Chin. Ann. of Math.
15B: 1(1994),43-52.

INITIAL BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN LIPSCHITZ CYLINDERS**

GAO WENJIE*

Abstract

The initial-Dirichlet and initial-Neumann problems in Lipschitz cylinders are studied for the general second order parabolic equations of constant coefficients with squarely integrable boundary data. By layer potential method developed in the past decade, the author proves that the double layer potential and the single layer potential operators are invertible and hence obtains the solvability of the initial boundary value problems. Also, the solutions can be represented by these operators.

Keywords Nonsmooth domains, Initial boundary value problems, Parabolic equations.
1991 MR Subject Classification 35K05, 31B25.

§1. Introduction

Boundary value problems for partial differential equations and systems on nonsmooth domains have attracted attentions of many mathematicians in recent years. Based on Calderon's theorem on the L^2-continuity of the Cauchy integral on Lipschitz curves which was finally proven by the authors of [1], one applied the classical layer potential method to the investigation of boundary value preblems of some partial differential equations on Lipschitz domains. The main difficulty in solving these problems is that the layer potential operators are no longer Fredholm operators and hence the classical method in solving the integral equation of Volterra type can not be applied here. In 1983, G. Verchota[2] studied the Laplace equation by utilizing Nečas-Rellich integral identity[3]. Afterwards, Dahlberg, Fabes, Kenig and Verchota studied the systems of elastostatics and the Stokes systems in [4,5]. By adopting a similar idea, R. Brown studied the heat equation in Lipschitz cylinders[6,7] and Z. Shen[8] solved some boundary value problems for parabolic Láme system and a nonstationary linearized system of Navier-Stokes equation. Some other works can be found in [9–12] and the related papers.

In this paper, we shall study the general parabolic equations in Lipschitz cylinders. We shall prove the existence of solutions for the initial boundary value problem and show that these solutions can be represented by layer potentials. We also show that the solution of the initial-Dirichlet problem with boundary data having first spatial derivatives and 1/2 order time derivative in L^2 can be represented by a single layer potential and hence the nontangential maximum functions of the first spatial derivatives and 1/2 order time derivative of the solution exist and are squarely integrable.

Manuscript received June 22, 1991.
*Institute of Mathematics, Jilin University, Changchun 130023, Jilin, China.
**Project supported by the National Natural Science Foundation of China.

The outline of this paper is as follows.

We give some brief definitions and notations in the first section. In Section 2, we will construct the layer potentials for parabolic equations and state some basic properties of these potentials. Section 3 contains some fundamental estimates and the invertibility results for a double layer potential operator constructed in Section 2. The invertibility of the single layer potential operator will be proven in Section 4, and the solvability and uniquess results for the initial-boundary value problems will be stated and proven in Section 5.

§1. Definitions and Notations

We will use some well-known notations and definitions without explanation.

Let Ω be a bounded connected open set of $\mathbf{R}^n$, we say that Ω is a Lipschitz domain if for each point $Q \in \partial\Omega$, the boundary of Ω, there is a system of coordinates of $\mathbf{R}^n$, isometric with the usual coordinate system and a sphere $B_\delta(Q)$ with center Q and radius $\delta > 0$ such that relative to this coordinate system Q is the origin and

$$\Omega \cap B_\delta(Q) = \{(x,t) : x \in \mathbf{R}^{n-1},\ t > \phi(x)\} \cap B_\delta(Q),$$

where ϕ is a Lipschitz continuous function on $\mathbf{R}^{n-1}$ and $\phi(0) = 0$. We use $S_T = \partial\Omega \times (0,T)$ to denote the lateral boundary of the cylinder $\Omega_T = \Omega \times (0,T)$. We denote $\Omega = \Omega^+$ and $\mathbf{R}^n \setminus \overline{\Omega} = \Omega^-$. We denote by u^* the nontangential maximal function of u on $\partial\Omega$ and denote by u^+ and u^- the nontangential limits of u on the boundary of the domain from the inside and the outside of Ω respectively. We write $L_1^p(\partial\Omega)$ as the space of functions in L^p with first derivatives in L^p and write $L^p_{1,\frac{1}{2}}(S_T)$ as the space of functions in L^p with first spatial derivatives and $1/2$ time derivative in L^p. The details of these definitions can be found in [2, 8].

Let $f \in C^\infty(-\infty, T)$ and $f(t) = 0$ for $t < 0$. We use $I_\sigma(f)$ to denote the fractional integral of f and define the fractional derivatives as $D_t^\sigma(f)(t) = D_t I_{1-\sigma}(f)(t),\ 0 < \sigma < 1$. The properites of I_σ and D_t^σ can be found in [8].

§2. Layer Potentials

We study the general parabolic equation

$$\frac{\partial u}{\partial t} - a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{2.1}$$

with constant coefficients a_{ij} satisfying $a_{ij} = a_{ji}$ and

$$a_{ij}\xi_i\xi_j \geq \mu|\xi|^2$$

for any $\xi \in \mathbf{R}^n$. We always use the summation convention on repeated indices.

It is well-known (e.g. see [13]) that the fundamental solution of this equation has the form

$$\Gamma(x,t) = ct^{-n/2} exp\{-(a^{ij}x_ix_j)/4t\},$$

where a^{ij}s are the entries of the inverse of the matrix $(a_{ij})_{n\times n}$.

We shall study the solution of (2.1) with the initial-Dirichlet boundary data

$$u\big|_{S_T} = g, \qquad u\big|_{t=0} = 0 \tag{2.2}$$

or with the initial-Neumann boundary data

$$\frac{\partial u}{\partial \nu} \stackrel{\text{def}}{=} a_{ij} n_i \frac{\partial u}{\partial x_j} = g, \qquad u\big|_{t=0} = 0 \tag{2.3}$$

where $N(Q) = (n_1(Q), \cdots, n_n(Q))$ is the outward unit normal to $\partial\Omega$ at Q.

We shall investigate the solution of above problems in the case when g is squarely integrable in S_T. As in the classical cases, we define the single layer potential of above parabolic equation as

$$\mathbf{S}f(x,t) = \int_0^t \int_{\partial\Omega} \Gamma(x-P, t-\tau) f(P,\tau)\, d\sigma_P d\tau$$

and the double layer potential as

$$\mathbf{D}f(x,t) = \int_0^t \int_{\partial\Omega} a_{ij} n_i(P) \frac{\partial}{\partial x_j} \Gamma(x-P, t-\tau) f(P,\tau)\, d\sigma_P d\tau.$$

It is clear that for any integrable function f, both $\mathbf{S}f(x,t)$ and $\mathbf{D}f(x,t)$ are the solutions of (2.1) with zero initial values. We shall prove that the solution for (2.1), (2.2) and for (2.1), (2.3) can be represented by either $\mathbf{S}f$ or $\mathbf{D}f$ for some f in $L^2(S_T)$. For this purpose, we must study the behavior of the trace of above operators. We define

$$\begin{aligned}\widetilde{\mathbf{K}}f(Q,t) =& p.v \int_0^t \int_{\partial\Omega} a_{ij} n_i(P) \frac{\partial}{\partial x_j} \Gamma(Q-P, t-\tau) f(P,\tau)\, d\sigma_P d\tau \\ =& \lim_{\epsilon\to 0} \int_0^{t-\epsilon} \int_{\partial\Omega} a_{ij} n_i(P) \frac{\partial}{\partial x_j} \Gamma(Q-P, t-\tau) f(P,\tau)\, d\sigma_P d\tau\end{aligned}$$

and

$$\mathbf{K}f(Q,t) = p.v \int_0^t \int_{\partial\Omega} a_{ij} n_i(Q) \frac{\partial}{\partial x_j} \Gamma(Q-P, t-\tau) f(P,\tau)\, d\sigma_P d\tau.$$

For simplicity, we will write $\|u\|$ as the $L^2(S_T)$ norm of the function u and write $\|u\|_\partial$ as the $L^2(\partial\Omega)$ norm of u. In general, we write $\|\cdot\|_X$ as the norm in the Banach space X.

The proof of the following theorem is standard by referring to [7, 8, 12], we omit the details.

Theorem 2.1. *The singular integral operators* $\mathbf{K}$ *and* $\widetilde{\mathbf{K}}$ *defined as above are bounded operators in* $L^p(S_T)$ *and the restriction of* $\mathbf{S}$ *on the boundary of the domain is a bounded mapping from* $L^p(S_T)$ *to* $L^p_{1,1/2}(S_T)$ *for any* $p > 1$ *and*

i) $\|(\nabla \mathbf{S}f)^*\|_p + \|(D_t^{\frac{1}{2}} \mathbf{S}f)^*\|_p \le C\|f\|_p$;

ii) $\|(\mathbf{D}f)^*\|_p \le C\|f\|_p$;

iii) $(\mathbf{D}f)^\pm = (\pm\frac{1}{2} + \widetilde{\mathbf{K}})f$;

iv) $\left(\dfrac{\partial}{\partial v}\mathbf{S}f\right)^\pm = \left(\pm\frac{1}{2} + \mathbf{K}\right) f$;

v) $\left(D_t^{\frac{1}{2}}\mathbf{S}f\right)^+ = \left(D_t^{\frac{1}{2}}\mathbf{S}f\right)^-$,
$(\nabla_T \mathbf{S}f)^+ = (\nabla_T \mathbf{S}f)^-, \qquad (\mathbf{S}f)^+ = (\mathbf{S}f)^-$,

where C *is independent of* f, $\|\cdot\|_p$ *is the* $L^p(S_T)$ *norm and* $\nabla_T u$ *is the tangential derivative of* u, *i.e.,*

$$\nabla_T u = \nabla u - \langle \nabla u, N\rangle N,$$

where $\langle\ ,\ \rangle$ *denotes the usual inner product in* $\mathbf{R}^n$.

As in [6,8], we consider the equation

$$a_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} = i\tau u \tag{2.4}$$

for each $\tau \in \mathbf{R}$. It is easy to see that $\widehat{\Gamma}(x,\tau) = \int_0^\infty e^{-it\tau}\Gamma(x,t)\,dt$ is a fundamental solution of the equation (2.4). Define, for $h \in L^2(\partial\Omega)$,

$$u_\tau(x) = \int_{\partial\Omega} \widehat{\Gamma}(x-Q,\tau)h(Q)\,d\sigma_Q. \tag{2.5}$$

Then $u_\tau(x)$ satisfies (2.1) for x in $\mathbf{R}^n \setminus \partial\Omega$. We have

Lemma 2.1. *Let $h \in L^2(\partial\Omega)$, $u_\tau(x)$ be defined above. Then*

i) $\|(\nabla u_\tau)^*\|_\partial \le C\|h\|_\partial$;

ii) $(\nabla_T u_\tau)^+ = (\nabla_T u_\tau)^-, \quad u_\tau^+ = u_\tau^- \qquad$ *a.e on* $\partial\Omega$;

iii) $\left(\dfrac{\partial u_\tau}{\partial \nu}\right)^{\pm}(P) = \left(\pm\frac{1}{2} + \widehat{\mathbf{K}}(\tau)\right)h(P)$,

where $\widehat{\mathbf{K}}$ is the singular integral operator bounded in $L^2(\partial\Omega)$ with kernal

$$\widehat{K}(P,Q,\tau) = a_{kj}n_k\frac{\partial}{\partial x_j}\widehat{\Gamma}(x-Q,\tau)\big|_{x=P}$$

and

$$\left|\widehat{K}(P,Q,\tau_1) - \widehat{K}(P,Q,\tau_2)\right| \le \frac{C|\tau_1-\tau_2|}{|P-Q|^{n-3}}$$

with C independent of τ_1, τ_2, P, Q.

Proof. In the case of $\tau = 0$, $\widehat{\Gamma}(x,0)$ is a fundamental solution of equation $a_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} = 0$. i)–iii) follow from the general results in [12] by using Calderon's Theorem[1]. For $\tau \neq 0$, these results follow from the case when $\tau = 0$ and a standard proof as in [8].

§3. Invertibility of Double Layer Potentials

To study the initial boundary value problems stated in Section 2, we need the so-called Rellich type inequalities. We shall establish these inequalities by using a revision of Nečas-Rellich integral identity[3] and some estimates for the solutions of (2.4).

Suppose that $u = u_\tau$ is the function defined by (2.5). Then

$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^{\pm}\bar{u}\,d\sigma = \pm\int_{\Omega_\pm}(a_{kj}\frac{\partial}{\partial x_k}\bar{u}\frac{\partial}{\partial x_j}u + i\tau|u|^2)dx. \tag{3.1}$$

By checking the real part and the image part and using the ellipticity condition of the equation, we have, for any complex valued C^1 function u,

$$a_{kj}\frac{\partial u}{\partial x_k}\frac{\partial \bar{u}}{\partial x_j} \ge \mu|\nabla u|^2.$$

Furthermore, for any Lipschitz continuous vector field h, we have

$$\begin{aligned}&\frac{\partial}{\partial x_k}\left[(h_k a_{sj} - h_s a_{kj} - h_j a_{sk})\frac{\partial u}{\partial x_s}\cdot\frac{\partial \bar{u}}{\partial x_j}\right]\\ =&b_{sj}\frac{\partial u}{\partial x_s}\cdot\frac{\partial \bar{u}}{\partial x_j} + 2\tau \mathrm{Im}(h_j\frac{\partial \bar{u}}{\partial x_j}\cdot u)\end{aligned} \tag{3.2}$$

where $b_{sj} = \frac{\partial}{\partial x_k}(h_k a_{sj} - h_s a_{kj} - h_j a_{sk})$ with $\|b_{sj}\|_{L^\infty} \le C < \infty$. Integrating both sides of (3.2) over $\Omega_\pm$ and using the Divergence Theorem, we have

$$\begin{aligned}&\int_{\partial\Omega} n_k(h_k a_{sj} - h_s a_{kj} - h_j a_{sk})\frac{\partial u^\pm}{\partial x_s}\cdot\frac{\partial \bar{u}^\pm}{\partial x_j}d\sigma\\ &= \pm\int_{\Omega_\pm}\Big(b_{sj}\frac{\partial u}{\partial x_s}\cdot\frac{\partial \bar{u}}{\partial x_j} + 2\tau\mathrm{Im}(h_j\frac{\partial \bar{u}}{\partial x_j}\cdot u)\Big)dx.\end{aligned} \tag{3.3}$$

Analogus to the method used in [8], we need the following lemmas:

Lemma 3.1. *Let u be the function defined in (2.5). Then*

$$\int_{\Omega_\pm}(|\nabla u|^2 + |\tau||u|^2)\,dx \le C\int_{\partial\Omega}|u||\frac{\partial u^\pm}{\partial\nu}|\,d\sigma$$

and

$$\int_{\Omega_\pm}|\tau||\nabla u||u|\,dx \le C\int_{\partial\Omega}|\tau|^{\frac{1}{2}}|u||\frac{\partial u^\pm}{\partial\nu}|\,d\sigma.$$

Proof. A consequence of (3.1) and the Cauchy inequality.

Lemma 3.2.

$$\int_{\partial\Omega}|u|^2 d\sigma \le C\int_{\Omega_\pm}(|u|^2 + |u||\nabla u|)dx.$$

Proof. An easy application of the Divergence Theorem.

Lemma 3.3.

i)

$$\int_{\partial\Omega}|\nabla u^\pm|^2\,d\sigma \le C\int_{\partial\Omega}\left(|\nabla_T u|^2 + |\tau||u|^2 + |u|^2\right)\,d\sigma$$

and

ii)

$$\int_{\partial\Omega}(|\nabla u^\pm|^2 + |\tau||u|^2)\,d\sigma \le C\int_{\partial\Omega}\left(|\frac{\partial u^\pm}{\partial\nu}| + |u|^2\right)\,d\sigma.$$

Proof. It is easy to prove[12] that for any Lipschitz domain Ω we can find a vector field h such that $\langle h(P), N(P)\rangle \ge C > 0$ with C independent of $P \in \partial\Omega$. By (3.3)

$$\begin{aligned}&\int_{\partial\Omega} a_{sj}n_k h_k\frac{\partial u^\pm}{\partial x_s}\cdot\frac{\partial \bar{u}^\pm}{\partial x_j}\,d\sigma\\ &= \int_{\partial\Omega}\left[(h_k a_{sj} - h_s a_{kj})n_k\frac{\partial u^\pm}{\partial x_s}\right]\frac{\partial \bar{u}^\pm}{\partial x_j}\,d\sigma - \int_{\partial\Omega}\left[(h_j a_{sk} - h_k a_{sj})n_k\frac{\partial \bar{u}^\pm}{\partial x_j}\right]\frac{\partial u^\pm}{\partial x_s}\,d\sigma\\ &\quad -\int_{\Omega_\pm}\left[b_{sj}\frac{\partial u}{\partial x_s}\cdot\frac{\partial \bar{u}}{\partial x_j} + 2\tau\mathrm{Im}\left(h_j\cdot\frac{\partial \bar{u}}{\partial x_j}\cdot u\right)\right]\,dx.\end{aligned} \tag{3.4}$$

For fixed j, the vector with $(h_k a_{sj} - h_s a_{kj})n_k$ as the sth entry is orthogonal to the normal. Hence, the first two terms on the right hand side of (3.4) can be controlled by $C\int_{\partial\Omega}|\nabla u^\pm||\nabla_T u|\,d\sigma$ with C independent of u and τ. By Lemma 3.1,

$$\left|\int_{\Omega_\pm}\left[b_{sj}\frac{\partial u}{\partial x_s}\frac{\partial \bar{u}}{\partial x_j} + 2\tau\mathrm{Im}(h_j\frac{\partial \bar{u}}{\partial x_j}u)\right]dx\right| \le C\int_{\partial\Omega}\left(|u|\left|\frac{\partial u^\pm}{\partial\nu}\right| + |\tau|^{\frac{1}{2}}|u|\left|\frac{\partial u^\pm}{\partial\nu}\right|\right)\,d\sigma. \tag{3.5}$$

By the ellipticity and the choice of h, we know from (3.4) and (3.5) that

$$\int_{\partial\Omega}|\nabla u^\pm|^2 d\sigma \le C\int_{\partial\Omega}\left(|\nabla_T u||\nabla u^\pm| + |u|\left|\frac{\partial u^\pm}{\partial\nu}\right| + |\tau|^{\frac{1}{2}}|u||\frac{\partial u^\pm}{\partial\nu}|\right)\,d\sigma.$$

Part i) follows from the inequality $ab \le \frac{1}{\epsilon}a^2 + \epsilon b^2$ with suitable choice of ϵ.

An easy computation and (2.4) yield

$$\begin{aligned}&\int_{\partial\Omega} \langle N, h\rangle a_{sj} \frac{\partial u^\pm}{\partial x_s} \cdot \frac{\partial \bar{u}^\pm}{\partial x_j} d\sigma \\ =&2\mathrm{Re} \int_{\Omega_\pm} h_k \left[\frac{\partial}{\partial x_s}\left(a_{sj}\frac{\partial u}{\partial x_k} \cdot \frac{\partial \bar{u}}{\partial x_j}\right) - i\tau u \frac{\partial u}{\partial x_k}\right] dx + \int_{\Omega_\pm} (\operatorname{div} h) a_{sj} \frac{\partial u}{\partial x_s} \cdot \frac{\partial \bar{u}}{\partial x_j} dx.\end{aligned} \tag{3.6}$$

By Lemma 3.1, the second and the third terms of (3.6) can be controlled by

$$\int_{\partial\Omega} \left(|\frac{\partial u^\pm}{\partial \nu}|^2 + |u|^2 + |\tau|^{\frac{1}{2}}|u||\frac{\partial u^\pm}{\partial \nu}|\right) d\sigma.$$

Using Lemma 3.1 again, we get

$$|\int_{\Omega_\pm} h_k a_{sj} \frac{\partial}{\partial x_s}\left(\frac{\partial u}{\partial x_k} \cdot \frac{\partial \bar{u}}{\partial x_j}\right) dx| \le C \int_{\partial\Omega} \left(|\frac{\partial u^\pm}{\partial \nu}||\nabla u| + |u||\frac{\partial u^\pm}{\partial \nu}|\right) d\sigma. \tag{3.7}$$

The second part of Lemma 3.3 follows easily.

Lemma 3.4. *The operator $\pm\frac{1}{2} + \widehat{\mathbf{K}}(\tau)$ defined in Lemma 2.1 is invertible from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$ and for $\tau \ne 0$,*

$$\|f\|_\partial \le C\left(1 + \frac{1}{|\tau|}\right) \| \left(\pm\frac{1}{2} + \widehat{\mathbf{K}}(\tau)\right) f\|_\partial$$

for $f \in L^2(\partial\Omega)$ with C independent of h and τ.

Proof. For $\tau \ne 0$, let $u = u_\tau$ be defined as in (2.5). Then by Lemmas 3.1 and 3.2,

$$\int_{\partial\Omega} |u|^2 d\sigma \le C \int_\Omega (|u|^2 + |\nabla u|^2)\, dx \le C(1 + \frac{1}{|\tau|}) \int_{\partial\Omega} |u||\frac{\partial u^+}{\partial \nu}|d\sigma.$$

By Lemma 3.3

$$\begin{aligned}\int_{\partial\Omega} |\frac{\partial u^-}{\partial \nu}|\, d\sigma \le& C \int_{\partial\Omega} (|\nabla_T u|^2 + |\tau||u|^2 + |u|^2)\, d\sigma \\ \le& C \int_{\partial\Omega} \left(|\frac{\partial u^+}{\partial \nu}|^2 + |u|^2\right) d\sigma \\ \le& C\left(1 + \frac{1}{|\tau|}\right)^2 \int_{\partial\Omega} |\frac{\partial u^+}{\partial \nu}|^2 d\sigma.\end{aligned}$$

Hence, by the jump relation in Lemma 2.1 iii),

$$\|f\|_\partial \le \|\frac{\partial u^+}{\partial \nu}\|_\partial + \|\frac{\partial u^-}{\partial \nu}\|_\partial \le C\left(1 + \frac{1}{|\tau|}\right) \| \left(\frac{1}{2} + \widehat{\mathbf{K}}(\tau)\right) f\|_\partial. \tag{3.8}$$

Thus, for $\tau \ne 0$, $\frac{1}{2} + \widehat{\mathbf{K}}(\tau)$: $L^2(\partial\Omega) \to L^2(\partial\Omega)$ is one to one with closed range in $L^2(\partial\Omega)$. To prove the invertibility, it remains to prove that the range of $\frac{1}{2} + \widehat{\mathbf{K}}(\tau)$ is dense in $L^2(\partial\Omega)$.

If Ω is a smooth domain, by Lemma 2.1 iii), it is obvious that $\widehat{\mathbf{K}}(\tau) - \widehat{\mathbf{K}}(0)$ is compact on $L^2(\partial\Omega)$. Notice that

$$\begin{aligned}&\left(\frac{1}{2} + \widehat{\mathbf{K}}(\tau)\right) - \left(\frac{1}{2} + \widehat{\mathbf{K}}(\tau)\right)^* \\ =&\left(\widehat{\mathbf{K}}(\tau) - \widehat{\mathbf{K}}(0)\right) - \left(\widehat{\mathbf{K}}(\tau) - \widehat{\mathbf{K}}^*(0)\right) + \left(\widehat{\mathbf{K}}(\tau) - \widehat{\mathbf{K}}(0)\right)^*.\end{aligned}$$

All the three terms above are compact by the results in [13]. Hence, by Lemma 2.3 in [5], $\frac{1}{2} + \widehat{\mathbf{K}}(\tau)$ is a Fredholm operator with index 0. The invertibility follows. For general

Lipschitz domain, the results follow from (3.8) and a standard approximation scheme as in [2]. The invertibility of $-\frac{1}{2}+\hat{\mathbf{K}}(\tau)$ may be proven similarily.

Theorem 3.1. *The operator* $\pm\frac{1}{2}+\mathbf{K}: L^2(S_T)\to L^2(S_T)$ *is invertible.*

Proof. Again, we only prove the theorem for $\frac{1}{2}+\mathbf{K}$. Suppose $f\in L^2(S_T)$ such that $(\frac{1}{2}+\mathbf{K})f=0$. Let $u(x,t)=\mathbf{S}f(x,t)$. Then $(\frac{\partial u}{\partial \nu})^+=0$ by Theorem 2.1. Hence

$$\begin{aligned}\pm\int_0^t\int_{\partial\Omega} u\left(\frac{\partial u}{\partial\nu}\right)^{\pm} d\sigma dt &= \int_0^t\int_{\Omega_\pm}\left(a_{sj}\frac{\partial u}{\partial x_s}\cdot\frac{\partial u}{\partial x_j}+\frac{\partial u}{\partial t}u\right)dxdt\\ &\geq \int_0^t\int_{\Omega_\pm}\mu|\nabla u|^2dxdt+\frac{1}{2}\int_{\Omega_\pm}|u(t)|^2\,dx.\end{aligned}\tag{3.9}$$

Hence $u=0$ in Ω_T. Therefore $u\big|_{S_T}=0$. Again by (3.9) $u=0$ in $\Omega^-\times(0,T)$. By the jump relation in Theorem 2.1, this implies that $f=0$ on S_T. Thus, $\frac{1}{2}+\mathbf{K}$ is one to one in $L^2(S_T)$.

As in [8], let $g\in L^2(S_T)$. We extend g to $\partial\Omega\times R$ by letting

$$h(P,t)=\begin{cases} g(P,t), & 0<t<T,\\ -g(P,t), & T<t<2T,\\ 0, & \text{elsewhere},\end{cases}$$

then $h\in L^2(\partial\Omega\times R)\bigcap L^1(\partial\Omega\times R)$. Let $\psi(P,\tau)$ be the partial Fourier transform of h,

$$\psi(P,\tau)=\int_0^\infty e^{-it\tau}h(P,t)dt.$$

Then $\psi(P,0)=0$ and $\psi(P,\tau)\in L^2(\partial\Omega)$ for each $\tau\in R$. By Parseval's Theorem

$$\|\psi\|_{L^2(\partial\Omega\times R)}=C\|h\|_{L^2(\partial\Omega\times R)}\leq C\|g\|.$$

Since $\frac{1}{2}+\widehat{\mathbf{K}}(\tau)$ is invertible on $L^2(\partial\Omega)$, there exists $\phi(P,\tau)\in L^2(\partial\Omega)$ such that

$$\left(\frac{1}{2}+\widehat{\mathbf{K}}(\tau)\right)\phi(P,\tau)=\psi(P,\tau)$$

for $\tau\neq 0$. If $\tau=0$, we may take $\phi(P,0)=0$ so that the above equality still holds. It is easy to check that $\|\widehat{\mathbf{K}}(\tau_1)-\widehat{\mathbf{K}}(\tau_2)\|\leq C|\tau_1-\tau_2|$ by Lemma 2.1 and that $\phi(P,\tau)$ is measurable on $\partial\Omega\times R$. By Lemma 3.4,

$$\int_{\partial\Omega}|\phi(P,\tau)|^2d\sigma_P\leq C\left(1+\frac{1}{|\tau|^2}\right)\int_{\partial\Omega}|\psi(P,\tau)|^2d\sigma_P.$$

An easy computation shows that $\phi\in L^2(\partial\Omega\times R)$ and $\|\phi\|\leq C\|g\|$.

Let $f\in L^2(\partial\Omega\times R)$ such that its partial Fourier transform is ϕ. Then for a.e. $(P,t)\in\partial\Omega\times R$,

$$h(P,t)=\frac{1}{2}f(P,t)+p.v\int_{-\infty}^t\int_{\partial\Omega}a_{kj}n_k(P)\frac{\partial}{\partial x_j}\Gamma(P-Q,t-\tau)f(Q,\tau)\,d\sigma_Q d\tau.$$

Since $h(P,t)=0$ for $t<0$, an argument similar to that in proving that $\frac{1}{2}+\mathbf{K}$ is one to one shows that $f(P,t)=0$ for $t<0$. Hence for a.e. $(P,t)\in S_T$

$$g(P,t)=\frac{1}{2}f(P,t)+p.v\int_0^t\int_{\partial\Omega}a_{kj}n_k(P)\frac{\partial}{\partial x_j}\Gamma(P-Q,t-\tau)f(Q,\tau)\,d\sigma_Q d\tau.$$

This proves that $\frac{1}{2}+\mathbf{K}$ is onto and hence invertible. The proof is complete.

§4. Invertibility of Single Layer Potential

As in [14], the invertibility of the single layer potential operator gives regularity results for the solution of the initial Dirichlet problem and also guarantees the existence of a Green's function with better properties than what we obtained for the solution of the Dirichlet problem in the class of $L^2(S_T)$. We will state these results in next section. The main result in this section is

Theorem 4.1. *The single layer potential operator defined in Section 2 is invertible from $L^2(S_T)$ to $L^2_{1,\frac{1}{2}}(S_T)$.*

The following lemmas will be needed in proving this theorem, Lemma 4.1 is a consequence of (3.9) and the proof of Lemma 4.2 can be found in [6].

Lemma 4.1. *Let $u = \mathbf{S}f$, for $f \in L^2(S_T)$. Then*

$$\int_0^T \int_{\Omega_\pm} \left|\nabla u\right|^2 dxdt \le C\|u\|\|\frac{\partial u^\pm}{\partial \nu}\|.$$

Lemma 4.2. *Let $f,\ g \in C^\infty(-\infty, T)$ and $f(t) = g(t) = 0$ for $t < 0$. Then*

$$|\int_0^T D_t^{\frac{1}{4}}(f)gdt| \le C\left(\int_0^T |f|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T |D_t^{\frac{1}{4}}g|^2 dt\right)^{\frac{1}{2}}$$

with C independent of T, f and g.

Lemma 4.3. *Let u be defined as above, h a vector field on $\partial\Omega$ with $\langle h, N\rangle > 0$. Then*

$$\int_0^T \int_{\Omega_\pm} \left(\left|D_t^{\frac{1}{4}}\nabla u\right|^2 + \left|D_t^{\frac{3}{4}}u\right|^2\right) dxdt \le C\|D_t^{\frac{1}{2}}u\|\|\frac{\partial u^\pm}{\partial \nu}\|$$

and

$$\int_0^T \int_{\Omega_\pm} h_j \frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x_j} dxdt \le C\|D_t^{\frac{1}{2}}u\|\|\frac{\partial u^\pm}{\partial \nu}\|.$$

Proof. Let W, Z be solutions of (2.1) with initial data 0. Integrating by parts, we get

$$\begin{aligned}\int_0^T \int_{\partial\Omega} W\frac{\partial Z^\pm}{\partial \nu} d\sigma dt &= \pm\int_0^T\int_{\Omega_\pm} \frac{\partial}{\partial x_s}\left(Wa_{sj}\frac{\partial Z}{\partial x_j}\right) dxdt \\ &= \pm\int_0^T\int_{\Omega_\pm}\left(\frac{\partial W}{\partial x_s}\cdot a_{sj}\cdot\frac{\partial Z}{\partial x_j} + W\cdot\frac{\partial Z}{\partial t}\right) dxdt.\end{aligned} \tag{4.1}$$

Setting $W = Z = D_t^{\frac{1}{4}}u$ and using Lemma 4.2, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega_\pm}\left|D_t^{\frac{1}{4}}\nabla u\right|^2 dxdt \le& \int_0^T\int_{\partial\Omega} D_t^{\frac{1}{4}}\left(\frac{\partial u}{\partial \nu}\right)^\pm D_t^{\frac{1}{4}} u d\sigma dt \\ \le& C\int_{\partial\Omega}\left(\int_0^T\left|D_t^{\frac{1}{2}}u\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T\left|\left(\frac{\partial u}{\partial\nu}\right)^\pm\right|^2 dt\right)^{\frac{1}{2}} d\sigma.\end{aligned}$$

If we set $Z = I_{\frac{1}{4}}u$ and $W = D_t^{\frac{3}{4}}u$, then

$$\int_0^T\int_{\Omega_\pm}\left|D_t^{\frac{3}{4}}\nabla u\right|^2 dxdt \le \|D_t^{\frac{1}{2}}u\|\|\frac{\partial u^\pm}{\partial\nu}\|.$$

The first part of the lemma follows. The second part follows from Lemma 4.2 and the first part.

Lemma 4.4. *Let $u = \mathbf{S}f$, for $f \in L^2(S_T)$. Then*

$$\|\frac{\partial u^{\pm}}{\partial \nu}\| \le C\left\{\|\nabla_T u\| + \|u\| + \|D_t^{\frac{1}{2}} u\|\right\}.$$

Proof. As in the proof of Lemma 3.3

$$\begin{aligned}&\int_0^T \int_{\partial\Omega} n_k(h_k a_{sj} - h_s a_{kj} - h_j a_{sk})\frac{\partial u^{\pm}}{\partial x_s}\cdot\frac{\partial u^{\pm}}{\partial x_j} d\sigma dt \\ &= \pm\int_0^T \int_{\Omega_\pm} b_{sj}\frac{\partial u}{\partial x_s}\cdot\frac{\partial u}{\partial x_j} - 2h_j\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x_j} dxdt,\end{aligned} \tag{4.2}$$

where $b_{sj} = \frac{\partial}{\partial x_k}(h_k a_{sj} - h_s a_{kj} - h_j a_{sk})$. Similar to (3.3), we get

$$\begin{aligned}&\int_0^T \int_{\partial\Omega} |\nabla u|^2 d\sigma dt \\ \le& C\left\{\int_0^T \int_{\partial\Omega} |\nabla_T u||u| d\sigma dt + \int_0^T \int_{\Omega_\pm} |\nabla u|^2 dxdt + \int_0^T \int_{\Omega_\pm} h_j\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x_j} dxdt\right\} \\ \le& C\left\{\|\nabla_T u\|\|u\| + \|u\|\|\frac{\partial u^{\pm}}{\partial \nu}\| + \|D_t^{\frac{1}{2}} u\|\|\frac{\partial u^{\pm}}{\partial \nu}\|\right\}.\end{aligned}$$

The Lemma follows easily.

Proof of Theorem 4.1. Note that for any $f \in L^2(S_T)$, by Lemma 4.4,

$$\|f\| \le \|\frac{\partial u^+}{\partial \nu}\| + \|\frac{\partial u^-}{\partial \nu}\| \le C\left\{\|\nabla_T u\| + \|D_t^{\frac{1}{2}} u\| + \|u\|\right\} \le C\|\mathbf{S}f\|_{L^2_{1,\frac{1}{2}}(S_T)}. \tag{4.3}$$

Hence, the single layer potential operator $\mathbf{S}$ is one to one with closed range in $L^2_{1,\frac{1}{2}}(S_T)$. We only need to show that the range is dense.

If Ω is smooth and $g \in C^\infty$, by the results in [13, Chapter 5], there exists a u such that

$$u\big|_{S_T} = g\big|_{S_T}, u\big|_{t=0} = 0 \ \ and \ \ u \in C^\infty(\overline{\Omega} \times [0, T]).$$

Hence $\frac{\partial u^+}{\partial \nu}$ exists and belongs to $L^2(S_T)$. By Theorem 3.1, we can find an $f \in L^2(S_T)$ such that $\frac{\partial(\mathbf{S}f)^+}{\partial \nu} = \frac{\partial u^+}{\partial \nu}$. Hence $u = \mathbf{S}f$ by the uniqueness results in [13, Chapter 5]. Therefore $\mathbf{S}f\big|_{S_T} = g\big|_{S_T}$. This proves that the single layer potential operator is invertible if the domain is smooth. For general Lipschitz domain, the theorem can be proven by the standard method in [2] and the inequality (4.3).

§5. Results for Initial Boundary Value Problems

In the first four sections, we constructed the layer potential operators for the parabolic equation (2.1) and studied the properties of these operators. In this section, we will sum up the results and apply these results to the parabolic equation. The invertibility of these operators guarantees the existence of solutions for the initial boundary value problems. We have

Theorem 5.1. *There exists a unique solution for the initial-Dirichlet problem with boundary data g in $L^2(S_T)$ and the solution u can be written as $u = \mathbf{D}f$ for some $f \in L^2(S_T), \|f\| \le C\|g\|$ with C independent of g.*

For the initial-Neumann problem, we have

Theorem 5.2. *There exists a unique solution for the initial-Neumann problem with boundary data g in $L^2(S_T)$ and the solution u can be written as $u = \mathbf{S}f$ for some $f \in L^2(S_T), \|f\| \le C\|g\|$ with C independent of g.*

The invertibility of the single layer potential operator gives

Theorem 5.3. *For $g \in L^2_{1,\frac{1}{2}}(S_T)$, the unique solution of the initial-Dirichlet problem with boundary data g can be written as $u = \mathbf{S}f$ for some $f \in L^2(S_T), \|f\| \le C\|g\|_{L^2_{1,\frac{1}{2}}}$ with C independent of g. Hence $\|(\nabla u)^*\| + \|\left(D_t^{\frac{1}{2}}\right)^*\| \le C\|g\|$.*

Proof of the Theorems. Theorem 5.2 is a consequence of Theorem 4.1 and Theorem 2.1. The uniqueness follows by a standard procedure. To prove Theorem 5.1, we only need to note that if we define $Rf(t) = f(T-t)$, then $R\left(\frac{1}{2} - \mathbf{K}\right)^* R = \frac{1}{2} + \widetilde{\mathbf{K}}$. Therefore the invertibility of $\frac{1}{2} - \mathbf{K}$ implies the invertibility of $\frac{1}{2} + \widetilde{\mathbf{K}}$ and hence the solvability of the initial-Dirichlet problem. The uniqueness follows by first constructing a Green's function G with property that $\|(\nabla G)^*\| + \|(D_t^{\frac{1}{2}} G)^*\| < \infty$ and then following the standard argument as in [15].

Theorem 5.3 is a combination of Theorem 4.1 and Theorem 5.1.

For the initial-boundary value problems in exterior domain, we may obtain similar results by using the results in Sections 3 and 4. We omit the details.

Acknowledgements. I thank Professor Wang Rouhuai for his constant encouragement and great help and thank Dr. R. Brown for his helpful suggestions.

References

[1] Coifman, R. R., David, G. & Meyer, Y., L'intégrale de Cauchy définite un opérateur borné sur L^2 pour les courbes Lipschitziennes, *Ann. of Math.,* **116** (1982), 361–388.
[2] Verchota, G. C., Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains, *J. Funct. Anal.,* **59** (1984), 572–611.
[3] Nečas, J., Les méthodes directes en théories d'équations elliptiques, Academia Aragues, 1967.
[4] Dahlberg, B. E. J., Kenig, C. E. & Verchota, G. C., Boundary value problems for the systems of elastostatics in a Lipschitz domain, *Duke Math. J.,* **57** (1988), 795–818.
[5] Fabes,E. B., Kenig, E. C. & Verchota, G. C., *The Dirichlet problem for the Stokes system on Lipschitz domains, Duke Math. J.,* **57** (1988), 769–793.
[6] Brown, R. M., Layer potentials and boundary value problems for the heat equation on Lipschitz cylinders, Thesis, Uiversity of Minnesota, 1987.
[7] Brown, R. M., The method of layer potentials for the heat equation in Lipschitz cylinders, *Amer. J. Math.,* **111** (1989), 339–379.
[8] Shen, Z., Boundary value problems for parabolic Láme Systems and a Nonstationary Linearized System of Navier-Stokes Equations in Lipschitz cylinders, *Amer. J. Math.,* **113** (1991), 293–373.
[9] Costable, M., Boundary integral operators for the heat equation, *Integral Equations and Operator Theory,* **13** (1990), 498–552.
[10] Costable, M., Boundary integral operators on Lipschitz domains: Elementary results, *SIAM J. Math. Anal.,* **19** (1988), 613–626.
[11] Dahlberg, B. E. J., Kenig, C. E. & Verchota, G. C., The Dirichlet problem for the biharmonic equation in a Lipschitz domain, *Ann. Inst. Fourier* (*Grenoble*)*,* **36** (1986), 109–135.
[12] Gao, W., Layer potentials and boundary value problems for elliptic systems in Lipschitz domains, *J. Funct. Anal.,* **95** (1991), 377–399.
[13] Friedman, A., Partial differential equations of parabolic type, Prentice–Hall, Inc, Englewood Cliffs, N.J., 1964.
[14] Gao, W., Boundary value problems on Lipschitz domains for general Elliptic systems, Thesis, University of Minnesota, 1988.
[15] Fabes, E. B. & Riviere, N. M., Dirichlet and Neumann problem for the heat equation in C^1-cylinders, *Proceeding of Symposia in pure Math.,* **35**.