Chin. Ann. of Math.
15B: 1(1994),23-34.

ON SOLUTIONS OF TWO-SCALE DIFFERENCE EQUATIONS**

WANG JIANZHONG*

Abstract

This paper is concerned with solutions of the following two-scale difference equations: $f(x) = \sum_{n=0}^{N} a_n f(2x-n)$. The conditions for the solvability and the iteractive solvability of this kind of equations in certain spaces are obtained respectively.

Keywords Two-scale difference equations, Iterative solutions, Wavelet analysis.
1991 MR Subject Classification 41A15, 41A05, 41A30.

§1. Introduction

This paper is concerned with solution of the two-scale difference equations of the following form:

$$u(x) = \sum_{n=0}^{N} a_n u(2x-n), \quad x \in \boldsymbol{R}, \tag{1.1}$$

where $a_0 a_N \neq 0$ and the a_n are complex numbers.

Note that a general equations of the type

$$u(x) = \sum_{n=-N_1}^{N_2} a_n u(2x-n)$$

can be reduced to the form (1.1) by the linear change of variable $x \mapsto x - N_1$.

Functions that satisfy two-scale difference equations are used for many purposes. de Rham[1] employed them to construct an example of a continuous, nowhere-differentiable function. Dubuc proposed in [2] a dyadic interpolation scheme where the "fundamental function" satisfies this kind of equations. Then this interpolation scheme was applied in [3] by Deslaurieers and Dubuc to the construction of fractal objects and functions with fractal properties. Recently a new motivation to study them arises from wavelet analysis, where one is involved in the construction of compactly supported wavelet basis, say $\{h_{mn}(x)\}_{m,n\in Z}$, generated by translating and dilating a single function $h(x)$ via $h_{mn}(x) = h(2^{-m}x - n)$. And the construction of such an $h(x)$ requires anxilary function which satisfies a two-scale difference equation (cf. [4-7]).

Manuscript received November 4, 1991.
*Department of Mathematics, Wuhan University, Wuhan 430072, Hubei, China.
**Project supported by the National Natural Science Foundation of China.

Of course, in all these situations, the first thing we are interested in is the solvability of the two-scale difference equations. Since the equations of form (1.1) have solutions with compact support, they are more important in most cases.

There are a few results about that in literatures. We state them here.

Let $I\!N$ denote the set of positive integers, $Z^+ = I\!N \cup \{0\}$. The collection of all polynomials is denoted by π. Then $\pi_k \subset \pi$ is the set of all polynomials of degree $\leq k$. If $c(z) = \sum_{j \in z^j} c_j z^j \in \pi$, its coefficient sequence is denoted by $c = (c_j)_{j \in Z}$. It is obvious that $c_j = 0$ for $j < 0$ and $j > n$ when $c(z) \in \pi_n$.

We write, for $p \in \pi$, $p_k^*(z) = \prod_{j=1}^{k} p(e^{-i2^{-j}z})$ and $p^k(z) = \prod_{j=0}^{k-1} p(z^{2^j})$. Then

$$p^k(e^{-i2^{-k}z}) = p_k^*(z). \tag{1.2}$$

Let $B_k(p) = \sup_{|z|=1} |p^k(z)|$.

In (1.1), we always write $a(z) = \frac{1}{2} \sum_{n=0}^{N} a_n z^n$, which is called the symbol of equation (1.1).

Y. Meyer[8] proved

Theorem A. *If* $\inf_{|\theta| \leq \frac{\pi}{2}} |a(e^{i\theta})| > 0$ *and* $\sup_{k \in I\!N} \int_0^{2^k} |a_k^*(2\pi\theta)| d\theta < \infty$, *then the equation* (1.1) *is solvable in* C_0.

Cavarettla, Dahmen and Micchelli[9] obtained

Theorem B. *In* (1.1) *suppose that for some* $m \in I\!N$,

$$a(z) = \Big(\frac{1+z^m}{2}\Big) b(z),$$

where $b(z) \in \pi$ *satisfies* (1) $b(1) = 1$, *and* (2)

$$B_k(b) \leq 1 \quad \text{for some } k \in I\!N, \tag{1.3}$$

then (1.1) *is solvable in* L^2.

[9] also pointed out that if (1.1) has a nontrivial solution such that $\hat{u} \in L^p$ $(1 \leq p \leq \infty)$, then the symbol $a(z)$ satisfies

$$G(a) \leq 2^{-\frac{1}{p}},$$

where $G(f)$ denotes the geometric mean of a complex valued function f on the unit circle:

$$G(f) = \exp\Big\{\int_0^1 \log |f(e^{i2\pi\theta})| d\theta\Big\}.$$

The following result belongs to Daubechies[6]:

Theorem C. *In* (1.1) *suppose that for some* $L \in I\!N$,

$$a(z) = \Big(\frac{1+z}{2}\Big)^L b(z)$$

where $b(z) \in \pi$ *satisfies* (1) $b(1) = 1$, *and* (2)

$$B_k(b) < 2^{(L-1)k} \quad \text{for some } k \in I\!N,$$

then (1.1) *is solvable in* C_0.

In order to solve the equation (1.1) in effectively numerical way, an iterative algorithm has been introduced (cf. [6]). Let T be the operator defined by

$$Tu(x) = \sum_{n=0} g_n u(2x - n). \tag{1.4}$$

Then the iterative preocess is started from an initiative function, say $\mu_0(x)$, and the successful solutions are obtained:

$$\mu_1(x) = T\mu_0(x), \cdots, \mu_m(x) = T\mu_{m-1}(x), \cdots.$$

If $\mu_m(x)$ converges to $u(x)$ in some space X, then $u(x) \in X$ is a solution of (1.1).

Choosing $\chi_{(0,1]}$ as the initiative function makes the iterative algorithm extremely easy to implement numerically. Its graphical illustration can be found in [6]. Now we give the following

Definition 1.1. *The equation* (1.1) *is said to be iteratively solvable in* $L^p(1 \le p \le \infty)$ *if there exists a function* $u \in L^p$ *such that*

$$\lim_{m\to\infty} \|T^m \chi_{(0,1]} - u\|_p = 0;$$

and iteratively solvable in C_0 *if there exists a function* $u \in C_0$ *such that*

$$\lim_{m\to\infty} \|T^m \chi_{(0,1]} - u\|_c = 0,$$

where $\|f\|_c = \sup_{x\in R} |f(x)|$.

Note that although $\mu_m(x) = (T^m \chi_{(0,1]})(x)$ are not in C_0, their limit $u(x)$ can be in C_0.

The solvability of (1.1) does not imply its iterative solvability. For example, the equation

$$u(x) = u(2x) + u(2x - 3)$$

is solvable in $L^p(1 \le p \le \infty)$ and $u(x) = \frac{1}{3}\chi_{(0,1]}$ is its solution with the normalization condition $\hat{u}(0) = 1$. But it is not iteratively solvable in L^p, since any $\mu_m(x) = (T^m \chi_{(0,1]})(x)$ takes its value only 0 or 1, there is therefore no pointwise convergence for any $x \in (0, 3]$.

It is worth to point out that the conditions in Theorem C also guarantee the iterative solvability of the equation (1.1) in C_0.

The results about solutions of two-scale difference equation in literatures seem scattered and not systematical, and some of them can be improved aslo. The objective of this paper is to make a more thorough study. In section 2, we discuss the solvability of the equation (1.1) in certain distribution spaces. One of the results there improves Theorem B. Section 3 deals with the iterative solvability of (1.1) not only in C_0 but also in $L^p(1 \le p < e\infty)$. The discussing of the relation between the equation (1.1) and the compactly supported wavelet basis is put in Final Remarks.

§2. Solvability of Two-Scale Difference Equations

We first consider the solvability of the equations (1.1) in certain distribution spaces. Some notations are introduced.

The notations $\mathcal{S} \equiv \mathcal{S}(R)$ and $\mathcal{S}'$ denote Schwartz space and its dual respectively. For any $u \in \mathcal{S}'$, $\hat{u}$ denotes its Fourier transform. Let $\mathcal{E} \equiv \mathcal{E}(\boldsymbol{R})$ be the space $C^\infty(\boldsymbol{R})$ equipped with the topology: $\phi \mapsto \sum_{\alpha\le k} \sup_{IK} |\phi^{(\alpha)}|$, where IK ranges over all compact subsets of $\boldsymbol{R}$ and k over all non-negative integers. Accordingly the space of distributions with compact support in

$\boldsymbol{R}$, i.e., the dual space of $\mathcal{E}$, is denoted by $\mathcal{E}' \equiv \mathcal{E}'(\boldsymbol{R})$. Let A be an arbitrary subset of $\boldsymbol{R}$ Then $\mathcal{E}'(A)$ denotes the set of distributions in $\mathcal{E}'$ with supports contained in A. Besides, $H^p_{(s)}(s \in \boldsymbol{R},\ 1 \le p \le \infty)$ are employed for the subspaces of $\mathcal{S}'$ containing all $u \in \mathcal{S}'$ such that

$$\|u\|_{(s),p} = \Big(\frac{1}{2\pi}\int_R |\hat{u}(\xi)|^p(1+|\xi|^p)^s d\xi\Big)^{\frac{1}{p}} < +\infty, \quad 1 \le p \le \infty.$$

Let $L^p_{(s)}$ be the L^p space with respect to the measure $\frac{1}{2\pi}(1+|\xi|^p)^s d\xi$. Then $u \in H^p_{(s)} \Leftrightarrow \hat{u} \in L^p_{(s)}$. For the properties of these distribution spaces, refer to [10].

Now we solve the equation (1.1) in $\mathcal{E}'$ under the normalization condition

$$\hat{u}(0) = 1. \tag{2.1}$$

We write $I_N = [0, N]$.

Theorem 2.1. *Suppose that the symbol of* (1.1) $a(z)$ *satisfies* $a(1) = 1$, *then* (1.1) *has a unique solution* $u \in \mathcal{E}'$ *with* $\hat{u}(0) = 1$; *and* supp$u \subset I_N$. *Futhermore, for any* p, $1 \le p \le \infty$, *there always exists an* $s \in \boldsymbol{R}$ *such that*

$$\inf \lim_{l \to +\infty} 2^{sl}\Big(\int_0^{2^l} |a_l^*(2\pi\theta)|^p d\theta\Big)^{\frac{1}{p}} < +\infty. \tag{2.2}$$

Then the solution u *is also in* $H^p_{(s)}$ *for such* s.

Proof. It is easy to verify that

$$\Phi(z) = \prod_{j=1}^{\infty} a(e^{-i2^{-j}z}) = \lim_{l \to \infty} a_l^*(z), \quad z \in \mathbb{C} \tag{2.3}$$

is an entire function. Let u be a distribution such that $\hat{u}(w) = \Phi(w)$. Then u is the solution of (1.1) with $u \in \mathcal{E}'(I_N)$ and $\hat{u}(0) = 1$ (cf. [6]).

Now we write $\max\limits_{|z|=1} a(z) = 2^r$. Since $a(1) = 1$, $r \ge 0$. Then

$$\sup_{w \in \boldsymbol{R}} |a_l^*(w)| = \sup_{|z|=1} \prod_{j=1}^{l} a(z^{2^{-j}}) \le 2^{lr}. \tag{2.4}$$

By (2.4), there always exist $s \in \boldsymbol{R}$ such that (2.2) holds. Let s be such a real number. Then setting $M = \max\limits_{|z|=1} |\Phi(z)|$, we have

$$\begin{aligned}\|u\|_{(s),p} &= \lim_{l \to \infty}\Big(\frac{1}{2\pi}\int_{-2^l}^{2^l} |\hat{u}(\theta)|^p(1+|\theta|^p)^s d\theta\Big)^{\frac{1}{p}} \\ &= \inf \lim_{l \to \infty}\Big(\frac{1}{2\pi}\int_{-2^l}^{2^l} |\hat{u}(2^{-l}\theta)|^p |a_l^*(\theta)|^p (1+|\theta|^p)^s d\theta\Big)^{\frac{1}{p}} \\ &\le \Big(\frac{1}{2\pi}\Big)^{\frac{1}{p}} M \inf \lim_{l \to \infty} (1+2^l)^s \Big(\int_{-2^l\pi}^{2^l\pi} |a_l^*(\theta)|^p d\theta\Big)^{\frac{1}{p}},\end{aligned}$$

i.e.,

$$\|u\|_{(s),p} \le c \inf \lim_{l \to \infty} 2^{ls}\Big(\int_0^{2l} |a_l^*(2\pi\theta)|^p d\theta\Big)^{\frac{1}{p}} < +\infty$$

$$\Rightarrow u \in H^p_{(s)}. \tag{2.5}$$

Remark 2.1. Since $\left(\int_0^{2^l}|a_l^*(2\pi\theta)|^p d\theta\right)^{\frac{1}{p}} = 2^{\frac{l}{p}}\left(\int_0^1|a^l(e^{i2\pi\theta})|^p d\theta\right)^{\frac{1}{p}}$, (2.2) can be replaced by

$$\inf\lim_{l\to+\infty} 2^{(s+\frac{1}{p})l}\Big(\int_0^1|a^l(e^{i2\pi\theta})|^p d\theta\Big)^{\frac{1}{p}} < +\infty. \tag{2.2$'$}$$

Remark 2.2. If $a(z)\neq 1$, there is no nontrivial solution of (1.1) with compact support (cf. [12]).

A necessary condition for the solution of (1.1) in the space $H^p_{(s)}$ is

Theorem 2.2. *If* (1.1) *has a nontrivial solution* $u\in H^p_{(s)}$, *then*

$$G(a)\le 2^{-s-\frac{1}{p}}. \tag{2.6}$$

Proof. We have

$$|\hat{u}(2^l w)| = \prod_{j=0}^{l-1}|a(e^{-i2^j w})||\hat{u}(w)|, \quad \forall l\in \mathbb{N}.$$

Then

$$\begin{aligned}&\sum_{j=0}^{l-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|a(e^{-i2^j w})|dw + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{p}\log(|\hat{u}(w)|^p(1+|w|^p)^s)dw\\&-\frac{s}{2\pi}\int_{-\pi}^{\pi}\log(1+|w|^p)^{\frac{1}{p}}dw\\=&\frac{1}{2\pi p}\int_{-\pi}^{\pi}\log|\hat{u}(2^l w)|^p(1+2^{pl}|w|^p)^s dw - \frac{s}{2\pi}\int_{-\pi}^{\pi}\log(1+2^{pl}|w|^p)^{\frac{1}{p}}dw.\end{aligned}$$

By Jensen's inequality, we obtain

$$\frac{1}{2\pi p}\int_{-\pi}^{\pi}\log(|\hat{u}(2^l w)|^p(1+2^{pl}|w|^p)^s)dw\le\log\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{u}(2^l w)|^p(1+2^{pl}|w|^p)^s dw\Big)^{\frac{1}{p}}.$$

Hence, considering that

$$\int_{-\pi}^{\pi}\log|a(e^{-i2^j w})|dw = \int_{-\pi}^{\pi}\log|a(e^{iw})|dw$$

for any $j\in Z^+$, we have

$$\begin{aligned}&\frac{l}{2\pi}\int_{-\pi}^{\pi}\log|a(e^{iw})|dw\\\le&-\Big(s+\frac{1}{p}\Big)l\log 2+\log\Big(\frac{1}{2\pi}\int_{-2^l\pi}^{2^l\pi}|\hat{u}(w)|^p(1+|w|^p)^s dw\Big)^{\frac{1}{p}}\\&-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{p}\log|\hat{u}(w)|^p(1+|w|^p)^s dw+\frac{s}{2\pi}\int_{-\pi}^{\pi}\log\Big(1+\frac{1-2^{-pl}}{|w|^p+2^{-pl}}\Big)^{\frac{1}{p}}dw.\end{aligned}$$

Dividing by l and letting $l\to+\infty$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|a(e^{iw})|dw\le-\Big(s+\frac{1}{p}\Big)\log 2,\quad 1\le p\le\infty.$$

Then (2.6) holds.

The following corallary is obvious.

Corallary 2.1. *In Theorem* 2.1, *if* (2.2) *is replaced by one of the following conditions:*

(1) *there exist an infinite set* $\mathbb{L}\subset\mathbb{N}$ *and a constant* $M>0$ *such that*

$$B_l(a)\le M2^{-(s+\frac{1}{p})l},\quad\forall l\in\mathbb{L};$$

(2) there exist an $m \in \mathbb{N}$ such that $B_m(a) \le 2^{-(s+\frac{1}{p})m}$;
then Theorem 2.1 still holds.

Proof. The conclusion can be derived by using Remark 2.1 and the inequality $B_{lm}(a) \le (B_m(a))^l$.

The gap between the sufficient condition (2.2) (or the conditions in Corallary 2.1 as well) and the necessary condition (2.6) seems quite "small", for we have

Theorem 2.3. *If $G(a) < c$, $a \in \pi$, then for any sufficient small $\varepsilon > 0$ there is a measurable set $A_\varepsilon \in [0,1]$ with $|A_\varepsilon| \le \varepsilon$ such that*

$$\lim_{l\to\infty} c^{-l}\Big(\int_{[0,1]\setminus A_\varepsilon} |a^l(e^{i2\pi\theta})|^p d\theta\Big)^{\frac{1}{p}} = 0, \quad \forall p,\ 1 \le p \le \infty. \tag{2.7}$$

Proof. By the Mean Ergodic Theorem[13],

$$\lim_{l\to\infty} \frac{1}{l}\sum_{j=0}^{l-1} \log|a(e^{i2\pi 2^j\theta})| = \int_0^1 \log|a(e^{2\pi i\theta})|d\theta$$

hold for almost all $\theta \in [0,1]$. Hence given an $\varepsilon > 0$, there is a positive integer l_ε and a set $A_\varepsilon \subset [0,1]$ with $|A_\varepsilon| \le \varepsilon$ such that for $l > l_\varepsilon$ and $\theta \in [0,1]\setminus A_\varepsilon$

$$\frac{1}{l}\sum_{j=0}^{l-1} \log|a(e^{i2\pi 2^j\theta})| \le \varepsilon + \int_0^1 \log|a(e^{2\pi i\theta})|d\theta.$$

Since $0 < G(a) < c$, we choose $\varepsilon = \frac{1}{2}\log\frac{c}{G(a)} > 0$. Then $\forall\theta \in [0,1]\setminus A_\varepsilon$,

$$|a(e^{i2\pi\theta})|^{\frac{1}{l}} \le e^\varepsilon G(a) = c\Big(\frac{G(a)}{c}\Big)^{\frac{1}{2}}$$

$$\Rightarrow c^{-l}\Big(\int_{[0,1]\setminus A_\varepsilon} |a^l(e^{i2\pi\theta})|^p d\theta\Big)^{\frac{1}{p}} \le \Big(\frac{G(a)}{c}\Big)^{\frac{l}{2}}, \quad \forall p,\ 1 \le p \le \infty;$$

i.e., (2.7) holds.

Remark 2.3. In Theorem 2.1, letting $s = 0$, we see that if $\inf \lim\limits_{l\to+\infty} (\int_0^{2^l} |a_l^*(2\pi\theta)|^p d\theta)^{\frac{1}{p}} < +\infty$ and $1 \le p \le \infty$, then (1.1) has a solution u with $\hat{u} \in L^p$. Therefore, $u \in C_0$ if $p = 1$ and $u \in L^{\frac{p}{p-1}}$ if $1 < p \le 2$.

When the symbol of (1.1) has a special form, we can get

Theorem 2.4. *Suppose that in* (1.1) $a(1) = 1$ *and*

$$a(z) = \prod_{j=1}^{k}\Big(\frac{1+z^{m_j}}{2}\Big)^{L_j} b(z),$$

where $m_j \in \mathbb{N}$ and $1 \le m_1 < \cdots < m_k$; $L_j \in Z^+$ and $L = \sum\limits_{j=1}^{k} L_j \ge 1$; besides, for some p, $1 \le p \le \infty$, $b(z) \in \pi$ satisfies

$$\inf \lim_{l\to\infty} 2^{(s-L)l}\Big(\int_0^{2^l} |b_l^*(2\pi\theta)|^p d\theta\Big)^{\frac{1}{p}} < +\infty, \tag{2.9}$$

then the solution of (1.1) $u \in H^p_{(s)}$.

Proof. We have

$$\hat{u}(w) = \prod_{j=1}^{\infty} a(e^{-i2^{-j}w}) = \prod_{j=1}^{k}\Big(\frac{1-e^{-im_j w}}{im_j w}\Big)^{L_j} \prod_{j=1}^{\infty} b(e^{-i2^{-j}w}).$$

By Theorem 2.1, $\prod_{j=1}^{\infty} b(e^{-i2^{-j}w}) \in L^p_{(s-L)}$.

Write $\mu(w) = \prod_{j=1}^{k} \left(\frac{1-e^{-im_j w}}{im_j w}\right)^{L_j}$, then $\sup_{w\in R} |\mu(w)| < +\infty$ and $\sup_{w\in R} |\mu(w)(1+|w|^p)^{\frac{L}{p}}| < +\infty$.

Hence $\mu(w) \prod_{j=1}^{\infty} b(e^{-i2^{-j}w}) \in L^p_{(s)}$, i.e., $u \in H^p_{(s)}$.

Corollary 2.2. *In Theorem 2.4, if (2.9) is replaced by one of the following conditions:*

(1) *there exist an infinite set* $I\!K \subset I\!N$ *and a constant* $M > 0$ *such that for some* p, $1 \le p \le \infty$,

$$B_k(b) \le M 2^{k(L-s-\frac{1}{p})}, \ \forall k \in I\!K, \tag{2.10}$$

(2) *there exist an* $m \in I\!N$ *such that for some* p, $1 \le p \le \infty$,

$$B_m(b) \le 2^{m(L-s-\frac{1}{p})}, \tag{2.11}$$

then the solution of (1.1) $u \in H^p_{(s)}$.

Remark 2.4. By Corollary 2.2, if $a(z) = (\frac{1+z^m}{2})b(z)$ and $B_k(b) \le 2^{\frac{k}{2}}$, then the solution of (1.1) is in L^2; that improves Theorem B.

§3. Iterative Solvability of Two-Scale Difference Equation

At first we introduce the mth order cardinal B-spline $N_m(x)$ with knot sequence Z. $N_m(x)$ is defined recursively by $N_1(x) = \chi_{(0,1]}(x)$, and

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)dt.$$

Hence $N_m \in W^{m-1,\infty}$ and $\mathrm{supp} N_m = [0, m]$.

The reason of our introducing N_m here is that they satisfy the special two-scale difference equations:

$$N_m(x) = 2^{-m+1} \sum_{j=0}^{m} \binom{m}{j} N_m(2x-j), \quad m \in I\!N. \tag{3.1}$$

(3.1) is equivalent to

$$\widehat{N}_m(w) = \left(\frac{1+e^{-i\frac{w}{2}}}{2}\right)^m \widehat{N}_m\left(\frac{w}{2}\right). \tag{3.2}$$

(3.1) (or (3.2)) tells us that if $a(z) = (\frac{1+z}{2})^m$ in (1.1), then the solution of (1.1) is $N_m(x)$. Hence we only consider $a(z) = (\frac{1+z}{2})^m b(z)$ with $\deg b \ge 1$ in this section. We begin with

Lemma 3.1. *Let* G_m *be the operater defined by*

$$(G^m \varphi)(x) = \sum_{j=0}^{J} g_j^m \varphi(2^m x - j). \tag{3.3}$$

If $\mathrm{supp}\varphi \in I_N$, *then* $\mathrm{supp} G^m\varphi = [0, \frac{J+N}{2^m}]$ *and*

$$\begin{cases} \|G^m\varphi\|_p \le N^{1-\frac{1}{p}} 2^{-\frac{m}{p}} \|g^m\|_{l_p} \|\varphi\|_p, & 1 \le p \le \infty, \\ \|G^m\varphi\|_p \le N \|g^m\|_{l_\infty} \|\varphi\|_c. \end{cases} \tag{3.4}$$

Proof. It is obvious that $\mathrm{supp} G^m\varphi \subset [0, \frac{J+N}{2^m}]$. We prove (3.4). Let $\varphi^k(x) = \varphi(x)\chi_{(k,k+1]}(x)$.

Then for $1 \le p \le \infty$,

$$\begin{aligned}\|G^m\varphi^k\|_p =&\Big(\int_{-\infty}^{+\infty}|\sum_{j=0}^{J} g_j^m\varphi^k(2^m x - j)|^p dx\Big)^{\frac{1}{p}}\\ =&\Big(\sum_{l\in Z}\int_l^{l+1}|\sum_{j=0}^{J} g_j^m\varphi^k(2^m x - j)|^p dx\Big)^{\frac{1}{p}}\\ \le&\Big(\sum_{j=0}^{J}2^{-m}|g_j^m|^p\Big)^{\frac{1}{p}}\|\varphi^k\|_p\\ =&2^{-\frac{m}{p}}\|g^m\|_{l_p}\|\varphi^k\|_p.\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad \|G^m\varphi^k\|_p \le& \sum_{k=0}^{N-1}\|G^m\varphi^k\|_p \le 2^{-\frac{m}{p}}\|g^m\|_{l_p}\sum_{k=0}^{N-1}\|\varphi^k\|_p\\ \le& N^{1-\frac{1}{p}}2^{-\frac{m}{p}}\|g^m\|_{l_p}\|\varphi\|_p.\end{aligned}$$

Similarly, $\|G^m\varphi\|_c \le N\|g^m\|_{l_\infty}\|\varphi\|_c$.

Lemma 3.2. *Let* $\alpha(z) = \sum\limits_{j=0}^{N}\alpha_j z^j$, $\beta(z) = \sum\limits_{j\in Z}\beta_j z^j$, *and* $d(z) \equiv \sum\limits_{j=0}^{N} d_j z^j = \alpha^k(z)\beta(z^{2^k})$. *Then*

$$\|d\|_{l_p} \le C_{Np}\|\alpha^k\|_{l_p}\|\beta\|_{l_p},\quad 1 \le p \le \infty, \tag{3.5}$$

where

$$C_{Np} = \begin{cases}(N+1)^{\frac{2}{p}}, & 1 \le p < 2,\\ N+1, & 2 \le p \le \infty.\end{cases} \tag{3.6}$$

Proof. Letting $\alpha^k(z)(= \prod\limits_{j=0}^{k-1}\alpha(z^{2^j})) = \sum\limits_{j\in Z}\alpha_j^k z^j$, we have $d_j = \sum\limits_{n\in K_j}\alpha^k_{j-2^{-k}n}\beta_n$, where $K_j = \{n;\ j-(2^k-1)N \le 2^k n \le j\}$. Write $\frac{1}{q}+\frac{1}{p} = 1$. Then, since $|K_j| \le N$,

$$\begin{aligned}|d_j| \le&\Big(\sum_{n\in K_j}|\alpha^k_{j-2^k n}|^q\Big)^{\frac{1}{q}}\Big(\sum_{n\in K_j}|\beta_n|^p\Big)^{\frac{1}{p}}\\ \le& C_{Np}(N+1)^{-\frac{2}{p}}\Big(\sum_{n\in K_j}|\alpha^k_{j-2^k n}|^p\Big)^{\frac{1}{q}}\Big(\sum_{n\in K_j}|\beta_n|^p\Big)^{\frac{1}{p}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \sum_{j\in Z}|d_j|^p \le& C_{Np}^p(N+1)^{-2}\sum_{j\in Z}\Big(\sum_{n\in K_j}|\alpha^k_{j-2^k n}|^p\sum_{n\in K_j}|\beta_n|^p\Big)\\ \le& C_{Np}^p(N+1)^{-2}\sum_{s\in Z}\Big(\sum_{i=0}^{2^k-1}\Big(\sum_{0\le s-n\le N}|\alpha^k_{2^k(s-n)+i}|^p\sum_{0\le s-n\le N}|\beta_n|^p\Big)\Big)\\ \le& C_{Np}^p(N+1)^{-2}\sum_{s\in Z}\Big[\Big(\sum_{0\le s\le N}\|\alpha^k\|_{l_p}^p\Big)\Big(\sum_{0\le s-n\le N}|\beta_n|^p\Big)\Big]\\ \le& C_{Np}^p\|\alpha^k\|_{l_p}^p\|\beta\|_{l_p}^p.\end{aligned}$$

Hence (3.6) holds.

Theorem 3.1. *In (1.1), suppose that* $N > 1$, $a(1) = 1$, *and* $a(z) = \big(\frac{1+z}{2}\big)b(z)$, *where*

$b \in \pi$ satisfies

$$\inf \lim_{k\to\infty} 2^{-\frac{k}{p}} \|b^k\|_{l_p} = 0 \quad \textit{for some } p,\ 1 \le p \le \infty. \tag{3.7}$$

Then (1.1) is iteratively solvable in L^p for $1 \le p < \infty$ and in C_0 for $p = \infty$.

Proof. By (3.7), for some p, $1 \le p < \infty$, there is an integer $k_0 > 0$ such that $2^{-\frac{k_0}{p}} C_{N-1,p} \|b^{k_0}\|_{l_p} = \delta < 1$. Then by Lemma 3.2,

$$\begin{aligned} 2^{-\frac{k_0 m}{p}} \|b^{k_0 m}\|_{l_p} &\le 2^{-\frac{k_0(m-1)}{p}} \|b^{(m-1)k_0}\|_{l_p} (2^{-\frac{k_0}{p}} \|b^{k_0}\|_{l_p} C_{N-1,p}) \\ &\le 2^{-\frac{k_0(m-1)}{p}} \|b^{(m-1)k_0}\|_{l_p} \delta \le \delta^m. \end{aligned}$$

Write $M_{k_0,p} = \max_{0\le n\le k_0} \{\|b^n\|_{l_p}\}$, where $b^0 = (\delta_{0s})_{s\in Z}$. Since $b(1) = 1$, there is a polynomial $r(z) \in \pi_{N-2}$ such that $r(z)(1-z) = b(z) - 1$.

Now let $g_k(x) = (T^k N_1)(x) - (T^{k-1} N_1)(x)$. Then

$$\begin{aligned} \hat{g}_k(w) &= b^*_{k-1}(w) \prod_{j=1}^{k} \Big(\frac{1+e^{-i2^{-j}w}}{2}\Big) (b(e^{-i2^k w}) - 1) \widehat{N}_1(2^{-k} w) \\ &= (1 - e^{-iw}) \frac{1}{2^{k-1}} b^*_{k-1}(w) \frac{1}{2} r(e^{-i2^{-k}w}) \widehat{N}_1(2^{-k} w), \end{aligned}$$

$$\Rightarrow\ g_k(x) = \sum_{j\in Z} b_j^{k-1} (\varphi(2^{k-1}x - j) - \varphi(2^{k-1}(x-1) - j)), \tag{3.8}$$

where $\varphi(x) = \sum_{j=0}^{N-2} r_j N_1(2x - j)$.

By using Lemma 3.1 and Lemma 3.2, from (3.8) we obtain

$$\|g_k\|_p \le 2 \cdot 2^{-\frac{k}{p}} \|b^{k-1}\|_{l_p} \|r\|_{l_p}.$$

Let $k = m_k k_0 + n$, $0 \le n < k_0$. Then

$$\|g_k\|_p \le 2^{-m_k k_0/p} \|b^{m_k k_0}\|_p 2^{1-\frac{n}{p}} M_{k_0} \|r\|_{l_p} \le c\delta^{m_k},$$

where $\delta < 1$, $c = 2M_{k_0}\|r\|_{l_p}$ is a constant independent of k. This implies that $g_k = (T^k N_1 - T^{k-1} N_1)$ is a Cauchy sequence in L^p $(1 \le p \le \infty)$. Hence there is a function $u \in L^p$ such that

$$\lim_{k\to\infty} \|T^k N_1 - u\|_p = 0.$$

In the case of $p = \infty$, by Lemma 3.1, we have $\text{supp}(T^k N_1) \subset [0, 1]$, $\forall k \in I\!N$. Then $\text{Supp} g_k \subset [0, N]$. Similar to the proof above, we have $\|g_k\|_c \le c\delta^{m_k}$. Then there exists a bounded function u with $\text{supp} u \subset [0, N]$ such that $\lim_{k\to\infty} \|T^k N_1 - u\|_c = 0$.

Now we prove that $u \in C_0$. Taking arbitrary $x, y \in [0, N]$, we get

$$|u(x) - u(y)| \le |T^k N_1(x) - u(x)| + |T^k N_1(y) - u(y)| + |T^k N_1(x) - T^k N_1(y)|,\ \forall k \in I\!N$$

$\forall \varepsilon > 0$, there exists an $n_0 \in I\!N$ such that $2M_{k_0}\delta^{n_0} < \frac{\varepsilon}{3}$ and

$$\|T^k N_1 - u\|_c < \frac{\varepsilon}{3},\quad k \ge n_0 k_0.$$

Now we choose $\delta = 2^{-n_0 k_0}$. When $0 < |x - y| < \delta$, $x > y$, there exists an $k \ge n_0 k_0$ such that $|2^k x| - |2^k y| = 1$. For this k, letting $h_k(x) = T^k N_1(x) - T^k N_1(y)$, we have $h_k(x) = (T^k N_1)(x) - (T^k N_1)(x - 2^{-k})$ and $\hat{h}_k(w) = (1 - e^{-iw}) 2^{-k} b^*_k(w) \widehat{N}_1(2^{-k} w)$. Hence,

$$|T^k N_1(x) - T^k N_1(y)| \le \|h_k\|_c \le 2\|b^k\|_\infty \le 2\delta^{n_0} M_{k_0} < \frac{\varepsilon}{3}.$$

$$\Rightarrow \quad |u(x)-u(y)| < \varepsilon, \quad \forall 0<|x-y|<\delta.$$

Corollary 3.1. *In (1.1) if $a(1)=1$ and*

$$a(z) = \Big(\frac{1+z}{2}\Big)^L b(z) \quad \textit{for some } L \in \mathbb{N},$$

where $b(z)\in\pi$ satisfies (3.7), the (1.1) is iteratively solvable in L^p for $1\le p<\infty$ and in C_0 for $p=\infty$.

Furthermore, the solution of (1.1) $u\in W^{L-1,p}$ for $1\le p<\infty$ and in C_0^{L-1} for $p=\infty$.

Proof. Let $\tilde b(z) = \big(\frac{1+z}{2}\big)^{L-1} b(z)$. Then we can prove that $\tilde b(z)$ also satisfies (3.7). Hence (1.1) is iteratively solvable in L^p or C_0 by Theore 3.1. Now write $b^\nabla(z) = (1+z)b(z)$ and set

$$v(x) = \sum_{j\in Z} b_j^\nabla v(2x-j). \tag{3.9}$$

The solution of (3.9) v is in L^p for $1\le p<\infty$ and in C_0 for $p=\infty$. Letting $u(x) = (N_{L-1}*v)(x)$, we have

$$\hat u(x) = \hat N_{L-1}(w)\hat v(w)$$

$$= \Big(\frac{1+e^{-i\frac{w}{2}}}{2}\Big)^{L-1}\hat N_{L-1}\Big(\frac{w}{2}\Big)\frac{1}{2}b^\nabla(e^{-i\frac{w}{2}})\hat v\Big(\frac{w}{2}\Big)$$

$$= \Big(\frac{1+e^{-i\frac{w}{2}}}{2}\Big)^{L} b(e^{-i\frac{w}{2}})\hat u\Big(\frac{w}{2}\Big).$$

$$\Rightarrow \quad u(x) = \sum_{j\in Z} a_j u(2x-j).$$

Since $N_{L-1}\in W^{L-2,\infty}$, we have $u\in W^{L-1,p}$ for $1\le p<\infty$ and $u\in C_0^{L-1}$ for $p=\infty$.

Corollary 3.2. *In (1.1), if $a(1)=1$ and for some $L\in\mathbb{N}$,*

$$a(z) = \Big(\frac{1+z}{2}\Big)^L b(z),$$

where $b(z)\in\pi$ ($\deg b\ge 1$) satisfies one of the following conditions:

(1) *There exist an infinite set $\mathbb{K}\subset\mathbb{N}$ and a constant $M>0$ such that for some p, $1\le p\le\infty$,*

$$B_k(b)\le M 2^{\frac{k}{p}}, \ \forall k\in\mathbb{K} \tag{3.10}$$

and

$$G(b) < 2^{\frac{1}{p}}; \tag{3.11}$$

(2) *There is some $m\in\mathbb{N}$ such that*

$$B_m(b)\le 2^{\frac{m}{p}} \quad \textit{for some } p,\ 2\le p\le\infty; \tag{3.12}$$

then for $2\le p<\infty$, (1.1) is iteratively solvable in L^p and the solution is in $W^{L-1,p}$; for $p=\infty$, (1.1) is iteratively solvable in C_0 and the solution is in C_0^{L-1}.

Proof. At first we consider the condition (1). By Theorem 2.3, (3.11) implies that for any $\varepsilon>0$, there is a set $A_\varepsilon\subseteq[0,1]$ with $|A_\varepsilon|<\varepsilon$ such that

$$\lim_{k\to\infty} 2^{-\frac{k}{p}}\Big(\int_{[0,1]\setminus A_\varepsilon}|b^k(e^{i2\pi\theta})|^q d\theta\Big)^{\frac{1}{q}} = 0 \tag{3.13}$$

holds for any q, $1 \le q \le \infty$. Letting $\frac{1}{q} + \frac{1}{p} = 1$. By (3.10), since $B_k(b) \le M2^{\frac{k}{p}}$, $\forall k \in I\!K$,

$$2^{-\frac{k}{p}} \Big(\int_{A_\varepsilon} |b^k(e^{i2\pi\theta})|^q d\theta \Big)^{\frac{1}{q}} \le \varepsilon M. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain, since $\varepsilon > 0$ is arbitrary,

$$\inf \lim_{k \to \infty} 2^{-\frac{k}{p}} \Big(\int_0^1 |b^k(e^{i2\pi\theta})|^q d\theta \Big)^{\frac{1}{q}} = 0.$$

But for $2 \le p \le \infty$,

$$\|b^k\|_{l_p} \le c \Big(\int_0^1 |b^k(e^{i2\pi\theta})|^q d\theta \Big)^{\frac{1}{q}}, \quad \frac{1}{q} + \frac{1}{p} = 1,$$

where c is a constant. Then the conclusions can be drawn from Theorem 3.1 and Corollary 3.1.

Now we turn to the condition (2). Because $b^m(1) = 1 < 2^{\frac{m}{p}}$ for $2 \le p < \infty$,

$$\int_0^1 |b^m(e^{i2\pi\theta})| d\theta < 2^{\frac{m}{p}}.$$

Hence $G(b) < 2^{\frac{1}{p}}$ for $2 \le p < \infty$.

As to the case of $p = \infty$, $B_m(b) \le 1$ still implies $G(b) < 1$, since $\deg b \ge 1$. In fact, if it were not so, we would have $G(b) = 1$, then $|b^m(z)| = 1$ for $|z| = 1$. Hence $B^m(z) = z^\alpha$ for some $\alpha \in I\!N$, since $b^m(1) = 1$ and $\deg b \ge 1$. Note that $b^m(0) = 2^L a(0) = 2^{L-1} a_0 \ne 0$. The contradiction implies that $G(b) < 1$. Besides, $B_m(b) \le 2^{\frac{m}{p}}$ also implies that for any p, $2 \le p \le \infty$,

$$B_{mk}(b) \le (B_m(b))^k \le 2^{mk/p}.$$

Then the condition (2) $\Rightarrow$ the condition (1).

§4. Final Remarks

As we said in introduction, solutions of two-scale difference equations can be used, as auxilary functions, to construct wavelet bases. The main idea is as follows: first we form a multiresolution approximation by an auxilary function, then use the space decomposition techniqe to construct a wavelet basis from this multiresolution analysis. If the auxilary function, say Φ, has orthonormal integer translates (i.e., $\Phi \in L^2$ and $\langle \Phi(\cdot - m), \Phi(\cdot - n) \rangle = \delta_{m,n}$, $m, n \in Z\!\!\!Z$) and satisfies

$$\Phi(x) = \sum_{n \in Z} c_n \Phi(2x - n), \tag{4.1}$$

then with $\Psi(x) = \sum_{n \in Z} (-1)^n c_{1-n} \Phi(2x - n)$, the collection $\{2^{\frac{j}{2}} \Psi(2^j x - k)\}_{j,k \in Z\!\!\!Z}$ forms an orthonormal basis for L^2(cf. [1-6]). In non-orthonormal cases, the construction of wavelet basis is more complicated (cf. [7]).

A function generating a multiresolution approximation must have stable integer translates.

Let $\Phi \in L^p$, $(1 \le p \le \infty)$. We say that integer translates $\Phi(\cdot, -j) (j \in Z\!\!\!Z)$ are l_p-stable, if there exist poistive constants m and M such that for any sequence $a \in l_p$,

$$m\|a\|_{l_p} \le \|\sum_{j \in Z} a_j \Phi(\cdot - j)\|_p \le M\|a\|_{l_p}.$$

Then the multiresolution approximation can be difined briefly as follows.

A subspace nest of $L^p(1 \le p \le \infty)$, $\{V_j\}_{j\in Z}$:

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \text{ with } \bigcap_{j\in Z} V_j \text{ and } \overline{\bigcup_{j\in Z} V_j} = L^p$$

is called a multiresolution approximation of L^p, if there is a function $\Phi \in L^p$ such that Φ has l^p-stable integer translates and

$$V_j = \Big\{ \sum_{k\in Z} a_k \Phi(2^j x - k); \quad a \in l_p \Big\}, \quad \forall j \in Z. \tag{4.2}$$

From (4.2) we know that Φ must satisfy an equation of form (4.1). If supp$\Phi \subset [0, N]$ is required, then (4.1) degenerated into (1.1). In this caes, $\overline{\bigcup_{j\in Z} V_j} = L^p$ is equivalent to that $a(x)$ has the factor $(1+z)$ (cf. [7]), that is why we assume $a(z) = \big(\frac{1+z}{2}\big)b(z)$ in Theorem 3.1.

[14] pointedout that l_p-stability for any p, $1 \le p \le \infty$ is equivalent to each other, so we can say stable instead of l_p-stable. [14] also proved

Theorem D. *The integer translates of u are stable if and only if the symbol of* (1.1) $a(z)$ *satisfies the following two conditions:*

(1) $a(z)$ *does not have any symmetric zeros on the unit circle* $|z| = 1$;

(2) *For any odd integer* $m > 1$ *and a primitive mth root w of unity, there exists an integer* $d, 0 \le d$ *such that* $a(-w^{2^d}) \ne 0$.

If $a(z)$ has the factor $(1+z^m)$ with $m \ge 2$, then $a(z)$ does not satisfy one of these conditions. Hence, when $a(z)$ has the form in Theorem 2.4, it does not supply a solution of (1.1) generating a multiresolution approximation, unless $k = 1$ and $m_1 = 1$ there (cf. Theorem 2.4 in Section 2).

References

[1] de Rham, G., Sur un example de founction continue same de 'rivée, *Ens. Math.*, **3** (1957), 71-72.

[2] Dubuc, S., Interpolation through an iterative scheme, *J. Math. Anal. & Appl.*, **114** (1986), 185-204.

[3] Deslauriers, G. & Dubuc, S., Interpolation dyadique, in "Fractals: dimensions non entiéres et applications", ed. G. Cherbit (Massor, Paris), 1987.

[4] Meyer, Y., Ondelettes et opérateurs, Vol. 1, Hermann, Paris, 1990.

[5] Mallat, S. G., Multiresolution approximations and wavelet orthonormal bases of $L^2(\boldsymbol{R})$, *Trans. Amer. Math. Soc.*, **315** (1989), 69-87.

[6] Daubechies, I., Orthonormal bases of compactly supported wavelets, *Comm. Pure and Appl. Math.*, **41** (1988), 909-996.

[7] Chui, C. K. & Wang, J. Z., A general framework of compactly supported splines and wave-lets, CAT Report # 219, Texas A & M University, College Stations, 1990 (to appear in *J. Appox. Theory*).

[8] Meyer, Y., Analysis multi-échelles et fractals, Unpublished memorandum, 1987.

[9] Cavarretta, A. S., Dahmen, W. & Micchelli, C. A., Stationary subdevision, IBM Research Report, Yorktown Heights, N. Y., 1989 (to appear in *Mem. Amer. Math. Soc.*).

[10] Hömander, L., The analysis of linear partial differential operators I, Springer-Verlag, 1983.

[11] Stein, E. & Weiss, G., Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, 1973.

[12] Halmos, P. R., Lectures on ergodic theory, Chelsea Publishing Co. New York, 1956.

[13] Daubechies, I. & Lagarias, J. C., Two-scale difference equations I: existence and global regularity of solutions, *SIAM J. Math. Anal.* (to appear).

[14] Jia, R. Q. & Wang, J. Z., Orthogonality and stability associated with wavelet decompositions, *Proc. Amer. Math. Soc.* (to appear).