Chin. Ann. of Math.
15B: 1(1994),53-58.

SOME RESULTS ABOUT COVERING PROPERTIES OF PRODUCTS**

JIANG JIGUANG*   TENG HUI*

Abstract

The following results are proved:

1. Let X be a screenable P-space and Y a metric space. Then $X \times Y$ is screenable.
2. Let X be a strongly submetacompact P-space and Y a metacompact Σ-space. Then $X \times Y$ is strongly submetacompact.
3. Suppose that $X = \prod_{n<\omega} X_n$ is countably paracompact and for each $n < \omega$, $\prod_{i<n} X_i$ is submeta-Lindelöf. Then X is submeta-Lindelöf.

Keywords Product space, Covering properties, Screenable P-space, Metric space.
1991 MR Subject Classification 54E18, 54E35.

§1. Introduction

The study of covering properties of products is an essential subject in the theory of product spaces. For understanding the development of this aspect, see D. K. Burke's survey paper [2]. Most recently, K. Chiba made a further discussion and posed several questions, two of which are:

A ([3], Question 3). Let X be a screenable P-space and Y a metric space. Is $X \times Y$ screenable?

B ([3], Question 5). Suppose that $X = \prod_{n<\omega} X_n$ is countably paracompact and for each $n < \omega$, $\prod_{i<n} X_i$ is submeta-Lindelöf. Is X submeta-Lindelöf?

In this paper we shall prove three theorems; the first and the last ones give answers to the two questions A and B respectively while the second one is about strong submetacompactness which improves the corresponding result of [3].

A space X is called screenable [1] if every open cover of X has a σ–mutually disjoint open refinement; A space X is said to be strong submetacompact[6] if for every open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{V}_n : n < \omega\}$ of open covers of X such that each $\mathcal{V}_n$ refines $\mathcal{U}$ and if $x \in X$, there is an $n_x < \omega$ such that Ord $(x, \mathcal{V}_n) < \omega$ whenever $n \geq n_x$. Such a sequence is called a strong θ–refine sequence of $\mathcal{U}$.

For convenience we shall not distinguish between "refinement" and "partial refinement" whenever no confusion is possible.

Manuscript received July 8, 1991.
*Institute of Mathematics, Sichuan University, Chengdu 610064, Sichuan, China.
**Project supported by the National Natural Science Foundation of China.

§2. Theorems and Proofs

Theorem 2.1. *Let X be a screenable P–space and Y a metric space. Then $X \times Y$ is screenable.*

To prove this theorem, we need the following well-known lemma.

Lemma 2.1. *Suppose that Y is a metric space. Then X has a collection $\{M(s) : s \in \Omega^{<\omega}\}$ of open subsets, where Ω is an index set, which satisfies*

1° *For each $n < \omega$, $\{M(s) : s \in \Omega^n\}$ is a σ-discrete open cover of Y. Denote $\Omega^n = \bigcup\{A_{n,m} : m < \omega\}$ such that each $\{M(s) : s \in A_{n,m}\}$ is discrete.*

2° *For $s \in \Omega^n$,*

$$M(s) = \bigcup\{M(s) : s \vee \alpha) : \alpha \in \Omega\}.$$

Here if $s = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, then $s \vee \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha)$.

3° *For $y \in Y$, there is a $t \in \Omega^\omega$ such that $\{M(t \upharpoonright n) : n < \omega\}$ is a local base of Y at point y. Here if $t = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$, then $t \upharpoonright n = (\alpha_1, \cdots, \alpha_n)$.*

Proof of Theorem 2.1. Let $\{M(s) : s \in \Omega^{<\omega}\}$ be a collection of open sets of Y as that in Lemma 2.1. Let $\mathcal{U}$ be an arbitrary open cover of $X \times Y$ and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$. For $s \in \Omega^{<\omega}$ and $\lambda \in \Lambda$, let

$$H(s, \alpha) = \bigcup\{G : G \text{ is open in} X \text{and } G \times M(s) \subseteq U_\lambda\}$$

and

$$H(s) = \bigcup\{H(s, \lambda) : \lambda \in \Lambda) : \lambda \in \Lambda\}.$$

Then $H(s) \subseteq H(s \vee \alpha)$ for $s \in \Omega^{<\omega}$ and $\alpha \in \Omega$. Since X is a P-space, there is a closed collection $\{D(s) : s \in \Omega^{<\omega}\}$ such that

1) $D(s) \subseteq H(s)$ for each $s \in \Omega^{<\omega}$,

2) if $t \in \Omega^\omega$ with $\bigcup\{H(t \upharpoonright n) : n < \omega\} = X$, then $\bigcup\{D(t \upharpoonright n) : n < \omega\} = X$.

Since X is screenable, it is easy to see that for each $s \in \Omega^{<\omega}$ there is a σ–mutually disjoint collection $\mathcal{W}_s = \bigcup\{\mathcal{W}_{s,l} : l < \omega\}$ of open sets of X covering $D(s)$ and refining $\{H(s, \lambda) : \lambda \in \Lambda\}$. For m, n, $l < \omega$, let

$$\mathcal{V}_{m,n,l} = \{W \times M(s) : w \in W_{s,l} \text{ and } s \in A_{m,n}\}.$$

Then

3) $\mathcal{V}_{m,n,l}$ is a disjoint collection of open sets of $X \times Y$.

Let $\mathcal{V}_m = \bigcup\{\mathcal{V}_{m,n,l} : n, l < \omega\}$. Then

4) $\mathcal{V}_m$ is a σ-mutually disjoint open collection which covers

$$T_m = \bigcup\{D(s) \times M(s) : s \in \Omega^m\}.$$

Let $\mathcal{V} = \bigcup\{\mathcal{V}_m : m < \omega\}$. It remains to prove that $\mathcal{V}$ is an open cover of $X \times Y$ and refines $\mathcal{U}$. The latter is obvious. For proving the former it is sufficient to prove that

$$\bigcup\{T_m : m < \omega\} = X \times Y.$$

Let $(x_0, y_0) \in X \times Y$ be arbitrary. By Lemma 2.1 there is a $t \in \Omega^\omega$ such that $\{M(t \upharpoonright n) : n < \omega\}$ is a local base of Y at point y_0. So for each $x \in X$, there are an open neighbourhood G of x and an $n < \omega$ and a $\lambda \in \Lambda$ such that $G \times M(t \upharpoonright n) \subseteq U_\lambda$. By the definitions of $H(s, \lambda)$ and $H(s)$, we have $x \in H(t \upharpoonright n)$ and thus

5) $\bigcup\{H(t\restriction n) : n<\omega\}=X$, and

6) $\bigcup\{D(t\restriction n) : n<\omega\}=X$ by 2).

So there is an $n_0<\omega$ such that

$$(x_0,y_0)\in D(t\restriction n_0)\times M(t\restriction n_0)\subseteq T_{n_0}.$$

This proves that $\bigcup\{T_m : m<\omega\}=X\times Y$. Thus $X\times Y$ is screenable.

Lemma 2.2[5]. *Assume Y is a Σ–space. Then Y has a collection $\{F(s) : s\in\Omega^{<\omega}\}$ of closed sets, where Ω is an index set, which satisfies:*

a) *Each $\mathcal{F}_n=\{F(s) : s\in\Omega^n\}$ is a locally finite cover of Y.*

b) *For $s\in\Omega^{<\omega}$, $F(s)=\bigcup\{F(s\vee\alpha) : \alpha\in\Omega\}$.*

c) *For $y\in Y$, there is a $t\in\Omega^\omega$ such that $\{F(t\restriction n) : n<\omega\}$ converges to $\bigcap\{F(t\restriction n) : n<\omega\}$ which is countably compact and contains y.*

Theorem 2.2. *Let X be a strongly submetacompact P-space and Y a meta-compact Σ–space. Then $X\times Y$ is strongly submetacompact.*

For the proof of this theorem, we need yet another lemma about strong submetacompactness.

Lemma 2.3. *Let X be a space and $\mathcal{U}$ an open cover of it. If there is a closed cover $\{A_n : n<\omega\}$ such that for each $n<\omega$ there is a sequence $\{\mathcal{W}_{n,m} : m<\omega\}$ of open collections satisfying: each $\mathcal{W}_{n,m}$ covers A_n and refines $\mathcal{U}$, and for any fixed $n<\omega$ and any $x\in X$ there is an $m_x<\omega$ such that* ord $(x,\mathcal{W}_{n,m})<\omega$ *whenever $m\ge m_x$, then $\mathcal{U}$ has a strong θ–refine sequence.*

Proof. For $n,m<\omega$, let

$$\mathcal{V}_{0,m}=\mathcal{W}_{0,m},$$
$$\mathcal{V}_{n,m}=\{W\setminus\bigcup_{l=0}^{l=n-1}A_l : W\in\mathcal{W}_{n,m}\}$$

for $n>0$ and $\mathcal{V}_m=\bigcup\{\mathcal{V}_{n,m} : n<\omega\}$. Then $\mathcal{V}_m$ is an open cover of X and refines $\mathcal{U}$. We can prove that $\{\mathcal{V}_m : m<\omega\}$ is a strong θ–refine sequence of $\mathcal{U}$. In fact, assume $x\in X$. Denote $n_0=\min\{n<\omega : x\in A_n\}$. For $l\in\{0,1,\cdots,n_0\}$, there is an $m_l<\omega$ such that if $m\ge m_1$, ord $(x,\mathcal{W}_{l,m})<\omega$. Let $m_x=\max\{m_l : l=0,1,\cdots,n_0\}$. If $m\ge m_l$,

$$\text{ord }(x,\mathcal{V}_m)\le\sum_{l=0}^{l=n_0}\text{ord }(x,\mathcal{V}_{l,m}\le\sum_{l=0}^{l=n_0}\text{ ord }(x,\mathcal{W}_{l,m})<\omega.$$

The Lemma is thus proved.

Proof of Theorem 2.2. Let $\mathcal{U}$ be an open cover of $X\times Y$, $\{F(s) : s\in\Omega^{<\omega}\}$ a collection of closed sets of space Y as in Lemma 2.2. By Lemma 2.2, each $\mathcal{F}_n=\{F(s) : s\in\Omega^n\}$ is locally finite. Since Y is metacompact, there is a point-finite collection $\mathcal{G}_n=\{G(s) : s\in\Omega^n\}$ of open sets of Y such that $F(s)\subseteq G(s)$ for each $s\in\Omega^n$. For $s\in\Omega^n$, let $\mathcal{H}(s)=\{H : H$ is open in X and there exists an open set $W(H)$ of Y with $F(s)\subseteq W(H)\subseteq G(s)$ such that $H\times W(H)$ is covered by some finite subfamily $\mathcal{U}(H)$ of $\mathcal{U}\}$. Let $H(s)=\bigcup\mathcal{H}(s)$. Then $H(s)\subseteq H(s\vee\alpha)$ for every $s\in\Omega^{<\omega}$. Since X is a P-space, there is a collection $\{D(s) : s\in\Omega^{<\omega}\}$ of closed sets such that

7) $D(s)\subseteq H(s)$ for each $s\in\Omega^{<\omega}$,

8) if for some $t \in \Omega^\omega$,

$$\bigcup\{H(t \upharpoonright n) : n < \omega\} = X,$$

then

$$\bigcup\{D(t \upharpoonright n) : n < \omega\} = X.$$

Since X is strongly submetacompact, it is easy to prove that for each $s \in \Omega^{<\omega}$, there is a sequence $\{\mathcal{A}_{s,m} : m < \omega\}$ of open collections of X such that each $\mathcal{A}_{s,m}$ covers $D(s)$ and refines $\mathcal{H}(s)$ and if $x \in X$, there is an $m_x < \omega$, ord $(x, \mathcal{A}_{s,m}) < \omega$ whenever $m \geq m_x$. We can assume that

$$\mathcal{A}_{s,m} = \{A(s,m,H) : H \in \mathcal{H}(s)\}$$

and $A(s,m,H) \subseteq H$ for $H \in \mathcal{H}(s)$. Let

$$A_n = \bigcup\{D(s) \times F(s) : s \in \omega^n\}.$$

Then

9) $\{A_n : n < \omega\}$ is a closed cover of $X \times Y$.

In fact, let $(x_0, y_0) \in X \times Y$ be arbitrary. By Lemma 2.2, there is a $t \in \Omega^\omega$ such that $\{F(t \upharpoonright n) : n < \omega\}$ converges to the compact set $\bigcap\{F(t \upharpoonright n) : n < \omega\}$ which contains y_0. For any $x \in X$, there are an open neighbourhood H of x, $n < \omega$, an open set W of Y with $F(t \upharpoonright n) \subseteq W$ and a finite subfamily $\mathcal{U}(H)$ of $\mathcal{U}$ such that $H \times W$ is covered by $\mathcal{U}(H)$. Let $W(H) = W \cap G(t \upharpoonright n)$. Then by the definition of $H(s)$, $x \in H \subseteq H(t \upharpoonright n)$. It is proved that $\bigcup\{H(t \upharpoonright n) : n < \omega\} = X$. By 8), $\bigcup\{D(t \upharpoonright n) : n < \omega\} = X$. So there is an $n_0 < \omega$ such that $x_0 \in D(t \upharpoonright n_0)$ and thus

$$(x_0, y_0) \in D(t \upharpoonright n_0) \times F(t \upharpoonright n_0) \subseteq A_{n_0}.$$

This proves 9).

Let

$$\mathcal{V}'_{s,m} = \{A(s,m,H) \times W(H) : H \in \mathcal{H}(s)\}$$

and

$$\mathcal{V}'_{n,m} = \bigcup\{\mathcal{V}'_{s,m} : s \in \Omega^n\}.$$

Then

10) each $\mathcal{V}'_{n,m}$ covers A_n,

11) if $(x,y) \in X \times Y$, then there is an $m_{(x,y)} < \omega$ such that when $m \geq m_{(x,y)}$, ord $((x,y), \mathcal{V}'_{n,m}) < \omega$.

10) follows immediately from the definition of $\mathcal{V}'_{n,m}$. Let $(x,y) \in X \times Y$. Since $\mathcal{G}_n$ is point–finite, there are $s_1, s_2, \cdots, s_1 \in \Omega^n$ such that, if $s \in \Omega^n \backslash \{s_1, s_2, \cdots, s_1\}$, $y \notin G(s)$. For each s_i there is an $m_i < \omega$ such that, when $m \geq m_i$, ord $(x, \mathcal{A}_{s_i,m}) < \omega$ and thus ord $((x,y), \mathcal{V}'_{s_i,m}) < \omega$. Let $m_{(x,y)} = \max\{m_1, m_2, \cdots, m_l\}$. Then if $m \geq m_{(x,y)}$,

$$\text{ord } ((x,y), \mathcal{V}'_{n.m}) \leq \sum_{i=1}^{i=l} \text{ord } ((x,y), \mathcal{V}'_{s_i,m}) < \omega.$$

Now let

$$\mathcal{V}_{s,m} = \{(A(s,m,H) \times W(H)) \cap U : U \in \mathcal{U}(H) \text{ and } H \in \mathcal{H}(s)\}$$

and

$$\mathcal{V}_{n,m} = \bigcup\{V_{s,m} : s \in \Omega^n\}.$$

Then

12) each $V_{n,m}$ covers A_n and refines $\mathcal{U}$.

13) if $(x, y) \in X \times Y$, there is an $m_{(x,y)} < \omega$ such that, if $m \geq m_{(x,y)}$,

$$\mathrm{ord}((x, y), \mathcal{V}_{n,m}) < \omega.$$

By Lemma 2.3, $\mathcal{U}$ has a strong θ-refine sequence. So $X \times Y$ is strongly submetacompact.

Theorem 2.2 much improves the corresponding results of [3].

Corollary 2.1 ([3], Theorem 5). *Let X be a strongly submetacompact normal P–space and Y a metric space. Then $X \times Y$ is strongly submetacompact.*

Corollary 2.2 ([3], Theorem 4). *Let X be a strongly submetacompact normal P-space and Y a strongly submetacompact M–space.* (*i.e. paracompact M–space*). *Then $X \times Y$ is strongly submetacompact.*

The last theorem is about submeta–Lindelöfness[7] of infinite products.

Theorem 2.3. *Suppose $X = \prod_{n<\omega} X_n$ is countably paracompact and for each $n < \omega$, $\prod_{i<n} X_i$ is submeta–Lindelöf. Then X is submeta–Lindelöf.*

Lemma 2.4. *Let X be a space and $\mathcal{U}$ be an open cover of X. If there is a closed cover $\{A_n : n < \omega\}$ of X such that for each $n < \omega$, there is a sequence $\{\mathcal{W}_{n,m} : m < \omega\}$ of open collections such that each $\mathcal{W}_{n,m}$ refines $\mathcal{U}$ and covers A_n and for every point $x \in A_n$, there is an m_x with* $\mathrm{ord}(x, \mathcal{W}_{n,m_x}) \leq \omega$, *then there is a sequence $\{\mathcal{V}_n : n < \omega\}$ of open refinements of $\mathcal{U}$ such that for any $x \in X$, there is an n_x with* $\mathrm{ord}(x, \mathcal{V}_{n_x}) \leq \omega$.

Proof. For n, $m < \omega$, let $\mathcal{V}_{n,m} = \mathcal{W}_{n,m} \cup \mathcal{U}|_{X \setminus A_n}$. Then $\mathcal{V}_{n,m}$ is an open cover of X and refines $\mathcal{U}$. Renumerate $\{\mathcal{V}_{n,m} : n, m < \omega\}$ as $\{\mathcal{V}_n : n < \omega\}$. Then this sequence is the required one.

Proof of Theorem 2.3. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of X. For $n < \omega$, denote by $p_n : X \to \prod_{i<n} X_i$ the projection. For $n < \omega$ and $\lambda \in \Lambda$, let

$$W_{n,\lambda} = \bigcup\{W : W \text{ is open in } \prod_{i<n} X_i \text{ and } p_n^{-1}(W) \subseteq U_\lambda\}.$$

Let $G_n = \bigcup\{W_{n,\lambda} : \lambda \in \Lambda\}$. It can be proved that $\{p_n^{-1}(G_n) : n < \omega\}$ is an increasing open cover of X. Since X is countably paracompact, there is an increasing open cover $\{H_n : n < \omega\}$. of X such that $\overline{H}_n \subseteq p_n^{-1}(G_n)$ for each $n < \omega$. Let $T_n = \prod_{i<n} X_i \setminus p_n(X \setminus \overline{H}_n)$. Then T_n is a closed set of $\prod_{i<n} X_i$ and $\{p_n^{-1}(T_n) : n < \omega\}$ is a cover of X. Since $T_n \subseteq G_n$ and $\prod_{i<n} X_i$ is submeta–Lindelöf, there is a sequence $\{\mathcal{W}'_{n,m} : m < \omega\}$ of open collections of $\prod_{i<n} X_i$ such that each $\mathcal{W}'_{n,m}$ refines $\{W_{n,\lambda} : \lambda \in \Lambda\}$ and covers T_n and for every point $y \in T_n$, there is an m_y such that ord $(y, \mathcal{W}'_{n,m_y}) \leq \omega$. Let

$$\mathcal{W}_{n,m} = \{p_n^{-1}(W) : W \in \mathcal{W}'_{n.m}\}, \quad A_n = p_n^{-1}(T_n).$$

Then $\{A_n : n < \omega\}$ and for each $n < \omega$ the sequence $\{\mathcal{W}_{n,m} : m < \omega\}$ is as that in Lemma 2.4. This proves that X is submeta-Lindelöf.

References

[1] Bing, R. H., Metrization of topological spaces, *Canad. J. Math.,* **3** (1951), 175-186.

[2] Burke, D. K., Covering properties, in Handbook of set–theoretic topology, Chapter 9, K. Kunen & J. Vaughan, editors, North Holland, Amsterdam, 1984, 347-422.

[3] Chiba, K., Covering properties in products, *Math. Japonica,* **34** (1989), 693-713.

[4] Morita. K., Products of normal spaces with metric spaces, *Math. Anal.,* **154** (1964), 365-382.

[5] Nagami, K., Σ-spaces, *Fund. Math.,* **65** (1969), 169-192.

[6] Uemura, Y., On products of generalized paracompact spaces, *Memories of Osaka Kyoiku Univ.,* **24** (1975), 21-24.

[7] Worrel, J. M., & Wicke, H. H., A covering property which implies isocompactness I, *Proc. Amer. Math. Soc.,* **79** (1980), 331-334.