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A NEW LAPLACIAN COMPARISON THEOREM AND THE ESTIMATE OF EIGENVALUES**

Ding Qing*

Abstract

This paper establishes a new Laplacian comparison theorem which is specially useful to the manifolds of nonpositive curvature. It leads naturally to the corresponding heat kernel comparison and eigenvalue comparison theorems. Furthermore, a lower estimate of L^2-spectrum of an n-dimensional non-compact complete Cartan-Hadamard manifold is given by $(n-1)k/4$, provided its Ricci curvature $\leq -(n-1)k$ ($k = \text{const.} \geq 0$).

Keywords Laplacian operator, Comparison theorem, Heat kernel, Eigenvalue.
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§1. Introduction

Let M be an n-dimensional Riemannian manifold. It is well known that Ranch, Hessian and Laplacian comparison theorems on M are important and fundamental results which have deep applications in Riemannian geometry. Laplacian comparison theorem holds by assuming that the Ricci curvature is bounded below. Naturally, we want to know what may happen if we replace the lower bound of Ricci curvature of M by upper bound of its Ricci curvature. Motivated by this idea and eigenvalue problems, we will study this interesting problem in this paper. The main result is Theorem 2.1 in section 2. This new kind Laplacian comparison theorem allows us to compare the Laplacians between Cartan-Hadamard manifolds, where a so-called Cartan-Hadamard manifold is a manifold of non-positive curvature. There are lots of Riemannain manifolds which satisfy the assumptions in our new comparison theorem. So this comparison theorem provides a new tool in studying the geometry of those manifolds and we believe that the discovery of this new Laplacian comparison theorem is interesting and useful.

As the applications of Theorem 2.1, in section 3 we will deduce a heat kernel comparison theorem firstly. In contrast with the result obtained by Debiard, Gaveau and Mazet (see [4]), we replace their assumptions of upper bound of sectional curvature by upper bound of Ricci curvature, it may be regarded as a corresponding result to Cheeger and Yau's (see [3]) in the case that $\mathrm{Ric}(M)$ is upper bounded. Secondly, we lead to an eigenvalue comparison theorem, this is a corresponding result to Cheng's (see [1]). For the further applications of those comparison results, we give a lower estimate of L^2-spectrum of non-compact complete

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*Institute of Mathematics, Fudan University, Shanghai 200433, China.
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Cartan-Hadamard manifold. Contrasting this estimate with Mckean's (see [6]), we also replace the assumption of upper bound of sectional curvature by upper bound of Ricci curvature. In this sense, we generalize Mckean's result.

§2. Comparison Theorem

The following new kind Laplacian comparison theorem between two n-dimensional Cartan-Hadamard manifolds is the main result in this section.

Theorem 2.1. *Let M, $\widetilde{M}$ be two n-dimensional Riemannian manifolds,*

$$\gamma : [0,b] \to M \text{ and } \tilde{\gamma} : [0,b] \to \widetilde{M}$$

are normal geodesics. Let $x = \gamma(0), \tilde{x} = \tilde{\gamma}(0)$, ρ and $\tilde{\rho}$ be the distance functions from x, $\tilde{x}$ in M, $\widetilde{M}$; Δ and $\widetilde{\Delta}$, Ric and $\widetilde{\mathrm{Ric}}$ be the Laplacians, the Ricci curvatures of M and $\widetilde{M}$ respectively. Suppose that

(1) *For any $t \in [0,b], \mathrm{Ric}(\dot{\gamma},\dot{\gamma})(t) \le \frac{1}{n-1}\mathrm{Ric}(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}})(t)$;*

(2) *M, $\widetilde{M}$ are Cartan-Hadamard manifolds.*

Then

$$\Delta\rho(\gamma(t)) \ge \frac{1}{n-1}\widetilde{\Delta}\tilde{\rho}(\tilde{\gamma}(t)), \quad \forall t \in (0,b]. \tag{2.1}$$

We know that the essential point in proving Rauch, Hessian and Laplacian comparison theorems is to compare two Jacobi fields which along the geodesic (e.g. see [8], §8). Since the Jacobi fields are the solutions of Jacobi equation, those comparisonal properties must be deduced from the equation directly. We would like to prove Theorem 2.1 exactly in this way. Let us now do some preparations firstly.

Let $gl(n-1,\mathbb{R})$ be the set of all $(n-1)\times(n-1)$ matrices, $K = (k_{ij}) : [0,b) \to gl(n-1,\mathbb{R})$ be a smooth mapping satisfying $K^T = K$, where K^T denotes the adjoint matrix of K, and

$$A : [0,b) \to gl(n-1,\mathbb{R})$$

be the solution of the following equation system:

$$\begin{cases} A_{tt} + AK = 0, \\ A(0) = 0, A_t(0) = I \text{ (the identity)}. \end{cases} \tag{2.2}$$

Here t is the natural parameter of $[0,b]$.

Similarly, let $\widetilde{A} : [0,b) \to gl(n-1,\mathbb{R})$ satisfy

$$\begin{cases} \widetilde{A}_{tt} + \widetilde{A}\widetilde{K} = 0, \\ \widetilde{A}(0) = 0, \widetilde{A}_t(0) = I, \end{cases} \tag{2.2'}$$

where $\widetilde{K} : [0,b) \to gl(n-1,\mathbb{R})$ and $\widetilde{K}^T = \widetilde{K}$.

Lemma 2.1.

(1) *If A^{-1} exists in $(0,b]$, then*

$$(A^{-1}A_t)^T = A^{-1}A;$$

(2) *If both A^{-1} and $\widetilde{A}^{-1}$ exist in $(0,b]$ and $K \ge \widetilde{K}$, then*

$$A^{-1}A_t \le \widetilde{A}^{-1}\widetilde{A}_t.$$

Here $C \geq D$ for symmetric matrices C, D means that for any $(\alpha_1, \cdots, \alpha_{n-1})$, $(\beta_1, \cdots, \beta_{n-1}) \in \mathbb{R}^{n-1}$ satisfying $\sum \alpha_i = \sum \beta_i$, we have

$$(\alpha_1, \cdots, \alpha_{n-1}) C (\alpha_1, \cdots, \alpha_{n-1})^T \geq (\beta_1, \cdots, \beta_{n-1}) D (\beta_1, \cdots, \beta_{n-1})^T.$$

The proof of this lemma is referred to [8], pp.163-165.

Lemma 2.2. *Suppose that both A^{-1} and $\widetilde{A}^{-1}$ exist in $(0, b]$, $A^{-1}A_t$ and $\widetilde{A}^{-1}\widetilde{A}_t$ are positive definite, and $\operatorname{tr} K \leq \frac{1}{n-1} \operatorname{tr} \widetilde{K}$. Then $\forall t \in (0, b]$ we have*

$$\operatorname{tr}(A^{-1}A_t) \geq \frac{1}{n-1} \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t). \tag{2.3}$$

Proof. Because A and $\widetilde{A}$ are solutions of (2.2) and (2.2′), we get $A, \widetilde{A} \in C^\infty[0, b]$ from the linearity of the equations. Noting the initial values of system (2.2), we have $A(0) = 0$, $A_t(0) = I$, $A_{tt}(0) = 0$, $A_{ttt}(0) = K(0)$. Therefore, when $t \to 0^+$, $A \sim tI$, $A_t \sim I$ and $A^{-1} \sim I/t - tK(0)/6$. Similarly, when $t \to 0^+$, we also have $\widetilde{A} \sim tI$, $\widetilde{A}_t \sim I$ and $\widetilde{A}^{-1} \sim I/t - t\widetilde{K}(0)/6$. So we have

$$(n-1)\operatorname{tr}(A^{-1}A_t) - \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t) \sim \begin{cases} \frac{(n-2)(n-1)}{t}, & \text{when } n \geq 3; \\ 0, & \text{when } n = 2. \end{cases}$$

Thus, when $n \geq 3$, there exists a small $\epsilon_0 > 0$ such that for $t \in (0, \epsilon_0)$

$$\operatorname{tr}(A^{-1}A_t) \geq \frac{1}{n-1} \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t). \tag{2.4}$$

When $n = 2$, we let $\epsilon_0 = 0$. Summarily, when $t \in (0, \epsilon_0)$, (2.4) holds and $t = \epsilon_0$,

$$(n-1)\operatorname{tr}(A^{-1}A_t) - \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t)|_{t=\epsilon_0} \geq 0. \tag{2.5}$$

Next, we prove that (2.4) holds in the whole interval $(0, b]$. For this purpose, we consider:

$$\begin{aligned} & [(n-1)A^{-1}A_t - \widetilde{A}^{-1}\widetilde{A}_t)]_t \\ = & -(n-1)A^{-1}A_tA^{-1}A_t + (n-1)A^{-1}A_{tt} + \widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t - \widetilde{A}^{-1}\widetilde{A}_{tt} \\ = & \widetilde{K} - (n-1)K + \widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t - (n-1)A^{-1}A_tA^{-1}A_t. \end{aligned}$$

Thus, we have

$$\begin{aligned} & [\operatorname{tr}((n-1)A^{-1}A_t - \widetilde{A}^{-1}\widetilde{A}_t)]_t \\ = & \operatorname{tr}(\widetilde{K} - (n-1)K) + \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t) - (n-1)\operatorname{tr}(A^{-1}A_tA^{-1}A_t) \\ \geq & \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t) - (n-1)\operatorname{tr}(A^{-1}A_tA^{-1}A_t). \end{aligned} \tag{2.6}$$

Since $\widetilde{A}^{-1}\widetilde{A}_t$ is a positive definite matrix, we have

$$\operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t) \geq \frac{1}{n-1}[\operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t)]^2,$$

and $A^{-1}A_t$ is also a positive definite matrix, so

$$\operatorname{tr}(A^{-1}A_tA^{-1}A_t) \leq [\operatorname{tr}(A^{-1}A_t)]^2.$$

Let $p(t) = \operatorname{tr}(A^{-1}A_t), q(t) = \operatorname{tr}(\widetilde{A}^{-1}\widetilde{A}_t)$ and $h(t) = (n-1)p(t) - q(t)$. Thus, the above inequality (2.6) can be rewritten by

$$h_t(t) + h(t)[(n-1)p(t) + q(t)] \geq 0. \tag{2.7}$$

Now, we come to treat the differential inequality (2.7) as follows. Multiply the two sides of (2.7) by a positive factor $\exp(\frac{1}{n-1}\int_{\epsilon_0}^t[(n-1)p(\tau)+q(\tau)]d\tau)$ $(t>\epsilon_0)$, we still have

$$\{h_t(t)+\frac{1}{n-1}h(t)[(n-1)p(t)+q(t)]\}\exp(\frac{1}{n-1}\int_{\epsilon_0}^t[(n-1)p(\tau)+q(\tau)]d\tau)\geq 0$$

or

$$\frac{d}{dt}\{h(t)\exp(\frac{1}{n-1}\int_{\epsilon_0}^t[(n-1)p(\tau)+q(\tau)]d\tau)\}\geq 0. \tag{2.8}$$

Integrating the two sides of (2.8) over $[\epsilon_0,t]$ $(t>\epsilon_0)$, we have

$$h(t)exp(\frac{1}{n-1}\int_{\epsilon_0}^t[(n-1)p(\tau)+q(\tau)]d\tau)-h(\epsilon_0)\geq 0$$

or

$$h(t)\geq h(\epsilon_0)exp(-\frac{1}{n-1}\int_{\epsilon_0}^t[(n-1)p(\tau)+q(\tau)]d\tau). \tag{2.9}$$

But, we know $h(\epsilon_0)\geq 0$ from (2.5). Thus, (2.9) implies that when $t>\epsilon_0$

$$h(t)=(n-1)p(t)-q(t)\geq 0,$$

i.e.,

$$(n-1)\mathrm{tr}(A^{-1}A_t)-\mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t)\geq 0. \tag{2.10}$$

Combining (2.4) with (2.10), we get the desired inequality:

$$\mathrm{tr}(A^{-1}A_t)\geq\frac{1}{n-1}\mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t),\quad \forall t\in(0,b].$$

Proof of Theorem 2.1. We choose a parallel normalized frame fields $\{e_1(t),\cdots,e_n(t)\}$ along γ in M such that $e_n(t)=\dot\gamma(t)$. Let $J_1(t),\cdots,J_{n-1}(t)$ be normal Jacobi fields along γ such that

$$J_i(0)=0,\dot J_i(0)=e_i(0),\quad i=1,\cdots,n-1,$$

and write $\{J_i(t)\}$ by

$$\begin{bmatrix} J_1(t)\\ \cdot\\ \cdot\\ \cdot\\ J_{n-1}(t)\end{bmatrix}=A(t)\begin{bmatrix} e_1(t)\\ \cdot\\ \cdot\\ \cdot\\ e_{n-1}(t)\end{bmatrix}, \tag{2.11}$$

where $A:[0,b]\to gl(n-1,\mathbb{R})$. Thus the Jacobi field equation becomes

$$\begin{cases} A_{tt}+AK=0,\\ A(0)=0, A(0)=I,\end{cases} \tag{2.12}$$

where $K=(k_{ij})_{1\leq i,j\leq n-1}$, $k_{ij}=\langle R(\dot\gamma,e_i)\dot\gamma,e_j\rangle$. Obviously, $K^T=K$.

Similarly, we choose a parallel normalized frame fields $\{\tilde e_1(t),\cdots,\tilde e_n(t)\}$ along $\tilde\gamma$ in $\widetilde M$ such that $\tilde e_n(t)=\dot{\tilde\gamma}(t)$, and let $\widetilde J_i(t)$, $i=1,\cdots,n-1$, be normal Jacobi fields along $\tilde\gamma$, s.t.

$$\widetilde J_i(0)=0,\ \dot{\widetilde J}_i(0)=\tilde e_i(0),\quad i=1,\cdots,n-1.$$

We also write $\{\widetilde J_i(t)\}$ by

$$\begin{bmatrix} \widetilde J_1(t)\\ \cdot\\ \cdot\\ \cdot\\ \widetilde J_{n-1}(t)\end{bmatrix}=\widetilde A(t)\begin{bmatrix} \tilde e_1(t)\\ \cdot\\ \cdot\\ \cdot\\ \tilde e_{n-1}(t)\end{bmatrix}. \tag{2.11'}$$

Then $\widetilde{A} : [o, b] \to gl(n-1, \mathbb{R})$ satisfies

$$\begin{cases} \widetilde{A}_{tt} + \widetilde{A}\widetilde{K} = 0 \\ \widetilde{A}(0) = 0, \widetilde{A}_t(0) = I, \end{cases} \tag{2.12'}$$

where $\widetilde{K} = (\widetilde{k}_{ij}), \widetilde{k}_{ij} = \langle \widetilde{R}(\dot{\tilde{\gamma}},\ \tilde{e}_i)\dot{\tilde{\gamma}},\ \tilde{e}_j \rangle$ and $\widetilde{K}^T = \widetilde{K}$.

For applying Lemma 2.2, we must check if the conditions in Lemma 2.2 are satisfied. Firstly, we prove that A^{-1} and $\widetilde{A}^{-1}$ exist in $(0, b]$. With the same argument, one can get $\widetilde{A}^{-1}$ exists as we show A^{-1} exists as following: In fact, we want to prove that $\forall t \in (0, b], |A(t)| \neq 0$. If this does not hold, then there is a $t_0 \in (0, b]$ such that $|A(t_0)| = 0$. From (2.11), we know that $J_1(t_0), \cdots, J_{n-1}(t_0)$ are linear dependent. Thus, there are $n-1$ constants $a_1, \cdots, a_{n-1}$ not all zero such that $\sum\limits_1^{n-1} a_i J_i(t_0) = 0$. Let $U(t) = \sum\limits_{i=1}^{n-1} a_i J_i(t)$. Obviously, $U(t) \not\equiv 0$ and $U(t)$ is also a normal Jacobi field along γ for Jacobi equation is linear. But $U(0) = 0$ and $U(t_0) = 0$, so t_0 is a conjugate point on γ. It contradicts the fact that there is no conjugate point on γ, for M is a Cartan-Hadamard manifold. Hence A^{-1} exists in $(0, b]$. Next, from $\mathrm{Riem}(M) \leq 0$ and $\mathrm{Riem}(\widetilde{M}) \leq 0$, we can see that $A^{-1}A_t$ and $\widetilde{A}^{-1}\widetilde{A}_t$ are positive definite matrices from Lemma 2.1. Finally, it is easy to get $\mathrm{tr}K \leq \frac{1}{n-1}\mathrm{tr}\widetilde{K}$, from condition (1). Therefore, by Lemma 2.2, we have

$$\mathrm{tr}(A^{-1}A_t) \geq \frac{1}{n-1}\mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t), \quad \forall t \in (0, b].$$

Calculating directly (see [8], §8), we have

$$\Delta\rho(\gamma(t)) = \mathrm{tr}(A^{-1}A_t),$$
$$\widetilde{\Delta}\tilde{\rho}(\tilde{\gamma}(t)) = \mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t).$$

Hence, we get the desired inequality:

$$\Delta\rho(\gamma(t)) \geq \frac{1}{n-1}\widetilde{\Delta}\tilde{\rho}(\tilde{\gamma}(t)), \quad \forall t \in (0, b].$$

Remark 2.1. In contrast to the well known Laplacian comparison theorem, all the inequalities are inverse in this new Laplacian comparison theorem and now it has two new characters: i) either M or $\widetilde{M}$ needs not to be a space form; ii) there is a constant factor in comparison inequality and this factor can not be removed essentially even when $\widetilde{M}$ is a space form.

The following Corollary 2.1 was proved sketchily in [5], here it is a direct corollary of Theorem 2.1.

Corollary 2.1. *The assumptions are the same as in Theorem* 2.1 *except that* $\widetilde{M}$ *is a space form with constant curvature* $-k$ $(k \geq 0)$, *and* $\mathrm{Ric}(M) \leq -k$. *Then*

$$\Delta\rho \geq \frac{1}{n-1}\widetilde{\Delta}\tilde{\rho}.$$

Remark 2.2. There are lots of Cartan-Hadamard manifolds with Ricci curvature $\leq -k$ ($k = \mathrm{const.} > 0$) in geometry, which have theoretical and practical sense. For example, from the theory in several complex analysis, it is well known that a classical domain Ω endowed with Bergmann metric satisfies $\mathrm{Riem}(\Omega) \leq 0$ and $\mathrm{Ric}(\Omega) = -1$ (see [9]). So our theorem is very useful for those manifolds.

§3. The Lower Estimate of L^2-Spectrum

Let M be an n-dimensional Riemannian manifold. We define

$$\lambda(M) = \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla \phi|^2}{\int_M |\phi|^2}.$$

Then $\lambda(M) \geq 0$ is called the supremum of L^2-spectrum of M.

Proposition 3.1.[1] *Assume that $\{M_i\}$ are exhaustion compact domains of M. Then*

$$\lambda(M) = \lim_{i \to \infty} \lambda_1(M_i), \tag{3.1}$$

where $\lambda_1(M_i)$ is the first Dirichlet eigenvalue of M_i.

For a simply connected complete noncompact Riemannian manifold M, an important problem is (see [7], pp.117-118): In what weakly conditions does there exist $\lambda(M)$? We try to establish a lower estimate of $\lambda(M)$ by using the upper bound of its Ricci curvature as an application of Theorem 2.1. However, we first deduce a heat kernel comparison theorem. The following statement is, no more than another way of expressing Corollary 2.1, the key for us to establish the heat kernel comparison theorem.

Theorem 3.2. *Assume that M is an n-dimensional Cartan-Hadamard manifold with* $\mathrm{Ric}(M) \leq -(n-1)k$ $(k \geq 0)$. *Let $\widetilde{M}$ be an $n-$dimensional space form of constant curvature $-\frac{k}{n-1}$. Then*

$$\Delta\rho \geq \widetilde{\Delta}\tilde{\rho}. \tag{3.2}$$

Proof. Let M_1 be an n-dimensional space form of curvature $-(n-1)k$. Then $\Delta\rho \geq \frac{1}{n-1}\Delta_1\rho_1$ from Corollary 2.1. One may note that there is a scaling between $\widetilde{M}$ and M_1, i.e., $\tilde{g} = (n-1)g_1$, where $\tilde{g}, g_1$ denote the metrics on $\widetilde{M}, M_1$ respectively. Calculating directly, we get

$$\widetilde{\Delta}\tilde{\rho} = \frac{1}{n-1}\Delta_1\rho_1.$$

This proves the above theorem.

Let $B(x_0, r)$ be an open geodesic ball in M, $V_n(k, r)$ an open geodesic ball in the space form $\widetilde{M}$. The heat kernel on $B(x_0, r)$ (with boundary condition) is denoted by $H(x.y;t)$. The heat kernel on $V_n(k, r)$ (with the same boundary condition) is denoted by $E(x, y; t) = E(\rho(x, y); t)$, it may be regarded as a function on $B(x_0, r)$. In 1976, Debiard, Gaveau and Mazet (see [4]) established the following upper estimate of $H(x, y; t)$ by using Hessian comparison theorem:

$$H(x, y; t) \leq E_1(\rho(x, y); t),$$

provided the sectional curvature of $M \leq k$ $(k \leq 0)$. In 1981, Cheeger and Yau (see [3]) obtained the following lower estimate by using the well known Laplacian comparison theorem:

$$H(x, y; t) \geq E_1(\rho(x, y); t),$$

provided the Ricci curvature of $M \geq (n-1)k$. Here $E_1(\rho(x, y); t)$ is the heat kernel on $V_n(-(n-1)k, r)$, and the boundary conditions are Dirichlet or Neumann.

Now, we can prove a heat kernel comparison theorem as follows.

Theorem 3.1. *Let M be a Cartan-Hadamard manifold of dimension n with* $\mathrm{Ric}(M) \leq$

$-(n-1)k$ $(k \geq 0)$. *Then*

$$H(x,y;t) \leq E(\rho(x,y);t), \tag{3.3}$$

where the boundary condition is either Dirichlet or Neumann.

Proof. Following the ideas used in [3], we have

$$\begin{aligned}
&H(x,y;t) - E(\rho(x,y);t) \\
=&\int_0^t \int_{B(x_0,r)} [E(\rho(x,z);t-s)H(z,y;s)]_s dzds \\
=&\int_0^t \int_{B(x_0,r)} [E(\rho(x,z);t-s)]_s H(z,y;s)dzds \\
&+\int_0^t \int_{B(x_0,r)} E(\rho(x,z);t-s)[H(z,y;s)]_s dzds \\
=&-\int_0^t \int_{B(x_0,r)} \widetilde{\Delta}E(\rho(x,z);t-s)H(z,y;s)dzds \\
&+\int_0^t \int_{B(x_0,r)} E(\rho(x,z);t-s)\Delta H(z,y;s)dzds \\
=&\int_0^t \int_{B(x_0,r)} [(\Delta-\widetilde{\Delta})E(\rho(x,z);t-s)]H(z,y;s)dzds,
\end{aligned} \tag{3.4}$$

where the last equality is obtained by using Green formula and the boundary conditions.

Choose a normalized geodesic coordinates (ρ,ξ) at x, $\xi \in S_0^{n-1}$. Then

$$\widetilde{\Delta} = \frac{\partial^2}{\partial\rho^2} + \widetilde{m}(\rho)\frac{\partial}{\partial\rho}, \quad \widetilde{m}(\rho) = \frac{d\log\sqrt{\tilde{g}}}{d\rho};$$
$$\Delta = \frac{\partial^2}{\partial\rho^2} + m(\rho,\xi)\frac{\partial}{d\rho}, \quad m(\rho,\xi) = \frac{d\log\sqrt{g}}{d\rho},$$

where $\tilde{g}, g$ are metrics on $\widetilde{M}$, M respectively. Because of (3.2), similar to [3], it is easy to carry out that

$$m(\rho,\xi) \geq \widetilde{m}(\rho).$$

From the Lemmas 1.1 and 2.3 in [3], we have $H(x.y;t) > 0$ and $\frac{\partial}{\partial\rho}E(\rho;t) < 0$. Thus

$$m(\rho,\xi)\frac{\partial E}{\partial\rho} \leq \widetilde{m}(\rho)\frac{\partial E}{\partial\rho}$$

or

$$(\Delta-\widetilde{\Delta})E(\rho;t-s) \leq 0.$$

Substituting this into (3.4), we obtain (3.3).

It is not difficult to carry out eigenvalue comparison theorem from the above heat kernel comparison.

Theorem 3.2. *Let M be a Cartan-Hadamard manifold of dimension n with* $\mathrm{Ric}(M) \leq -(n-1)k$ $(k \geq 0)$. *Then, for the first eigenvalue with Dirichlet condition, we have*

$$\lambda_1(B(x_0,r)) \geq \lambda_1(V_n(k,r)).$$

Proof. With the same argument as in [7], pp. 115-116, we can obtain the above result.

Remark 3.1. Cheng in [1] showed that $\lambda_1(B(x_0,r)) \leq \lambda_1(V_n(-(n-1)k,r))$, provided $\mathrm{Ric}(M) \geq (n-1)k$. Theorem 3.2 may be regarded as a corresponding result in case $\mathrm{Ric}(M) \leq -(n-1)k$ $(k \geq 0)$.

Return to the problem discussed at the beginning of this section. Now we have

Theorem 3.3. *Let M be an n-dimensional non-compact complete* (*simple connected*) *Cartan-Hadamard manifold, and* $\mathrm{Ric}(M) \leq -(n-1)k$ $(k \geq 0)$. *Then*

$$\lambda(M) \geq \frac{n-1}{4}k.$$

Proof. Since M is complete, we see that $B(x_0,R)$, $R = 1,2,\cdots$, are exhausting compact domains of M , $x_0 \in M$. From (3.1), we know

$$\lambda(M) = \lim_{R\to\infty} \lambda_1(B(x_0,R)),$$

but we have

$$\lambda_1(B(x_0,R)) \geq \lambda_1(V_n(k,R))$$

by Theorem 3.2. From [2], pp. 95-96, we get

$$\lambda_1(V_n(k,R)) \geq \frac{(n-1)^2}{4}\frac{k}{n-1} = \frac{n-1}{4}k.$$

Therefore, $\lambda(M) \geq \frac{n-1}{4}k$.

Remark 3.2. H. P. Mckean[6] obtained: If $\mathrm{Riem}(M) \leq -k$ $(k \geq 0)$, then

$$\lambda(M) \geq \frac{(n-1)^2}{4}k.$$

For the existence of a lower bound of L^2-spectrum, Theorem 3.3 is more general than that of Mckean's, e.g., for a classical domain Ω in Remark 2.2 we only get $\lambda(\Omega) \geq 0$ by Mckean's result, but we have $\lambda(\Omega) \geq \frac{1}{4}$ by our Theorem 3.3. And the following corollary is a startpoint for us to discuss the existence of Martin boundary of manifolds.

Corollary 3.1. *Let M be a complete Cartan-Hadamard manifold with* $\mathrm{Ric}(M) \leq -k$ $(k = \mathrm{const.} > 0)$. *Then there exist gloabl Green functions on M.*

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