Chin. Ann. of Math.
15B: 1(1994),59-68.

OSCILLATION OF A FORCED SECOND ORDER NONLINEAR EQUATION***

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Abstract

This paper gives several criteria on the oscillatory behavior of solutions of the forced second order equation $x'' + a(t)f(x) = g(t)$, where $g(t)$ is oscillatory, by using a geometric idea. As special cases these results include and improve some recent results, given by J. S. Wong. The criteria also solve the problem posed by H. Onose in Mathematical Reviews, 1986.

Keywords Oscillation, Forced, Nonlinear Differential equations.
1991 MR Subject Classification 34C10.

§1. Introduction

We are concerned here with the second order nonlinear differential equation

$$x'' + a(t)f(x) = g(t), \quad t \in [0, \infty), \tag{1.1}$$

where $a(t), g(t)$ are real-valued piecewise continuous functions on $[0, \infty)$ and $f(x)$ is a continuous and nondecreasing function of x on $(-\infty, \infty)$. We assume further that the functions $a(t), g(t)$ and $f(x)$ are so smooth that Equation (1.1) always has solutions which are continuable throughout $[0, \infty)$. Equation (1.1) is said to be oscillatory if every solution $x(t)$ of (1.1) is oscillatory, i.e., there is a sequence $t_n \to \infty$ such that $x(t_n) = 0$.

There is a lot of work done on the oscillation of Equation (1.1) (see [2-9]). As in many papers we use Kartsatos's technique introduced by Kartsatos[2,3], i.e., assume that there exists a function $h(t)$ such that $h''(t) = g(t)$, and hence reduce Equation (1.1) to a homogeneous equation. There have been some results by Wong[9] which cover and improve a series of previous results with the same technique, but they are only concerned with the case that $h(t)$ is oscillatory and some exacting restrictions are posed on $h(t)$. Here from a geometric thought, rather than an analytic one, we obtain several criteria for the more general case that $g(t)$ is oscillatory no matter whether $h(t)$ is oscillatory or nonoscillatory.

By examples we see that under the above general assumptions our conditions are sharper than those by Wong[9]. Just as indicated by Onose, MR86c34133, although equation $x'' + 4x = 1/2 + \sin t$ is oscillatory, there are very few results which can be applied to this equation, we will also show by an example that one of our results applies to it.

Manuscript received June 4, 1991.
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***Project supported by the Nationa Natural Science Foundation of China.

§2. Preliminaries

Assume that $h \in C^2[t_0,\infty)$ is oscillatory for $t_0 \geq 0$, $h(t) \not\equiv 0$ on any $[T,\infty) \subset [t_0,\infty)$. Let $s_n\,(n=1,2,\cdots,)$ be the local maximal points of h on $[t_0,\infty)$, $s_1 < s_2 < \cdots$; and $\tau_n\,(n=1,2,\cdots)$be the local minimal points of h on $[t_0,\infty)$, $\tau_1<\tau_2<\cdots$. We give several curves related to the function h and their properties which are useful in determining the oscillation of (1.1).

Definition 2.1. $y=\phi(t)$, $t\in[s_0,\infty)$, *is called the upper broken-line* (UBL) *of* h *on* $[t_0,\infty)$ *provided*

(i) $\phi(s_n)=h(s_n)$, $n=1,2,\cdots$,

(ii) $\phi(t)=\phi(s_n)+\frac{\phi(s_{n+1}-\phi(s_n)}{s_{n+1}-s_n}(t-s_n)$, $\ t\in(s_n,s_{n+1})$.

Definition 2.2. $y=\phi(t)$, $t\in[s_1,\infty)$, *is called the increasing upper broken-line* (IUBL) *of* h *on* $[t_0,\infty)$ *provided*

(i) $\phi(s_1)=h(s_1)$;

(ii) *suppose* $\phi(s_i)=h(s_i)$ *for some* $i\geq 1$; *if* $h(s_n)\leq h(s_i)$ *for all* $n>i$, *then* $\phi(t)=h(s_i)$, $t>s_i$; *otherwise, let* $s_k=\min\{s_n: h(s_n)>h(s_i)\}$, *then*

$$\phi(t)=\phi(s_i)+\frac{\phi(s_k)-\phi(s_i)}{s_k-s_i}(t-s_i),\ t\in(s_i,s_k].$$

Definition 2.3. *For any constant* α, $y=\phi(t)$, $t\in[s_1,\infty)$, *is called the* α*-upper broken-curve* ($\alpha-$UBC) *of* h *on* $[t_0,\infty)$ *provided*

(i) *if* $s_{n+1}\geq s_n+\alpha(s_{n+1}-s_n)$, *then*

$$\phi(t)=\max\{h(t),h(s_n)+\alpha(t-s_n)\},\ t\in[s_n,s_{n+1}];$$

(ii) *if* $s_{n+1}<s_n+\alpha(s_{n+1}-s_n)$, *then*

$$\phi(t)=\max\{h(t),h(s_{n+1})+\alpha(t-s_{n+1})\},\ t\in[s_n,s_{n+1}],\quad n=1,2,\cdots.$$

Definition 2.4. *For any constant* $k>0$, $y=\phi(t)$, $t\in[s_1,\infty)$ *is called the* k*-upper parabolic curve* ($k-$UPC) *of* h *on* $[t_0,\infty)$ *provided*

(i) *for* $n=1,2,\cdots$, *there exist* b_n *and* c_n *such that* $\phi_n(t)=kt^2/2+b_nt+c_n$ *satisfying* $\phi_n(t)\geq h(t)$, $t\in[s_n,s_{n+1}]$, *and* $y=\phi_n(t)$ *is tangent to* $y=h(t)$ *at two points* s'_n, s'_{n+1}, *where* $s_n\leq s'_n<s'_{n+1}\leq s_{n+1}$;

(ii)

$$\phi(t)=\begin{cases}\phi_n(t), & t\in[s'_n,s'_{n+1}],\\ h(t), & t\in[s_n,s'_n)\cup(s'_{n+1},s_{n+1}],\quad n=1,2,\cdots.\end{cases}$$

Definition 2.5. $y=\psi(t)$, $t\in[\tau_1,\infty)$, *is called the lower broken-line* (LBL)*, the decreasing lower broken-line* (DLBL)*, the* α*-lower broken-curve* ($\alpha-$LBC)*, or the* k*-lower parabolic curve* ($k-$LBC) *of* h *on* $[t_0,\infty)$, *provided* $y=\psi(t)$ *is the* UBL*, the* $\alpha-$UBL *or the* $k-$UPC *of* $-h$ *on* $[t_0,\infty)$, *respectively.*

Definition 2.6. IUBL $y=\phi(t)$ *or* DLBL $y=\psi(t)$ *of* h *on* $t\in[t_0,\infty)$ *is called eventually constant if there exists an* s_n *or a* τ_n *such that* $\phi(t)\equiv h(s_n)$, $t\geq s_n$, *or* $\psi(t)\equiv h(\tau_n)$, $t\geq\tau_n$, *respectively.*

Obviously, except for the IUBL and DLBL of h, every one of UBL, $\alpha-$UBC, $k-$UPC, and LBL, $\alpha-$LBC, $k-$LPC intersects $y=h(t)$ infinitely many times.

Lemma 2.1. *Suppose that for any $T \geq t_0$ the* IUBL $y = \phi(t)$ *of h on $[T, \infty)$ is eventually constant and $y(t),\ t \in [t_0, \infty)$, is a function satisfying $y(t) > h(t),\ y'(t) > 0$ eventually. Then there is a constant $c > 0$ such that $y(t) - h(t) \geq c$ eventually.*

Proof. Without loss of generality we may assume $y(t) > h(t),\ y'(t) > 0,\ t \in [T, \infty)$, and $\phi(t) \equiv \phi(s_1) = h(s_1),\ t \in [s_1, \infty)$, where s_1 is the least local maximum point of h. Then $\phi(t) > h(t),\ t \in [s_1, \infty)$. Since $y(t)$ is increasing, $y(t) \geq y(s_1) > h(s_1),\ t \geq s_1$. Hence for $t \geq s_1$,

$$y(t) - h(t) \geq y(t) - \phi(t) = y(t) - h(s_1) \geq y(s_1) - h(s_1) = c > 0.$$

Lemma 2.2. *Suppose that $y = \phi(t)$ is the* UBL *or* IUBL (*not eventually constant*) *of h on $[t_0, \infty)$ and $y(t),\ t \in [t_0, \infty)$, is a function satisfying $y(t) > h(t),\ y''(t) \leq 0$ eventually. Then $y(t) \geq \phi(t)$ eventually.*

Proof. If $y(t) \leq \phi(t)$ eventually, noting that $y = \phi(t)$ intersects $y = h(t)$ infinitely many times, we see that either $y(t) \leq h(t)$ eventually or $y = y(t)$ intersects $y = h(t)$ infinitely many times, contradicting the assumption.

If there exist $t_n \to \infty$ such that $y(t_n) < \phi(t_n)$ and $y = y(t)$ crosses $y = \phi(t)$ infinitely many times, then for a sufficiently large n there must be at least two cross points t_n' and t_n'' such that

(i) $s_n < t_n' < t_n < t_n'' < s_{n+1}$,

(ii) $y'(t_n') \leq \phi'(t_n')$ and $y'(t_n'') \geq \phi'(t_n'')$,

(iii) $\phi'(t) \equiv \phi'(t_n')$ for $t \in [t_n', t_n'']$.

Hence $y'(t_n') \leq y'(t_n'')$. Noting that $y''(t) \leq 0$ we have $y'(t) \equiv y'(t_n')$ for $t \in [t_n', t_n'']$, so $y(t) \equiv \phi(t),\ t \in [t_n', t_n'']$, contradicting $y(t_n) < \phi(t_n)$.

Lemma 2.3. *Suppose that $y = \psi(t)$ is the $\alpha-$*LBC *of h on $[t_0, \infty)$ for some $\alpha > 0$, and $y(t),\ t \in [t_0, \infty)$, is a function satisfying $y(t) < h(t)$ and $y'(t) < \alpha$ eventually. Then $y(t) \leq \psi(t)$ eventually.*

Proof. Without loss of generality we assume $y(t) < h(t),\ y'(t) < \alpha,\ t \geq t_0$. Then $y(\tau_n) < h(\tau_n),\ n = 1, 2, \cdots$. In view of that $y'(t) < \alpha$ we see that $y(t) < h(\tau_n) + \alpha(t - \tau_n)$ and

$$y(t) < h(\tau_{n+1}) + \alpha(t - \tau_{n+1}),\ t \in [\tau_n, \tau_{n+1}],\ n = 1, 2, \cdots .$$

Therefore $y(t) \leq \psi(t),\ t \geq t_0$.

Lemma 2.4. *Suppose that $y = \psi(t)$ is the $k-$*UBC *of h on $[t_0, \infty)$ for some $k > 0$, and $y(t),\ t \in [t_0, \infty)$, is a function satisfying $y(t) < h(t)$ and $y''(t) \leq k$ eventually. Then $y(t) \geq \phi(t)$ eventually.*

Proof. Similar to the proof of Lemma 2.2 we see that it is impossible that $y(t) \leq \phi(t)$ eventually.

If there exist $t_n \to \infty$ such that $y(t_n) < \phi(t_n)$ and $y = y(t)$ crosses $y = \phi(t)$ infinitely many times, then for a sufficiently large n there must be at least two cross points t_n' and t_n'' such that

(i) $s_n < t_n' < t_n < t_n'' < s_{n+1}$,

(ii) $y'(t_n') \leq \phi'(t_n')$ and $y'(t_n'') \geq \phi'(t_n'')$,

(iii) $\phi''(t) \equiv k$ for $t \in [t_n', t_n'']$.

Hence

$$\phi'(t''_n) - \phi(t'_n) = k(t''_n - t'_n)$$

and

$$y'(t''_n) - y'(t'_n) \geq \phi'(t''_n) - \phi'(t'_n).$$

Noting that $y''(t) \leq k$ we have

$$y'(t) - y'(t'_n) = k(t - t'_n),\ t \in [t'_n, t''_n].$$

Thus $y'(t) \equiv \phi'(t)$ and hence $y(t) \equiv \phi(t)$, for $t \in [t'_n, t''_n]$, since $y(t'_n) = \phi(t'_n)$, contradicting $y(t_n) < \phi(t_n)$.

Similarly we can get the corresponding results to Lemmas 2.1-2.4 for DLBL, LBL, $\alpha-$UBC, and $k-$LPC of h on $[t_0, \infty)$, respectively. We omit them here.

§3. Main Results

Now we turn to the discussion of the oscillation of the forced second order Equation (1.1)

$$x'' + a(t)f(x) = g(t), \quad t \in [0, \infty),$$

which satisfies the hypothesis given at the beginning of this paper and also the hypothesis

(H_0) $a(t)$ is nonnegative but not eventually zero on $[0, \infty)$, $f(x)$ is continuous and nondecreasing satisfying $xf(x) > 0,\ x \neq 0$.

We will give several different oscillation criteria for Equation (1.1) based on different oscillatory properties of function g. We will discuss three cases for (1.1).

(A) At first we post the following hypothesis:

(H_1) There exists an $h \in C^2[0, \infty)$ such that $h''(t) = g(t)$ and $h(t)$ is oscillatory. Let $x(t) = y(t) + h(t)$; then Equation (1.1) can be rewritten as a homogeneous equation

$$y'' + a(t)f(y + h(t)) = 0. \tag{3.1}$$

As shown in [9], if $x(t) > 0$ for $t \geq t_0 \geq 0$, then there exists a large t, say t_0, such that $y''(t) \leq 0,\ y'(t) > 0$ and $y(t) > 0$ on $[t_0, \infty)$; and if $x(t) < 0$ for $t \geq t_0 \geq 0$, then there exists a large t, also say t_0, such that $y''(t) \geq 0,\ y'(t) < 0$ and $y(t) < 0$ on $[t_0, \infty)$.

Theorem 3.1. *Assume that* (H_1) *holds. Then under any one of the following conditions Equation* (1.1) *has no eventually positive solution:*

(I) *For any $a > 0$, there exist $t_n \to \infty$ such that $-h(t_n) = at_n$;*

(II) *There exists an $a > 0$ such that $-h(t) < at$ for $t \geq t_0 \geq 0$, and for any $T \geq t_0$ the IUBL of $-h$ on $[T, \infty)$ is eventually constant, and*

$$\int_{t_0}^{\infty} a(t)dt = \infty; \tag{3.2}$$

(III) *There exists an $a > 0$ such that $-h(t) < at$ for $t \geq t_0 \geq 0$, $y = \phi(t)$ is the* IUBL *of $-h$ on $[t_0, \infty)$ which is not eventually constant, and*

$$\int_{t_0}^{\infty} a(t)f(\phi(t) + h(t))dt = \infty. \tag{3.3}$$

Proof. Assume that $x(t)$ is an eventually positive solution of Equation (1.1). Then without loss of generality assume $y''(t) \leq 0,\ y'(t) > 0$ and $y(t) > 0$ for $t \geq t_0$.

(I) There exists an $a > 0$ such that $0 < y(t) < at,\ t \geq t_0$. Thus

$$h(t) + at > h(t) + y(t) = x(t) > 0 \text{ for } t \geq t_0,$$

contradicting $-h(t_n) = at_n$.

(II) Since $y(t) \geq -h(t),\ y'(t) > 0$ for $t \geq t_0$, according to Lemma 2.1 there exist a $c > 0$ and a $T \geq t_0$ such that

$$y(t) + h(t) \geq c,\ t \geq T. \tag{3.4}$$

Integrating Equation (2.1) from T to t we have

$$y'(t) - y'(T) + \int_T^t a(s)f(y(s) + h(s))ds = 0. \tag{3.5}$$

In view of (3.4) we get

$$y'(t) - y'(T) + f(c)\int_T^t a(s)ds \leq 0.$$

Letting $t \to \infty$ and noting $\int_T^\infty a(t)dt = \infty$ we obtain $y'(t) \to -\infty$ as $t \to \infty$. This contradicts $y'(t) > 0$.

(III) Since $y(t) \geq \phi(t)$ and $y''(t) \leq 0,\ t \geq t_0$, according to Lemma 2.2, there exists a $T \geq t_0$ such that $y(t) \geq \phi(t),\ t \geq T$. Then from (3.5)

$$y'(t) - y'(T) + \int_T^t a(s)f(\phi(s) + h(s))ds \leq 0.$$

Letting $t \to \infty$ we also have $y'(t) \to -\infty$, as $t \to \infty$, and this contradicts $y'(t) > 0$.

Theorem 3.2. *Assume that* (H_1) *holds. Then under any one of the following conditions Equation* (1.1) *has no eventually negative solution:*

(I$'$) *For any* $b < 0$, *there exist* $t_n \to \infty$ *such that* $-h(t_n) = bt_n$;

(II$'$) *There exists a* $b < 0$ *such that* $-h(t) > bt$ *for* $t \geq t_0 \geq 0$, *and for any* $T \geq t_0$ *the* DLBL *of* $-h$ *on* $[T, \infty)$ *is eventually constant, and*

$$\int_{t_0}^\infty a(t)dt = \infty; \tag{3.6}$$

(III$'$) *There exists a* $b < 0$ *such that* $-h(t) > bt$ *for* $t \geq t_0 \geq 0,\ y = \psi(t)$ *is the* DLBL *of* $-h$ *on* $[t_0, \infty)$ *which is not eventually constant, and*

$$\int_{t_0}^\infty a(t)f(\psi(t) + h(t))dt = -\infty. \tag{3.7}$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it here.

Theorem 3.3. *Assume that* (H_1) *holds. Then any one of* (I), (II) *and* (III) *together with any of the* (I$'$), (II$'$) *and* (III$'$) *guarantees that Equation* (1.1) *is oscillatory.*

Remark 3.1. Theorem 3.3 includes Theorems 1, 2 and 4 and their corresponding corollaries in [9] as its special cases.

Besides, Theorem 3.3 also improves the condition and the conclusion of Theorem 3 in [9], where the oscillation problem is not solved thoroughly.

Furthermore, Theorem 3.3 is a sharper result than Theorems 1-4 in [9]. For example, we have the following corollary which can not be covered by any result in [9].

Corollary 3.1. *Assume that* (H_1) *holds. If*

$$\lim_{t\to\infty} \text{ approx } h(t) = 0,^{1)}$$

$$\lim_{t\to\infty} \sup h(t) > 0, \quad \lim_{t\to\infty} \inf h(t) < 0$$

and

$$\int_{t_0}^{\infty} a(t) = \infty,$$

then Equation (1.1) *is oscillatory.*

Proof. From $\lim\limits_{t\to\infty} \sup[-h(t)] > 0$, we see that there are $t_n \to \infty$ such that $-h(t_n) \geq a > 0\,(n = 1, 2, \cdots)$. Hence the UBL or IUBL $\phi(t)$ of $-h(t)$ on $[t_1, \infty)$ satisfies that $\phi(t) \geq a,\ t \geq t_1$. In view of $\lim\limits_{t\to\infty}$ approx $h(t) = 0$, we get that mess $\mu < \infty$, where set $\mu = \{t : h(t) < -a/2\}$. Hence on $[t_0, \infty)\backslash\mu$, $\phi(t) + h(t) \geq a/2$. Then

$$\begin{aligned}\int_{t_0}^{\infty} a(t)f(\phi(t) + h(t))dt \geq& \int_{[t_0,\infty)\backslash\mu} a(t)f(\phi(t) + h(t))dt\\ \geq& f(a/2)\int_{[t_0,\infty)\backslash\mu} a(t)dt = \infty.\end{aligned}$$

Similarly we can show that if $\psi(t)$ is the LBL or the DLBL of $-h(t)$, then

$$\int_{t_0}^{\infty} a(t)f(\psi(t) + h(t))dt = -\infty.$$

By Theorem 3.3 Equation (1.1) is oscillatory.

(B) For the second case we consider Equatioin (1.1) under the hypothesis

(H_2) There exist $h_1, h_2 \in C^2[0, \infty)$ satisfying that

$$h_1''(t) + h_2''(t) = g(t),\ h_1(t) > 0,\ h_1''(t) \leq 0$$

and

$$\lim_{t\to\infty} \inf h_2(t) < 0 < \lim_{t\to\infty} \sup h_2(t). \tag{3.8}$$

Without loss of generality we may assume

(a) $\lim\limits_{t\to\infty} \sup h_1(t) = \infty$, and

(b) $\lim\limits_{t\to\infty} h_1'(t) = 0$.

Otherwise, if (a) does not hold, $\lim\limits_{t\to\infty} \sup h_1(t) = a < \infty$. Let $h_1^*(t) = h_1(t) - a$, then $\lim\limits_{t\to\infty} \sup h_1^*(t) = 0$. Because $\lim\limits_{t\to\infty} h_2(t)$does not exist, $h(t) = h_1^*(t) + h_2(t)$ is oscillatory and $h''(t) = g(t)$. This coincides with hypothesis (H_1). Since $h_1''(t) \leq 0,\ \lim\limits_{t\to\infty} h_1'(t) = b$ exists. If $h_1''(t) \equiv 0$ for $t \geq T \geq 0$, then we can replace $h_1(t)$ by 0. If $h''(t) \not\equiv 0$ eventually and $b \neq 0$, then $h_1'(t) > b$ for $t \geq 0$. Define

$$h_1^*(t) = h_1(t) - h_1(0) - bt.$$

Then

$$[h_1^*(t)]' = h_1'(t) - b > 0$$

1) see [1] for the definition.

and

$$\lim_{t\to\infty}[h_1^*(t)]' = 0.$$

Hence we have $h_1^*(t) > h_1^*(0) = 0,\ [h_1^*(t)]'' = h_1''(t) \le 0$, and

$$[h_1^*(t)]'' + h_2''(t) = g(t).$$

Combining (a) and (b) we get

$$\lim_{t\to\infty} h_1(t)/t = 0.$$

Under hypothesis (H_2) Equation (1.1) becomes

$$x'' + a(t)f(x) = h_1''(t) + h_2''(t). \tag{3.9}$$

Letting $x = y + h_2(t)$, we replace (3.9) by

$$(y - h_1(t))'' + a(t)f(y + h_2(t)) = 0. \tag{3.10}$$

Theorem 3.4. *Assume that* (H_2) *holds,* $y = \phi(t)$ *is the* UBL *of* $-h_2$ *on* $[t_0, \infty)$, *and for a sufficiently small* $\alpha > 0,\ y = \psi_\alpha(t)$, *the* $\alpha-$LBC *of* $-h_2$ *on* $[t_0, \infty)$, *exists. If*

$$\int_{t_0}^{\infty} a(t)f(\phi(t) + h_2(t))dt = \infty \tag{3.11}$$

and

$$\int_{t_0}^{\infty} a(t)f(\psi(t) + h_2(t))dt = -\infty, \tag{3.12}$$

then Equation (1.1) *is oscillatory.*

Proof. Assume that $x(t)$ is an eventually positive solution of Equation (1.1). Without loss of generality we assume $x(t) > 0,\ t \ge t_0$. Then from (3.10). $(y - h_1)'' \le 0,\ t \ge t_0$. We will show $(y - h_1)'' \ge 0,\ t \ge 0$. Otherwise, there exists a $t_1 \ge t_0$ and constants c and $d,\ c < 0$, such that $(y - h_1)'_{t_0} = c$, and

$$y(t) - h_1(t) \le d + ct, \quad t \ge t_1.$$

Hence

$$x(t) \le d + ct + h_1(t) + h_2(t), \quad t \ge t_1,$$

which would contradict $x(t) > 0$ since $\lim\limits_{t\to\infty} h_1(t)/t = 0$ and $h_2(t)$ is oscillatory.

Noting that $y'' = (y - h_1)'' + h_1'' \le 0$ and $y(t) \ge -h_2(t),\ t \ge t_0$, according to Lemma 3.2 there is a $T \ge t_0$ such that $y(t) \ge \phi(t)$ for $t \ge T$. From Equation (3.10), we have

$$(y - h_1)' - (y - h_1)'_T + \int_T^t a(s)f(y(s) + h_2(s))ds = 0. \tag{3.13}$$

Hence

$$(y - h_1)' - (y - h_1)'_T + \int_T^t a(s)f(\phi(s) + h_2(s))ds \le 0,\ t \ge T.$$

Letting $t \to \infty$ and noting (3.11) we get $(y - h_1)' \to -\infty$ as $t \to \infty$, contradicting $(y - h_1)' \ge 0$.

Similarly, we can show that Equation (1.1) has no eventually negative solutions.

(C) At last consider Equation (1.1) with the hypothesis

(H_3) There exist $h_1,\ h_2 \in C^2[0, \infty)$ satisfying

$$h_1''(t) + h_2''(t) = g(t),\ h_1(t) > 0,\ 0 \le h_1''(t) \le k$$

for some $k > 0$, and $h_2(t)$ is oscillatory.

Theorem 3.5. *Assume that* (H_3) *holds,* $y = \psi(t)$ *is the* LBL *of* $-h_2$ *on* $[t_0, \infty)$, *and for the constant* k *given by* (H_3), $y = \phi_k(t)$, *the* k-UPC *of* $-h_2$ *on* $[t_0, \infty)$, *exists. If*

$$\lim_{t\to\infty}\Big[\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)f(\phi(\tau)+h_2(\tau))d\tau ds - h_1(t)\Big]\Big/ t = \infty \tag{3.14}$$

and

$$\int_{t_0}^{\infty} a(s)f(\psi(s)+h_2(s))ds = -\infty, \tag{3.15}$$

then Equation (1.1) *is oscillatory.*

Proof. Assume that $x(t)$ is an eventually positive solution of Equation (1.1). Without loss of generality we assume $x(t) > 0,\ t \geq t_0$. Then from (3.10), $(y - h_1)'' \leq 0,\ t \geq t_0$. Hence $y(t)'' \leq h_1''(t) \leq k,\ t \geq t_0$. Noting that $y(t) > -h_2(t),\ t \geq t_0$, according to Lemma 2.4, there is a $T \geq t_0$ such that $y(t) \geq \phi_k(t)$ for $t \geq T$. Integrating both sides of (3.13) from T to t we have

$$(y-h_1) - (h-h_1)_T - (y-h_1)'_T(t-T) + \int_T^t\int_T^t a(\tau)f(y(\tau)+h_2(\tau))d\tau ds = 0.$$

Hence

$$(y-h_1) - (h-h_1)_T - (y-h_1)'_T(t-T) + \int_T^t\int_T^t a(\tau)f(\phi_k(\tau)+h_2(\tau))d\tau ds \leq 0.$$

Dividing both sides by t and letting $t \to \infty$ from (3.14) we get $\lim\limits_{t\to\infty} y(t)/t = -\infty$, hence $\lim\limits_{t\to\infty} y(t) = -\infty$, contradicting the assumption that $y(t) > -h_2(t)$ and $h_2(t)$ is oscillatory.

Assume that $x(t)$ is an eventually negative solution of Equation (1.1). Without loss of generality we asumme $x(t) < 0,\ t \geq t_0$. Then from (3.10), $(y - h_1)'' \geq 0,\ t \geq t_0$. We will show that $(y - h_1)' \leq 0,\ t \geq t_0$. Otherwise, there exist $t_1 \geq t_0$ and $c > 0$ such that $(y - h_1)'_{t_1} = c$. Hence $(y - h_1)' \geq c,\ t \geq t_1$. Integrating both sides we see that there is a constant d such that

$$y - h_1 \geq d + ct,\ \text{i.e., } y \geq d + ct + h_1,\ t \geq t_0.$$

Thus

$$x = y + h_2 \geq d + ct + h_1 + h_2,\ t \geq t_0.$$

Since there are $t_n \to \infty$ such that $h_2(t_n) = 0$, we have $x(t_n) > 0$ as n is sufficiently large, contradicting $x(t) < 0,\ t \geq t_0$. Noting that $y \leq -h_2(t)$ and

$$y'' = (y-h_1)'' + h_1'' \geq 0,$$

according to the result for LBL of $-h_2$ similar to that of the UBL of $-h_2$ in Lemma 3.2, there is a $T \geq t_0$ such that $y(t) \leq \psi(t)$ for $t \geq T$. From (3.13) we have

$$(y-h_1)' - (y-h_1)'_T + \int_T^t a(s)f(\psi(s)+h_2(s))ds \geq 0.$$

Letting $t \to \infty$, from (3.15) we get $(y(t) - h_1(t))' \to \infty$ as $t \to \infty$, contradicting $(y - h_1)' \leq 0$.

§4. Examples

Here we apply the above theorems to some particular equations.

Example 4.1. Consider the equation

$$x'' + a(t)f(x) = 2\cos t - t\sin t, \tag{4.1}$$

where (H_0) is satisfied. For any $\varepsilon > o$ denote $E = \sum_{n=1}^{\infty}(n\frac{\pi}{2} - \varepsilon, n\frac{\pi}{2} + \varepsilon)$. If for a sufficiently small $\varepsilon > 0$

$$\int_{[0,\infty)\setminus E} a(t) = \infty, \tag{4.2}$$

then Equation (1.1) is oscillatory.

Proof. Let $h(t) = t\sin t$. Then (H_1) holds and $y = \phi(t) = t$ and $y = \psi(t) = -t$ are the UBL and LBL of $-h$ on $[0,\infty)$, respectively. We see that for the ε in E there is a $\delta > 0$ such that the set

$$E_0 = \{t_0 : t \in [0,\infty) \text{ and } t + h(t) \le \delta\} \subset E.$$

Then on $[0,\infty)\setminus E_0,\ f(t + h(t)) \ge f(\delta) > 0$, so (4.2) implies that (3.3) holds. In the same way, (4.2) also implies that (3.7) holds. By Theorem 3.3 we know that Equation (4.1) is oscillatory.

Example 4.2. Consider the equation

$$x'' + 4x = 1/2 + \sin t. \tag{4.3}$$

We know that this equation is oscillatory. But before this paper there is almost no oscillation criterion to assure this fact. Now we show that Theorem 3.5 will do this. Here $h_1(t) = t^2/4,\ h_2(t) = -\sin t,\ a(t) = 4$ and $f(x) = x$. Hence (H_3) holds with $k = 1/2$ and $y = \psi(t_) = -1$ is the LBL of $-h_2$ on $[0,\infty)$. Obviously, $y = \phi_{1/2}(t)$, the $(1/2)-$UPC of $-h_2$ on $[0,\infty)$ exists, and

$$u(t) = \phi_{1/2}(t) + h_2(t) = \phi_{1/2}(t) - \sin t.$$

Let $v(t) = t^2/4 + \cos t$. Then $v(0.65\pi) \approx 0.246$, and

$$\int_{-0.65\pi}^{0.65\pi} [v(t) - v(0.65\pi)]dt \approx 1.623.$$

It is easy to see that for $n = 0, 1, 2, \cdots$,

$$\begin{aligned}
&\int_{2n\pi}^{(2n+1)\pi} a(t)f(\phi_1(t) + h_2(t))dt \\
=&4\int_{-\pi}^{\pi} (\phi_{1/2}(t) - \sin t)dt \\
\ge&4\int_{-0.65\pi}^{0.65\pi} [v(t) - v(0.5\pi)]dt \\
\approx&4 \times 1.623 = 6.492.
\end{aligned}$$

For any $t \ge 2\pi$, there is an $n > 0$ such that $t \in [2n\pi, 2(n+1)\pi)$. Thus

$$\begin{aligned}
&\int_0^t a(t)f(\phi_{1/2}(t) + h_2(t))dt \\
\ge&\sum_{i=1}^{n-1}\int_{2i\pi}^{2(i+1)\pi} a(t)f(\phi_{1/2}(t) + h_2(t))dt \\
\ge&6.492n \ge 6.492(t/2\pi - 1) \approx 1.033t - 6.492.
\end{aligned}$$

Hence

$$\int_0^t \int_0^s a(\tau)f(\phi_{1/2}(\tau)+h_2(\tau))d\tau ds \approx 0.517t^2 - 6.492t,$$

and this means that (3.14) holds.

It is clear that (3.15) holds also. Therefore we obtain the oscillatory property of Equation (4.3) by Theorem 3.5.

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