2^{1,1} = \mathrm{Ext}^1_S(N', \mathrm{Ext}^1_R(S, M)).$$

The short exact sequence (1.8) gives the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(S, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_R(xR, M) \longrightarrow \mathrm{Ext}^1_R(S, M) \longrightarrow 0.$$

But we have natural isomorphisms

$$\mathrm{Hom}_R(S, M) \cong (0 : x)_M,$$
$$\mathrm{Hom}_R(R, M) \cong M,$$
$$\mathrm{Hom}_R(xR, M) \cong xM.$$

Therefore $\mathrm{Ext}^1_R(S, M) \cong M/xM$, which means

$$E_2^{1,1} = \mathrm{Ext}^1_S(N', M/xM).$$

On the other hand,

$$E_2^{2,0} = \mathrm{Ext}^2_S(N', \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}^2_S(N', (0 : x)_M) = 0.$$

Thus

$$\mathrm{Ext}^2_R(N', M) \cong E_r^{1,1}$$

for sufficient large r. We can calculate $E_r^{1,1}$ as follows

$$0 = E_2^{-1,2} \xrightarrow{d^2} E_2^{1,1} \xrightarrow{d^2} E_2^{3,0},$$

where

$$E_3^{1,1} = \mathrm{Ext}_S^3(N', \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}_S^3(N', (0 : x)) = 0$$

(see the proof of Lemma 1.3). Thus $E_3^{1,1} = \mathrm{K}er\, d^2/\mathrm{Im}\, d^2 \cong E_2^{1,1}$. Similarly, $E_r^{1,1} \cong E_2^{1,1}$. Thus we have

$$\mathrm{Ext}_R^2(N', M) \cong \mathrm{Ext}_S^1(N', M/xM).$$

Corollary 1.5. *Let N' be σ-torsion S-module. Then for any τ-closed R-module M,* $\mathrm{Ext}_R^2(N', M) = 0$ *if and only if M/xM is τ-closed* (*or equivalently, σ-closed*).

Proof. Lemma 1.3 shows that M/xM is τ-torsionfree. Then M/xM is σ-injective if and only if $\mathrm{Ext}_S^1(N', M/xM) = 0$ for any σ-torsion S-module N'.

§2.

Lemma 2.1. *Let $\tau \in R$-tors, $\sigma = \varphi_*(\tau)$ which is perfect. Then for any S-module M, $Q_\sigma(M) = Q_\tau(M)$.*

Proof. This is an easy consequence of Lemma 1.2.

When τ is perfect, it is known that $\sigma = \varphi_*(\tau)$ is also perfect. For completeness, we give a simple homological proof here.

Lemma 2.2. *Let σ, τ be as above. If τ is perfect then σ is perfect.*

Proof. It is sufficient to show that for any σ-torsion module N and any σ-closed module M, $\mathrm{Ext}_S^2(N, M) = 0$. By Grothendieck cohomology 5-exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2,$$

where $H^2 = \mathrm{Ext}_R^2(N, M) = 0$ since τ is perfect, we have

$$E_2^{2,0} = \mathrm{Ext}_S^2(S \otimes_R N, M) = \mathrm{Ext}_S^2(N, M),$$
$$E_2^{0,1} = \mathrm{Hom}_S(\mathrm{Tor}_1^R(S, N), M) = 0.$$

Hence $\mathrm{Ext}_S^2(N, M) = E_2^{2,0} = 0$.

The inverse of Lemma 2.2 is not true in general. Here we give a counterexample which is also a negative answer to a problem in [1] (p. 488).

Example. Let $R = Z[x]$, the polynomial ring with integer cofficients. The element x is neither a unit nor a zero divisor. $S = R/xR \cong Z$. Assume that τ is a torsion theory cogenerated by $E(R) \oplus E(R/xR)$, where E means injective hull. It is well known that $Q_\tau(R) = R$ and τ is the largest torsion theory for which R is τ-closed (see [4]). By [5], $(2, x)$ is a dense ideal of R. But $(2, x)R \neq R$, which means τ is not a perfect torsion theory. Now $S = Z$, so every torsion theory in S-tors is perfect. By Lemma 2.1, $Q_\sigma(M) = Q_\tau(M)$, the condition of Proposition 47.16 in [1] is satisfied. But τ is not perfect.

Next we will study when each of two functors $\mathrm{Hom}_R(S, -)$ and $S \otimes_R -$ commutes with localization functors.

Theorem 2.1. *Let $\tau \in R$-tors, $\sigma = \varphi_*(\tau)$ which is perfect. Then*

$$Q_\sigma \mathrm{Hom}_R(S, M) \cong \mathrm{Hom}_R(S, Q_\tau(M))$$

for any τ-torsionfree R-module M.

Proof. Let M be τ-torsionfree. Since $Q_\tau(M)$ is closed, $\mathrm{Hom}_R(S, Q_\tau(M)) \cong (0:x)_{Q_\tau(M)}$ is τ- closed by Lemma 1.1. On the other hand,

$$Q_\sigma(\mathrm{Hom}_R(S, M)) = Q_\sigma((0:x)_M).$$

Hence

$$Q_\sigma((0:x)_M) \subseteq (0:x)_{Q_\tau(M)}.$$

Furthermore, for any $a \in (0:x)_{Q_\tau(M)}$, $xa = 0$ and there is a dense left ideal I of R such that $aI \subseteq M$. Then $aI \subseteq (0:x)_M$, which implies that $(0:x)_{Q_\tau(M)}/(0:x)_M$ is τ-torsion. Therefore $(0:x)_{Q_\tau(M)} \subseteq Q_\sigma((0:x)_M)$. Thus

$$Q_\sigma \mathrm{Hom}_R(S, M) \cong \mathrm{Hom}_R(S, Q_\tau(M)).$$

The proof is completed.

Let $\sigma \in S$-tors. According to [6], we may define a torsion theory $\varphi^e(\sigma) \in R$-tors as follows. Let E_0 be the injective S-module which cogenerates σ, $E = E(E_0)$, the R-injective hull of E_0 (regarding E_0 as an R-module). Define $\varphi^e(\sigma)$ to be the torsion theory in R-tors which is cogenerated by E. Another torsion theory $\varphi^g(\sigma)$ can be defined as follows: the torsion class of $\varphi^g(\sigma)$ is generated by the torsion class of σ (regarding each σ-torsion S-module as R-module). We denote by Ψ the canonical ring homomorphism $R \to Q_\tau(R)$, $J(R)$ the Jacobson radical of R.

Lemma 2.3. *For any $\sigma \in S$-tors, if $\tau \in R$-tors and $\varphi_*(\tau) = \sigma$, then*

$$\varphi^g(\sigma) \le \tau \le \varphi^e(\sigma).$$

Proof. Let T_σ be the σ-torsion class of S-modules. By the definition of φ_*, if we regard every σ-torsion S-module as an R-module, then $T_\sigma \subseteq T_\tau$. This means $\varphi^g(\sigma) \le \tau$.

On the other hand, it is not difficult to see that $\varphi^e(\sigma)$ is cogenerated by F_σ, the σ-torsionfree class of S-modules. Nevertheless, φ is surjective, so every torsion theory in R-tors is compatible with φ. Let ${}_SM$ be any σ-torsionfree S-module, then M is τ-torsionfree as an R-module since $\varphi_*(\tau) = \sigma$. Hence we have $\tau \le \varphi^e(\sigma)$.

Recall that $\varphi_* \varphi^g(\sigma) = \varphi_* \varphi^e(\sigma) = \sigma$ (see [6]). Lemma 2.3 means that $\varphi^g(\sigma)$ is the smallest torsion theory τ in R-tors such that $\varphi_*(\tau) = \sigma$ while $\varphi^e(\sigma)$ is the largest torsion theory τ in R-tors such that $\varphi_*(\tau) = \sigma$.

Theorem 2.2. *Let τ be a torsion theory in R-tors which is perfect, $\sigma = \varphi_*(\tau)$. Then*

(1) $Q_\sigma(S) \cong Q_\tau(R)/xQ_\tau(R)$;

(2) $S \otimes_R Q_\tau(R) \cong Q_\sigma(S \otimes_R M) \cong Q_\sigma(S) \otimes_R M$;

(3) *If $\Psi(x) \in J(Q_\tau(R))$, then there is only one $\tau \in R$-tors such that $\sigma = \varphi_*(\tau)$. In other words, $\varphi^g(\sigma) = \varphi^e(\sigma) = \tau$.*

Proof. 1. Since τ is perfect, Q_τ is exact. Then the exact sequence

$$0 \longrightarrow xR \longrightarrow R \longrightarrow S \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow Q_\tau(xR) \longrightarrow Q_\tau(R) \longrightarrow Q_\tau(S) \longrightarrow 0.$$

By Lemma 2.1, $Q_\tau(S) = Q_\sigma(S)$, so that $Q_\sigma(S) = Q_\tau(S) \cong Q_\tau(R)/Q_\tau(xR)$. Now it is sufficient to show that $Q_\tau(xR) = xQ_\tau(R)$. But the multiplication of x induces an isomorphism

$\rho : R \to xR$. Hence $Q_\tau(\rho)$ is also an isomorphism from $Q_\tau(R) \to Q_\tau(xR)$. Moreover it is easy to see that $Q_\tau(\rho)$ is also a multiplication of x. Thus $Q_\tau(xR) = xQ_\tau(R)$.

2. By Lemma 2.1, $Q_\sigma(S \otimes M) = Q_\tau(S \otimes M)$, we need only to show

$$S \otimes Q_\tau(M) \cong Q_\tau(S \otimes M) \cong Q_\sigma(S) \otimes M. \tag{2.1}$$

Since τ is perfect,

$$Q_\tau(M) = Q_\tau(R) \otimes M, Q_\tau(S \otimes M) = Q_\tau(R) \otimes (S \otimes M). \tag{2.2}$$

Since x is a central element,

$$\begin{aligned} S \otimes Q_\tau(R) &= R/xR \otimes Q_\tau(R) \cong Q_\tau(R)/xQ_\tau(R) \\ &\cong Q_\tau(R) \otimes R/xR = Q_\tau(R) \otimes S. \end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned} S \otimes Q_\tau(M) &= S \otimes Q_\tau(R) \otimes M \\ &\cong Q_\tau(R)/xQ_\tau(R) \otimes M \cong Q_\sigma(S) \otimes M, \end{aligned} \tag{2.4}$$

$$\begin{aligned} Q_\tau(R) \otimes (S \otimes M) &\cong (Q_\tau(R) \otimes S) \otimes M) \\ &\cong Q_\tau(R)/xQ_\tau(R) \otimes M = Q_\sigma(S) \otimes M. \end{aligned} \tag{2.5}$$

Combining (2.2), (2.3), (2.4), (2.5), we get (2.1).

3. Let $\bar{x} = \Psi(x)$. Then $\bar{x} \in J(Q_\tau(R))$, the Jacobson radical of $Q_\tau(R)$. First we show that a left ideal I of R is τ-dense iff $I + xR/xR$ is σ-dense in S. Clearly, if I is a dense ideal of R, then $I + xR/xR$ is σ dense since $R/xR/I + xR/xR \cong R/I + xR$ which is τ-torsion or equivalently σ-torsion. Conversely, if $I + xR/xR$ is dense in S, then $I + xR$ is dense in R. Therefore $Q_\tau(I + xR) = Q_\tau(R)$. Since τ is perfect, $Q_\tau(I) + Q_\tau(xR) = Q_\tau(R)$. However, $Q_\tau(xR) = xQ_\tau(R)$, $\bar{x} \in J(Q_\tau(R))$. By Nakayama Lemma, $Q_\tau(I) = Q_\tau(R)$. Hence I is a τ dense left ideal of R.

Now we can prove that $\varphi^e(\sigma) = \tau$. Let E_0 be an injective S-module which cogenerates σ. $E = E(E_0)$, the R-injective hull of E_0. Let I be a $\varphi^e(\sigma)$-dense left ideal of R. Then $\mathrm{Hom}_R(R/I, E) = 0$. Since $\sigma = \varphi_* \varphi^e(\sigma)$ and $I + xR/xR$ is σ-torsion, $R/I + xR$ is τ-torsion. Hence $I + xR$ is a τ-dense left ideal of R. Thus $\varphi^e(\sigma) \le \tau$. According to Lemma 2.3, $\tau \le \varphi^e(\sigma)$. Therefore $\tau = \varphi^e(\sigma)$. Since $\varphi_* \varphi^g(\sigma) = \sigma$, the last statement follows from the fact $\varphi^g(\sigma) = \varphi^e(\sigma)$.

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