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ON PJATECKIǏ-ŠAPIRO PRIME NUMBER THEOREM**

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Abstract

It is proved that, for $1 < c < \frac{20}{17}$, there are infinitely many primes of the form $[n^c]$.

Keywords Pjateckiǐ-Šapiro theorem, Prime number, Sieve method.
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§1. Introduction

It is an interesting problem whether there are infinitely many primes of the form $[n^c]$. Let

$$\pi_c(x) = \sum_{\substack{n \le x \\ [n^c]=p}} 1. \tag{1.1}$$

Pjateckiǐ-Šapiro[1] proved that

$$\pi_c(x) \sim \frac{x}{c\log x}, \quad \text{as } x \to \infty \tag{1.2}$$

holds for any $c \in (1, \theta)$, where $\theta = \frac{12}{11} = 1.090909\cdots$.

Kolesnik[2] proved that when $\theta = \frac{10}{9} = 1.111111\cdots$, (1.2) holds. Leitmann[3] got $\theta = \frac{69}{62} = 1.112903\cdots$. Heath-Brown[4] improved it to $\theta = \frac{755}{662} = 1.140483\cdots$. Kolesnik[5] improved it again to $\theta = \frac{39}{34} = 1.147058\cdots$. Recently, Liu Hongquan[6] used the trigonometrical sum method of Fouvry and Iwaniec[7] and proved $\theta = \frac{15}{13} = 1.153846\cdots$.

In this paper, we lead the sieve method into this problem for the first time and prove

Theorem. *For* $1 < c < \frac{20}{17} = 1.176470\cdots$,

$$\pi_c(x) \ge \frac{\rho_0 x}{c\log x}$$

holds for sufficiently large x, *where* ρ_0 *is a definite positive constant. From this, there are infinitely many primes of the form* $[n^c]$ *within the given limit of* c.

In the following, we always assume that x and Z are sufficiently large, ε is a sufficiently small positive constant and $\delta = \varepsilon^2$. Assume that

$$\gamma = \frac{1}{c},\ ((a)) = a - [a] - \frac{1}{2},$$

$$N(d) = [-d^\gamma] - [-(d+1)^\gamma],\ E(d) = ((-(d+1)^\gamma)) - ((-d^\gamma)).$$

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§2. Leading of the Sieve Method

From §2 in [4], we know

$$\pi_c(x) = \sum_{p \le x^c} N(p) + O(1).$$

So it is enough that, for

$$0.85 + \varepsilon \le \gamma \le \frac{13}{15} + \varepsilon, \tag{2.1}$$

we prove that

$$\Phi_c(Z) = \sum_{Z < p \le 2Z} N(p) \ge 0.01 \frac{Z^\gamma}{\log Z} \tag{2.2}$$

holds for sufficiently large Z.

We consider $\Phi_c(Z)$ by means of the sieve method. Let

$$P(z) = \prod_{p<z} p, \quad A = \{n: \ Z < n \le 2Z, \ n \text{ repeates } N(n) \text{ times}\},$$

$$S(A, z) = \sum_{\substack{n \in A \\ (n, P(z))=1}} 1.$$

We have

$$\Phi_c(Z) = S(A, (2Z)^{\frac{1}{2}}). \tag{2.3}$$

Let $X = (2Z)^\gamma - Z^\gamma, \ \omega(d) = 1, \ r(d) = |A_d| - \frac{X}{d}$,

$$W(z) = \prod_{p<z} (1 - \frac{1}{p}) = \frac{e^{-\mu}}{\log z} (1 + O(\delta)),$$

where μ is Euler constant.

Lemma 2.1. i) *Suppose that* $M, \ N \ge 1, \ MN = D$. *When* $Z^\varepsilon \le z \le D^{\frac{1}{2}}$, *we have*

$$S(A, z) \ge \frac{X}{\log z} \Big\{ f\big(\frac{\log D}{\log z}\big) + O(\delta) \Big\} + \sum_{m<M} \sum_{n<N} a^-(n) b^-(m) r(mn),$$

where $|a^-(n)|, \ |b^-(m)| \le 1$. *When* $4 \le u \le 6$,

$$f(u) = \frac{2}{u} \Big\{ \log(u-1) + \int_3^{u-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(s-1)}{s} ds \Big\}. \tag{2.4}$$

ii) *Suppose that* $M(p), \ N(p) \ge 1, \ M(p)N(p) = D(p)$. *When* $Z^\varepsilon \le v(p) \le D^{\frac{1}{2}}(p)$, *we have*

$$S(A_p, v(p)) \le \frac{X}{p \log v(p)} \Big\{ F\big(\frac{\log D(p)}{\log v(p)}\big) + O(\delta) \Big\} + \sum_{m<M(p)} \sum_{n<N(p)} a^+(n) b^+(m) r(pmn),$$

where $|a^+(n)|,\ |b^+(m)| \le 1.$

$$F(u) = \begin{cases} \dfrac{2}{u}, & 2 \le u \le 3, \\ \dfrac{2}{u}\Big\{1 + \displaystyle\int_2^{u-1} \dfrac{\log(t-1)}{t} dt\Big\}, & 3 \le u \le 5. \end{cases} \tag{2.5}$$

Proof. By Theorem 1 in [8], we can reach the conclusions.

$$|A_d| = \sum_{Z<kd\le 2Z} N(kd) = \sum_{Z<kd\le 2Z} ((kd+1)^\gamma - (kd)^\gamma) + \sum_{Z<kd\le 2Z} E(kd).$$

$$\begin{aligned} \sum_{Z<kd\le 2Z} ((kd+1)^\gamma - (kd)^\gamma) &= \sum_{Z<kd\le 2Z} \gamma\cdot(kd)^{\gamma-1} + O(Z^{\gamma-1}) \\ &= \int_{\frac{Z}{d}}^{\frac{2Z}{d}} \gamma(dt)^{\gamma-1} dt + O(Z^{\gamma-1}) \\ &= \frac{X}{d} + O(Z^{\gamma-1}). \end{aligned}$$

Hence, $r(d) = O(Z^{\gamma-1}) + \sum\limits_{Z<kd\le 2Z} E(kd).$

$$\begin{aligned} &\sum_{m<M}\sum_{n<N} a^-(n)b^-(m)r(mn) \\ =&O(DZ^{\gamma-1}) + \sum_{m<M}\sum_{n<N} a^-(n)b^-(m) \sum_{Z<kmn\le 2Z} E(kmn). \end{aligned}$$

$$\begin{aligned} &\sum_{P<p\le 2P}\ \sum_{m<M(p)}\ \sum_{n<N(p)} a^+(n)b^+(m)r(pmn) \\ =&O\Big(\sum_{P<p\le 2P} D(p)Z^{\gamma-1}\Big) + \sum_{P<p\le 2P}\ \sum_{m<M(p)}\ \sum_{n<N(p)} a^+(n)b^+(m) \sum_{Z<kpmn\le 2Z} E(kpmn). \end{aligned}$$

Lemma 2.2. *Assume that* $H = Z^{1-\gamma+4\delta},\ \varepsilon(h) = O(1)$ *and*

$$\Psi(n) = \sum_{1\le h\le H} \varepsilon(h)e(hn^\gamma). \tag{2.6}$$

i) *Suppose that* $M,\ N \ge 1,\ MN = D,\ D < Z^{1-\varepsilon},\ Z^\varepsilon \le z \le D^{\frac{1}{2}}$. *When*

$$\sum_{m<M}\sum_{n<N} a(n)b(m) \sum_{Z<kmn\le 2Z} \Psi(kmn) \ll Z^{1-\delta}, \tag{2.7}$$

where $|a(n)|,\ |b(m)| \le 1$, *we have*

$$S(A,z) \ge \frac{X}{\log z} f\Big(\frac{\log D}{\log z}\Big) + O\Big(\frac{\varepsilon X}{\log Z}\Big). \tag{2.8}$$

ii) *Suppose that* $M(p),\ N(p) \ge 1,\ M(p)N(p) = D(p),\ D(p) < \frac{Z^{1-\varepsilon}}{p}$ *and* $Z^\varepsilon \le v(p) \le D^{\frac{1}{2}}(p)$. *When*

$$\sum_{P<p\le 2P}\ \sum_{m<M(p)}\ \sum_{n<N(p)} a(n)b(m) \sum_{Z<kpmn\le 2Z} \Psi(kpmn) \ll Z^{1-2\delta}, \tag{2.9}$$

where $|a(n)|,\ |b(m)| \le 1$, *we have*

$$\sum_{P<p\le 2P} S(A_p, v(p)) \le \sum_{P<p\le 2P} \frac{X}{p\log v(p)} F\Big(\frac{\log D(p)}{\log v(p)}\Big) + O\Big(\frac{\varepsilon X}{\log^2 Z}\Big). \tag{2.10}$$

Proof. i) By the above discussion, if

$$\sum_{m<M}\sum_{n<N} a(n)b(m) \sum_{Z<kmn\le 2Z} E(kmn) \ll Z^{\gamma-\delta}, \tag{2.11}$$

we can come to the conclusion.

By the discussion in §2 of [4], if

$$\sum_{m<M}\sum_{n<N} a(n)b(m) \sum_{Z<kmn\le 2Z} \Psi(kmn) \ll Z^{1-\delta}$$

is proved, we can obtain (2.11). So, the conclusion follows.

ii) It can be proved in the same way as in i).

By Buchstab identity,

$$\begin{aligned} &S(A,(2Z)^{\frac12})\\ =&S(A,Z^{0.15}) - \sum_{Z^{0.15}<p\le Z^{0.25}} S(A_p,p) - \sum_{Z^{0.25}<p\le Z^{0.35}} S(A_p,p)\\ &-\sum_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}} S(A_p,p) - \sum_{Z^{\frac{1.15}{3}}<p\le Z^{0.45}} S(A_p,p) - \sum_{Z^{0.45}<p\le (2Z)^{\frac12}} S(A_p,p)\\ =&\Sigma_1-\Sigma_2-\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6. \end{aligned} \tag{2.12}$$

§3. The Estimation for Trigonometrical Sums

Lemma 3.1. *Assume that* $a_1 \ne 1,\ a_1a_2a_3 \ne 0$. *Assume that* $M_1, M_2, M_3,\ x \ge 1$ *and that* $|\varphi_{m_1}| \le 1,\ |\psi_{m_2m_3}| \le 1$. *Then we have*

$$\begin{aligned} &\sum_{M_1<m_1\le 2M_1}\ \sum_{M_2<m_2\le 2M_2}\ \sum_{M_3<m_3\le 2M_3} \varphi_{m_1}\psi_{m_2m_3} e\Big(x\frac{m_1^{a_1}m_2^{a_2}m_3^{a_3}}{M_1^{a_1}M_2^{a_2}M_3^{a_3}}\Big)\\ \ll&\{x^{\frac34}M_1^{\frac12}(M_2M_3)^{\frac14} + M_1^{\frac{7}{10}}M_2M_3 + M_1(M_2M_3)^{\frac34} + x^{-\frac14}M_1^{\frac{11}{10}}M_2M_3\}(M_1M_2M_3)^{\delta}. \end{aligned}$$

See Theorem 3 in [7].

Lemma 3.2. *Assume that* $KL = Z,\ H = Z^{1-\gamma+4\delta}$. *Assume that* $|\varepsilon(h)|,\ |a(k)|,\ |b(l)| \le 1$. *Then when* $Z^{0.45} \ll K \ll Z^{0.55}$, *we have*

$$\sum_{1\le h\le H}\ \sum_{K\le k\le 2K}\ \sum_{L\le l\le 2L} \varepsilon(h)a(k)b(l)e(hk^{\gamma}l^{\gamma}) \ll Z^{1-4\delta}.$$

Proof. At first we suppose that $Z^{0.5} \ll K \ll Z^{0.55}$. In Lemmma 3.1, we take $m_1 = k,\ m_2 = h,\ m_3 = l$ and $x = JZ^{\gamma}$. Then when $J \le H$, we have

$$\sum_{J\le h\le 2J}\ \sum_{K\le k\le 2K}\ \sum_{L\le l\le 2L} \varepsilon(h)a(k)b(l)e(hk^{\gamma}l^{\gamma}) \ll Z^{1-5\delta}.$$

So, we can get the lemma.

When $Z^{0.45} \ll K \ll Z^{0.5}$, exchanging k and l, we can get the lemma.

Lemma 3.3. *Assume that* $KL = Z,\ H = Z^{1-\gamma+4\delta}$. *Assume that* $|\varepsilon(h)|,\ |a(k)|,\ |b(l)| \le 1$. *Then when* $Z^{0.15} \ll K \ll Z^{0.25}$, *we have*

$$\sum_{1\le h\le H}\ \sum_{K\le k\le 2K}\ \sum_{L\le l\le 2L} \varepsilon(h)a(k)b(l)e(hk^\gamma l^\gamma) \ll Z^{1-4\delta}.$$

It is Lemma 4 in [4] in essence.

Lemma 3.4. *Assume that* $KL = Z,\ H = Z^{1-\gamma+4\delta}$. *Assume that* $|\varepsilon(h)|,\ |a(k)|,\ |b(l)| \le 1$. *Then when* $1 \le K \ll Z^{0.55}$, *we have*

$$\sum_{1\le h\le H}\ \sum_{K\le k\le 2K}\ \sum_{L\le l\le 2L} \varepsilon(h)a(k)e(hk^\gamma l^\gamma) \ll Z^{1-4\delta}.$$

Proof. When $Z^{0.45} \ll K \ll Z^{0.55}$, it can be proved by Lemma 3.2.

When $1 \ll K \ll Z^{0.45}$, by Lemmas 11 and 12 in [6], we get the lemma.

§4. Upper and Lower Bounds of Some Sieve Functions

Lemma 4.1. $\Sigma_1 = S(A, Z^{0.15}) \ge 3.707032\frac{X}{\log Z}$.

Proof. Let $D = Z^{0.65},\ M = Z^{0.55},\ N = Z^{0.1}$. By Lemma 3.2, it is known that

$$\sum_{Z^{0.45}<m\le Z^{0.55}}\ \sum_{n<Z^{0.1}} a(n)b(m) \sum_{Z<kmn\le 2Z} \Psi(kmn) \ll Z^{1-\delta}.$$

Let $mn = r$. By Lemma 3.4, it is obtained that

$$\sum_{m\le Z^{0.45}}\ \sum_{n\le Z^{0.1}} a(n)b(m) \sum_{Z<kmn\le 2Z} \Psi(kmn) = \sum_{r\le Z^{0.55}} g(r) \sum_{Z<kr\le 2Z} \Psi(kr) \ll Z^{1-\delta}.$$

By i) of Lemma 2.2, we obtain

$$\begin{aligned}
&S(A, Z^{0.15})\\
\ge&\frac{X}{0.15\log Z} f\Big(\frac{0.65}{0.15}\Big) + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\
=&\frac{2X}{0.65\log Z}\Big\{\log\Big(\frac{0.65}{0.15}-1\Big) + \int_3^{\frac{0.65}{0.15}-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(s-1)}{s}ds\Big\} + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\
=&\frac{2X}{0.65\log Z}\Big\{\log\Big(\frac{0.65}{0.15}-1\Big) + \int_2^{\frac{0.65}{0.15}-2}\frac{\log(s-1)}{s}\log\Big(\frac{\frac{0.65}{0.15}-1}{(s+1)}\Big)ds\Big\} + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\
\ge&3.707032\frac{X}{\log Z}.
\end{aligned}$$

Lemma 4.2. $\displaystyle\sum_{Z^{0.25}<p\le Z^{0.35}} S\Big(A_p, \Big(\frac{Z^{0.65}}{p}\Big)^{\frac{1}{3}}\Big) \le 1.920475\frac{X}{\log Z}$.

Proof. Let $D(p) = \frac{Z^{0.65}}{p},\ M(p) = \frac{Z^{0.55}}{p},\ N(p) = Z^{0.1}$. Using the discussion in Lemma 4.1, we get

$$\sum_{Z^{0.25}<p\le Z^{0.35}}\ \sum_{m\le \frac{Z^{0.55}}{p}}\ \sum_{n<Z^{0.1}} a(n)b(m) \sum_{Z<kpmn\le 2Z} \Psi(kpmn) \ll Z^{1-\delta}.$$

By ii) of Lemma 2.2, we get

$$\begin{aligned}&\sum_{Z^{0.25}<p\le Z^{0.35}} S\Big(A_p,\Big(\frac{Z^{0.65}}{p}\Big)^{\frac13}\Big)\\ \le&\sum_{Z^{0.25}<p\le Z^{0.35}}\frac{X}{p\log\frac{Z^{0.65}}{p}}\cdot 3F(3)+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ =&\frac{X}{\log Z}\int_{0.25}^{0.35}\frac{2dt}{t(0.65-t)}+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ \le&1.920475\frac{X}{\log Z}.\end{aligned}$$

Lemma 4.3. $\sum\limits_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}} S\big(A_p,\frac{Z^{0.45}}{p}\big)\le 0.624946\frac{X}{\log Z}$.

Proof. At first we prove that, for $Z^{0.35}<P\le Z^{0.45}$,

$$\sum_{P<p\le 2P}\sum_{m\le\frac{Z^{0.55}}{p}}\sum_{n\le\frac{Z^{0.55}}{p}} a(n)b(m)\sum_{Z<kpmn\le 2Z}\Psi(kpmn)\ll Z^{1-2\delta}. \tag{4.1}$$

When $Z^{0.45}<pm\le Z^{0.55}$, (4.1) follows from Lemma 3.2. So, we assume $m\le\frac{Z^{0.45}}{p}$. In the same way, we can assume $n\le\frac{Z^{0.45}}{p}$. Hence, $pmn\le\frac{Z^{0.9}}{p}\le Z^{0.55}$. Using Lemma 3.4, we can get (4.1).

Let $D(p)=\frac{Z^{1.1}}{p^2},\ M(p)=\frac{Z^{0.55}}{p},\ N(p)=\frac{Z^{0.55}}{p}$. By ii) of Lemma 2.2, we obtain

$$\begin{aligned}&\sum_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}} S\Big(A_p,\frac{Z^{0.45}}{p}\Big)\\ \le&\sum_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}}\frac{X}{p\log\frac{Z^{0.45}}{p}}F\Big(\frac{\log\frac{Z^{1.1}}{p^2}}{\log\frac{Z^{0.45}}{p}}\Big)+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ =&\frac{X}{\log Z}\int_{0.35}^{\frac{1.15}{3}}\frac{2dt}{t(1.1-2t)}\Big(1+\int_2^{\frac{1.1-2t}{0.45-t}-1}\frac{\log(u-1)}{u}du\Big)+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ =&\frac{X}{\log Z}\int_4^5\frac{2dr}{r(0.45r-1.1)}\Big(1+\int_2^{r-1}\frac{\log(u-1)}{u}du\Big)+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ =&\frac{X}{\log Z}\int_4^5\frac{2dr}{r(0.45-1.1)}+\frac{X}{\log Z}\int_2^3\frac{\log(u-1)}{u}du\int_4^5\frac{2dr}{r(0.45r-1.1)}\\ &+\frac{X}{\log Z}\int_3^4\frac{\log(u-1)}{u}du\int_{u+1}^5\frac{2dr}{r(0.45r-1.1)}+O\Big(\frac{\varepsilon X}{\log Z}\Big)\end{aligned}$$

$$\leq 0.496898\frac{X}{\log Z}+0.073155\frac{X}{\log Z}$$
$$+\frac{X}{\log Z}\int_3^4\frac{\log(u-1)}{u}\Big\{\frac{2}{1.1}\log\Big(\frac{1.15}{(0.45u-0.65)}\Big)-\frac{2}{1.1}\log\Big(\frac{5}{(u+1)}\Big)\Big\}du$$
$$\leq 0.570053\frac{X}{\log Z}+0.104031\frac{X}{\log Z}-0.049138\frac{X}{\log Z}$$
$$=0.624946\frac{X}{\log Z}.$$

Lemma 4.4. $\sum_{Z^{\frac{1.15}{3}}<p\leq Z^{0.45}} S\Big(A_p,\big(\frac{Z^{1.1}}{p^2}\big)^{\frac{1}{5}}\Big)\leq 1.715866\frac{X}{\log Z}.$

Proof. By the discussion in Lemma 4.3, we get

$$\sum_{Z^{\frac{1.15}{3}}<p\leq Z^{0.45}} S\Big(A_p,\Big(\frac{Z^{1.1}}{p^2}\Big)^{\frac{1}{5}}\Big)$$
$$\leq \sum_{Z^{\frac{1.15}{3}}<p\leq Z^{0.45}}\frac{X}{p\log\frac{Z^{1.1}}{p^2}}\cdot 5F(5)+O\Big(\frac{\varepsilon X}{\log Z}\Big)$$
$$\leq 1.406093\frac{X}{\log Z}\int_{\frac{1.15}{3}}^{0.45}\frac{2dt}{t(1.1-2t)}$$
$$\leq 1.715866\frac{X}{\log Z}.$$

§5. Asymptotic Formulas

Lemma 5.1. *When x and z are sufficiently large, $z\leq x^{1-\varepsilon}$, we have*

$$\sum_{\substack{n\leq x\\(n,P(z))=1}}1=\Big(w\Big(\frac{\log x}{\log z}\Big)+O(\delta)\Big)\frac{x}{\log z},$$

where

$$\begin{cases} w(u)=\dfrac{1}{u}, & 1\leq u\leq 2,\\ (uw(u))'=w(u-1), & u>2.\end{cases}$$

See pages 239 and 240 in [9].

Lemma 5.2. i) *When $u\geq 2$,*

$$0.5\leq w(u)\leq\frac{1}{1.763}.$$

ii) *When $u\geq 2.4$,*

$$w(u)\geq 0.55641.$$

See Lemma 3 in page 239 of [9] and Lemma 12 of [10].

Lemma 5.3. *When* $1 \le S \le Z^{1-2\varepsilon}$, $Z^{\varepsilon} \le z \le \frac{Z^{1-\varepsilon}}{S}$, $|a(s)| \le 1$, *we have*

$$\sum_{S<s\le 2S} a(s) \sum_{\substack{\frac{Z}{s}<n\le\frac{2Z}{s}\\(n,P(z))=1}} ((ns+1)^{\gamma}-(ns)^{\gamma})$$
$$= \sum_{S<s\le 2S} \frac{a(s)}{s} w\Big(\frac{\log\frac{Z}{s}}{\log z}\Big)\frac{X}{\log z} + O\Big(\frac{\varepsilon X}{\log Z}\sum_{S<s\le 2S}\frac{|a(s)|}{s}\Big).$$

Proof.

$$\Gamma = \sum_{\substack{\frac{Z}{s}<n\le\frac{2Z}{s}\\(n,P(z))=1}} ((ns+1)^{\gamma}-(ns)^{\gamma})$$
$$= \sum_{\substack{\frac{Z}{s}<n\le\frac{2Z}{s}\\(n,P(z))=1}} \gamma(ns)^{\gamma-1} + O\Big(\frac{Z^{\gamma-1}}{S}\Big).$$

$$\sum_{\substack{\frac{Z}{s}<n\le\frac{2Z}{s}\\(n,P(z))=1}} \gamma(ns)^{\gamma-1} = \int_{\frac{Z}{s}}^{\frac{2Z}{s}} \gamma(st)^{\gamma-1} d\Big(\sum_{\substack{n\le t\\(n,P(z)=1}} 1\Big)$$
$$= \int_{\frac{Z}{s}}^{\frac{2Z}{s}} \gamma(st)^{\gamma-1}\Big\{w\Big(\frac{\log t}{\log z}\Big)\frac{1}{\log z} + w'\Big(\frac{\log t}{\log z}\Big)\frac{1}{\log^2 z}\Big\}dt$$
$$+ O\Big(\frac{\delta Z^{\gamma}}{S\log z}\Big).$$

Noting $w\big(\frac{\log t}{\log z}\big) - w\big(\frac{\log\frac{Z}{s}}{\log z}\big) = O\big(\frac{1}{\log z}\big)$,

we have

$$\Gamma = w\Big(\frac{\log\frac{Z}{s}}{\log z}\Big)\frac{X}{s\log z} + O\Big(\frac{\varepsilon X}{S\log z}\Big).$$

$$\sum_{S<s\le 2S} a(s) \sum_{\substack{\frac{Z}{s}<n\le\frac{2Z}{s}\\(n,P(z))=1}} ((ns+1)^{\gamma}-(ns)^{\gamma})$$
$$= \sum_{S<s\le 2S} \frac{a(s)}{s} w\Big(\frac{\log\frac{Z}{s}}{\log z}\Big)\frac{X}{\log z} + O\Big(\frac{\varepsilon X}{\log Z}\sum_{S<s\le 2S}\frac{|a(s)|}{s}\Big).$$

Lemma 5.4. $\Sigma_2 = \sum_{Z^{0.15}<p\le Z^{0.25}} S\big(A_p,p\big) \le 1.512574\frac{X}{\log Z}$.

Proof. Assume that $Z^{0.15} < W \le Z^{0.25}$,

$$\begin{aligned}\Sigma &= \sum_{W<p\le 2W} S(A_p,p) = \sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(p))=1}} N(pq)\\ &\le \sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(W))=1}} N(pq)\\ &= \sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(W))=1}} ((pq+1)^\gamma-(pq)^\gamma) + \sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(W))=1}} E(pq).\end{aligned}$$

By the discussion in §2 of [4], if we prove that

$$\sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(W))=1}} \Psi(pq) \ll Z^{1-2\delta}, \tag{5.1}$$

where $\Psi(n)$ is defined in (2.6), we can get

$$\sum_{W<p\le 2W} \sum_{\substack{Z<pq\le 2Z\\ (q,P(W))=1}} E(pq) \ll Z^{\gamma-2\delta}.$$

By Lemma 3.3, we get (5.1). Using Lemmas 5.2, 5.3 and the prime number theorem, we obtain

$$\Sigma \le \sum_{W<p\le 2W} \frac{X}{p\log p} w\Big(\frac{\log\frac{Z}{p}}{\log W}\Big) + O\Big(\frac{\varepsilon X}{\log^2 Z}\Big),$$

$$\begin{aligned}\Sigma_2 \le& \sum_{Z^{0.15}<p\le Z^{0.25}} \frac{X}{p\log p} w\Big(\frac{\log\frac{Z}{p}}{\log p}\Big) + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ \le& \frac{1}{1.763} \sum_{Z^{0.15}<p\le Z^{0.25}} \frac{X}{p\log p} + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ =& \frac{1}{1.763}\frac{X}{\log Z}\int_{0.15}^{0.25}\frac{dt}{t^2} + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ \le& 1.512574\frac{X}{\log Z}\end{aligned}$$

Lemma 5.5.

$$\begin{aligned}\Omega =& \sum_{Z^{0.25}<p\le Z^{0.35}} \sum_{(\frac{Z^{0.65}}{p})^{\frac13}<q<\min(p,(\frac{2Z}{p})^{\frac12})} S(A_{pq},q)\\ \ge& 0.625585\frac{X}{\log Z}.\end{aligned}$$

Proof.

$$\Omega \geq \sum_{Z^{0.25}<p\leq Z^{0.275}} \sum_{Z^{0.15}<q<p} S(A_{pq},q) + \sum_{Z^{0.275}<p\leq Z^{0.3}} \sum_{Z^{0.15}<q<\frac{Z^{0.55}}{p}} S(A_{pq},q)$$
$$+ \sum_{Z^{0.3}<p\leq Z^{0.35}} \sum_{\frac{Z^{0.45}}{p}<q<Z^{0.25}} S(A_{pq},q)$$
$$=\Omega_1+\Omega_2+\Omega_3.$$

By Lemma 3.3, it is known that

$$\sum_{Z^{0.25}<p\leq Z^{0.275}} \sum_{Z^{0.15}<q\leq Z^{0.25}} \sum_{Z<rpq\leq 2Z} c(r)\Psi(rpq) \ll Z^{1-2\delta}.$$

Let $pq=k,\ r=l$. By Lemma 3.2, it is known that

$$\sum_{Z^{0.25}<p\leq Z^{0.275}} \sum_{Z^{0.25}<q<p} \sum_{Z<rpq\leq 2Z} c(r)\Psi(rpq) \ll Z^{1-2\delta}.$$

In the same way as in Lemma 5.4, we get asymptotic formula

$$\begin{aligned}
\Omega_1 = &\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{Z^{0.15}<q<p} \frac{X}{pq\log q} w\Big(\frac{\log\frac{Z}{pq}}{\log q}\Big) + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\
=&\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{Z^{0.15}<q<(\frac{Z}{p})^{\frac{1}{3.4}}} \frac{X}{pq\log q} w\Big(\frac{\log\frac{Z}{pq}}{\log q}\Big)\\
&+\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{(\frac{Z}{p})^{\frac{1}{3.4}}<q\le(\frac{Z}{p})^{\frac{1}{3}}} \frac{X}{pq\log q} w\Big(\frac{\log\frac{Z}{pq}}{\log q}\Big)\\
&+\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{(\frac{Z}{p})^{\frac{1}{3}}<q<p} \frac{X}{pq\log q} w\Big(\frac{\log\frac{Z}{pq}}{\log q}\Big) + O\Big(\frac{\varepsilon X}{\log Z}\Big)\\
\ge&0.556401\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{Z^{0.15}<q<(\frac{Z}{p})^{\frac{1}{3.4}}} \frac{X}{pq\log q}\\
&+0.5\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{(\frac{Z}{p})^{\frac{1}{3.4}}<q<(\frac{Z}{p})^{\frac{1}{3}}} \frac{X}{pq\log q}\\
&+\sum_{Z^{0.25}<p\le Z^{0.275}} \sum_{(\frac{Z}{p})^{\frac{1}{3}}<q<p} \frac{X}{pq\log\frac{Z}{pq}}\\
\ge&0.5564\frac{X}{\log Z}\int_{0.25}^{0.275}\frac{dt}{t}\int_{0.15}^{\frac{1-t}{3.4}}\frac{dw}{w^2} + 0.5\frac{X}{\log Z}\int_{0.25}^{0.275}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\
&+\frac{X}{\log Z}\int_{0.25}^{0.275}\frac{dt}{t}\int_{\frac{1-t}{3}}^{t}\frac{dw}{w(1-t-w)}\\
\ge&0.109099\frac{X}{\log Z} + 0.025842\frac{X}{\log Z} + 0.012810\frac{X}{\log Z}\\
=&0.147751\frac{X}{\log Z}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\Omega_2 \ge&0.5564\frac{X}{\log Z}\int_{0.275}^{0.3}\frac{dt}{t}\int_{0.15}^{\frac{1-t}{3.4}}\frac{dw}{w^2} + 0.5\frac{X}{\log Z}\int_{0.275}^{0.3}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\
&+\frac{X}{\log Z}\int_{0.275}^{0.3}\frac{dt}{t}\int_{\frac{1-t}{3}}^{0.55-t}\frac{dw}{w(1-t-w)}\\
\ge&0.091765\frac{X}{\log Z} + 0.024420\frac{X}{\log Z} + 0.018826\frac{X}{\log Z}\\
=&0.135011\frac{X}{\log Z},
\end{aligned}$$

$$\begin{aligned}\Omega_3 \geq& 0.5564\frac{X}{\log Z}\int_{0.3}^{0.35}\frac{dt}{t}\int_{0.45-t}^{\frac{1-t}{3.4}}\frac{dw}{w^2}\\ &+0.5\frac{X}{\log Z}\int_{0.3}^{0.35}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\ &+\frac{X}{\log Z}\int_{0.3}^{0.35}\frac{dt}{t}\int_{\frac{1-t}{3}}^{0.25}\frac{dw}{w(1-t-w)}\\ \geq& 0.260123\frac{X}{\log Z}+0.045651\frac{X}{\log Z}+0.037049\frac{X}{\log Z}\\ =& 0.342823\frac{X}{\log Z}.\end{aligned}$$

Combining all of these, we get

$$\Omega \geq 0.625585\frac{X}{\log Z}.$$

Lemma 5.6.

$$\begin{aligned}\Omega =& \sum_{Z^{0.35}<p\leq Z^{\frac{1.15}{3}}}\ \sum_{\frac{Z^{0.45}}{p}<q<(\frac{2Z}{p})^{\frac{1}{2}}} S(A_{pq},q)\\ \geq& 0.409017\frac{X}{\log Z}.\end{aligned}$$

Proof. In the same way as in Lemma 5.5, we have

$$\begin{aligned}\Omega \geq& \sum_{Z^{0.35}<p\leq Z^{\frac{1.15}{3}}}\ \sum_{\frac{Z^{0.45}}{p}<q<Z^{0.25}} S(A_{pq},q)\\ \geq& 0.5564\frac{X}{\log Z}\int_{0.35}^{\frac{1.15}{3}}\frac{dt}{t}\int_{0.45-t}^{\frac{1-t}{3.4}}\frac{dw}{w^2}+0.5\frac{X}{\log Z}\int_{0.35}^{\frac{1.15}{3}}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\ &+\frac{X}{\log Z}\int_{0.35}^{\frac{1.15}{3}}\frac{dt}{t}\int_{\frac{1-t}{3}}^{0.25}\frac{dw}{w(1-t-w)}\\ \geq& 0.342130\frac{X}{\log Z}+0.028723\frac{X}{\log Z}+0.038164\frac{X}{\log Z}\\ =& 0.409017\frac{X}{\log Z}.\end{aligned}$$

Lemma 5.7.

$$\begin{aligned}\Omega =& \sum_{Z^{\frac{1.15}{3}}<p\leq Z^{0.45}}\ \sum_{(\frac{Z^{1.1}}{p^2})^{\frac{1}{5}}<q<(\frac{2Z}{p})^{\frac{1}{2}}} S(A_{pq},q)\\ \geq& 1.251837\frac{X}{\log Z}.\end{aligned}$$

Proof. In the same way as in Lemma 5.5, we have

$$\begin{aligned}\Omega \geq & \sum_{Z^{\frac{1.15}{3}}<p\leq Z^{0.4}} \sum_{(\frac{Z^{1.1}}{p^2})^{\frac{1}{5}}<q<Z^{0.25}} S(A_{pq},q)\\ &+\sum_{Z^{0.4}<p\leq Z^{0.45}} \sum_{(\frac{Z^{1.1}}{p^2})^{\frac{1}{5}}<q<\frac{Z^{0.55}}{p}} S(A_{pq},q)\\ &+\sum_{Z^{0.4}<p\leq Z^{0.45}} \sum_{Z^{0.15}<q<Z^{0.25}} S(A_{pq},q)\\ \geq & 0.5564\frac{X}{\log Z}\int_{\frac{1.15}{3}}^{0.4}\frac{dt}{t}\int_{\frac{1.1-2t}{5}}^{\frac{1-t}{3.4}}\frac{dw}{w^2}+0.5\frac{X}{\log Z}\int_{\frac{1.15}{3}}^{0.4}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\ &+\frac{X}{\log Z}\int_{\frac{1.15}{3}}^{0.4}\frac{dt}{t}\int_{\frac{1-t}{3}}^{0.25}\frac{dw}{w(1-t-w)}+0.5564\frac{X}{\log Z}\int_{0.4}^{0.45}\frac{dt}{t}\int_{\frac{1.1-2t}{5}}^{0.55-t}\frac{dw}{w^2}\\ &+0.5564\frac{X}{\log Z}\int_{0.4}^{0.45}\frac{dt}{t}\int_{0.15}^{\frac{1-t}{3.4}}\frac{dw}{w^2}+0.5\frac{X}{\log Z}\int_{0.4}^{0.45}\frac{dt}{t}\int_{\frac{1-t}{3.4}}^{\frac{1-t}{3}}\frac{dw}{w^2}\\ &+\frac{X}{\log Z}\int_{0.4}^{0.45}\frac{dt}{t}\int_{\frac{1-t}{3}}^{0.25}\frac{dw}{w(1-t-w)}\\ \geq & 0.241758\frac{X}{\log Z}+0.013991\frac{X}{\log Z}+0.023308\frac{X}{\log Z}+0.794005\frac{X}{\log Z}\\ &+0.049474\frac{X}{\log Z}+0.040958\frac{X}{\log Z}+0.088343\frac{X}{\log Z}\\ = & 1.251837\frac{X}{\log Z}.\end{aligned}$$

Lemma 5.8. $\Sigma_6 = \sum\limits_{Z^{0.45}<p\leq (2Z)^{\frac{1}{2}}} S(A_p,p) \leq 0.200671\frac{X}{\log Z}$.

Proof. By Lemma 3.2 and the discussion in Lemma 5.4, we get

$$\begin{aligned}\Sigma_6 = & \sum_{Z^{0.45}<p\leq (2Z)^{\frac{1}{2}}} \frac{X}{p\log\frac{Z}{p}}+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ = & \frac{X}{\log Z}\int_{0.45}^{0.5}\frac{dt}{t(1-t)}+O\Big(\frac{\varepsilon X}{\log Z}\Big)\\ \leq & 0.200671\frac{X}{\log Z}\end{aligned}$$

§6. The Proof of Theorem

By Lemma 4.1,

$$\Sigma_1 \geq 3.707032\frac{X}{\log Z}.$$

By Lemma 5.4,

$$\Sigma_2 \leq 1.512574\frac{X}{\log Z}.$$

By Lemmas 4.2 and 5.5,

$$\begin{aligned}\Sigma_3 =& \sum_{Z^{0.25}<p\le Z^{0.35}} S\Big(A_p, \Big(\frac{Z^{0.65}}{p}\Big)^{\frac13}\Big)\\ &- \sum_{Z^{0.25}<p\le Z^{0.35}} \sum_{(\frac{Z^{0.65}}{p})^{\frac13}<q<\min(p,(\frac{2Z}{p})^{\frac12})} S(A_{pq}, q)\\ \le& 1.920475\frac{X}{\log Z} - 0.625585\frac{X}{\log Z}\\ =& 1.294890\frac{X}{\log Z}.\end{aligned}$$

By Lemmas 4.3 and 5.6,

$$\begin{aligned}\Sigma_4 =& \sum_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}} S\Big(A_p, \frac{Z^{0.45}}{p}\Big)\\ &- \sum_{Z^{0.35}<p\le Z^{\frac{1.15}{3}}} \sum_{\frac{Z^{0.45}}{p}<q<(\frac{2Z}{p})^{\frac12}} S(A_{pq}, q)\\ \le& 0.624946\frac{X}{\log Z} - 0.409017\frac{X}{\log Z}\\ =& 0.215929\frac{X}{\log Z}.\end{aligned}$$

By Lemmas 4.4 and 5.7,

$$\begin{aligned}\Sigma_5 =& \sum_{Z^{\frac{1.15}{3}}<p\le Z^{0.45}} S\Big(A_p, \Big(\frac{Z^{1.1}}{p^2}\Big)^{\frac15}\Big)\\ &- \sum_{Z^{\frac{1.15}{3}}<p\le Z^{0.45}} \sum_{(\frac{Z^{1.1}}{p^2})^{\frac15}<q<(\frac{2Z}{p})^{\frac12}} S(A_{pq}, q)\\ \le& 1.715866\frac{X}{\log Z} - 1.251837\frac{X}{\log Z}\\ =& 0.464029\frac{X}{\log Z}.\end{aligned}$$

By Lemma 5.8,

$$\Sigma_6 \le 0.200671\frac{X}{\log Z}.$$

Combining all of these, we have

$$\begin{aligned}\Phi_c(Z) = S(A, (2Z)^{\frac12}) \ge& 0.018939\frac{X}{\log Z}\\ \ge& 0.01\frac{Z^{\gamma}}{\log Z}.\end{aligned}$$

So, (2.2) holds and Theorem follows.

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