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ON A CONJECTURE OF K. OGIUE FOR KAEHLER HYPERSURFACES**

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Abstract

An affirmative answer to a conjecture of K. Ogiue formulated in [2] is given, namely, the following result is proved:

Let M^n $(n \geq 2)$ be a complete Kaehler hypersurface immersed in a complex projective space CP^{n+1}. If every sectional curvature of M^n is positive, then M^n is totally geodesic in CP^{n+1}.

Keywords Kaehler hypersurfaces, Conjecture of K. Ogiue, Sectional curvature.
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§1. Introduction

In [2], K.Ogiue proposed a conjecture that a complete Kaehler hypersurface M^n immersed in a complex projective space CP^{n+1} with positive sectional curvature is totally geodesic. In [3], he proved that it is true for $n \geq 4$. Moreover, if M^n is imbedded in CP^{n+1}, then it is also true for $n \geq 2$. This paper solves completely the conjecture for immersed Kaehler hypersurfaces for $n \geq 2$, namely, we obtain the following

Theorem. *Let M^n $(n \geq 2)$ be a complete Kaehler hypersurface immersed in a complex projective space CP^{n+1}. If every sectional curvature of M^n is positive, then M^n is totally geodesic in CP^{n+1}.*

This theorem is closely related to another Ogiue's conjecture in [3] which says that a complete Kaehler submanifold M^n immersed in CP^{n+p} $(p < n(n+1)/2)$ with positive sectional curvature is totally geodesic. Our result can be regarded as some evidence that Ogiue's conjecture may be true.

§2. Basic Formulas

Let $CP^{n+1}(1)$ denote an $(n+1)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. Let M^n be a Kaehler hypersurface immersed in $CP^{n+1}(1)$. The Fubini-Study metric of $CP^{n+1}(1)$ and the induced metric on M^n both will be denoted by g. The complex structure of $CP^{n+1}(1)$ and the induced complex structure on M both will be denoted by J. Let $\overline{\nabla}$ and ∇ be respectively the Riemannian connections of $CP^{n+1}(1)$ and M^n, and let σ be the second fundamental form of M^n. By A and $\nabla^{\perp}$ denote the Weingarten endomorphism and the normal connection. Throughout this paper, X, Y, Z and W will be either vector fields on one of the special neighborhoods

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$U(x)$ of $x \in M^n$, or vectors tangent to M at a point of $U(x)$, unless otherwise specified. Let ξ and $J\xi$ be the unit normal vector fields on $U(x) \subset M$. On $U(x)$, we have[7]

$$\overline{\nabla}_x Y = \nabla_x Y + \sigma(X, Y) \tag{2.1}$$

$$= \nabla_x Y + h(X, Y)\xi + k(X, Y)J\xi, \tag{2.2}$$

where h and k are symmetric covariant tensor fields of degree 2 on $U(x)$ satisfying

$$h(X, J\xi) = -k(X, Y), \quad k(X, JY) = h(X, Y), \tag{2.3}$$

$$\overline{\nabla}_x \xi = -A_\xi(X) + \nabla_x^\perp \xi = -A_\xi(X) + s(X)J\xi, \tag{2.4}$$

where A_ξ and s are tensor fields on $U(x)$ of type (1.1) and $(0, 1)$ respectively. Furthermore A_ξ and JA_ξ are symmetric with respect to $g, A_\xi J = -JA_\xi$ and A_ξ satisfies

$$h(X, Y) = g(A_\xi X, Y), \quad k(X, Y) = h(JA_\xi X, Y), \tag{2.5}$$

$$A_{J\xi} = JA_\xi = -A_\xi J, \tag{2.6}$$

$$\nabla_x^\perp J\xi = J\nabla_x^\perp \xi. \tag{2.7}$$

Let $\overline{R}$, R and $R^\perp$ denote respectively the curvature tensors of the connections $\overline{\nabla}, \nabla$ and $\nabla^\perp$. Then we have

$$\begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} =& \frac{1}{4}\big\{g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y}) + g(J\overline{Y}, \overline{Z})J\overline{X} \\ & - g(J\overline{X}, \overline{Z})J\overline{Y} + 2g(\overline{X}, J\overline{Y})J\overline{Z}\big\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\overline{R}(X, Y)W =& R(X, Y)W - \big\{g(A_\xi Y, W)A_\xi X - g(A_\xi X, W)A_\xi Y\big\} \\ & - \big\{g(JA_\xi Y, W)JA_\xi X - g(JA_\xi X, W)JA_\xi Y\big\}.\end{aligned} \tag{2.9}$$

Let P be a 2−plane tangent to M at a point of $U(x)$. Then,

$$\begin{aligned}\overline{K}(P) =& K(P) - \big\{g(A_\xi X, X)g(A_\xi Y, Y) - g(A_\xi X, Y)^2\big\} \\ & - \big\{g(JA_\xi X, X)g(JA_\xi Y, Y) - g(JA_\xi X, Y)^2\big\},\end{aligned} \tag{2.10}$$

where $\{X, Y\}$ is an orthonormal basis of P and $\overline{K}(P)$ (resp. $K(P)$) is the sectional curvature is P considered as a 2−plane tangent to CP^{n+1} (resp. M).

The Ricci epuation of M is

$$g(R^\perp(X, Y)\xi, J\xi) = g(\overline{R}(X, Y)\xi, J\xi) + g([A_\xi, A_{J\xi}]X, Y),$$

which is epuivalent to

$$k(A_\xi X, Y) = \frac{1}{4}g(X, JY) + \frac{1}{2}[-X(s(Y) + Y(s(X)) + s([X, Y])]. \tag{2.11}$$

From (2.2) and the Codazzi epuation of M,we can obtain

$$\begin{aligned}&(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) \\ =& - h(R(X, Y)Z, W) - h(Z, R(X, Y)W),\end{aligned} \tag{2.12}$$

$$\begin{aligned}&(\nabla^2 k)(X, Y, Z, W) - (\nabla^2 k)(Y, X, Z, W) \\ =& - k(R(X, Y)Z, W) - k(Z, R(X, Y)W).\end{aligned} \tag{2.13}$$

Using the facts that $\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y)$ and

$$(\nabla\sigma)(JX, Y, Z) = (\nabla\sigma)(X, JY, Z) = (\nabla\sigma)(X, Y, JZ) = J(\nabla\sigma)(X, Y, Z),$$

we obtain

$$(\nabla\sigma)(Z,X,Y) =[(\nabla_z h)(X,Y) - h(X,Y)s(Z)]\xi \\ + [(\nabla_z k)(X,Y) + h(X,Y)s(Z)]J\xi, \tag{2.14}$$

$$(\nabla h)(JX,Y,Z) = -(\nabla k)(X,Y,Z) - h(Y,Z)s(X) + k(Y,Z)s(JX), \tag{2.15}$$

$$(\nabla k)(JX,Y,Z) =(\nabla h)(X,Y,Z) - h(Y,Z)s(JX) - k(Y,Z)s(X). \tag{2.16}$$

Finally, from (2.3) we obtain easily

$$(\nabla_z h)(JX,Y) = -(\nabla_z k)(X,Y), \quad (\nabla_z k)(JX,Y) = (\nabla_z h)(X,Y). \tag{2.17}$$

§3. A Lemma

Let M^n be a compact Kaehler hypersurface immersed in $CP^{n+1}(1)$. Let $\pi : UM \longrightarrow M^n$ and UM_p be the unit tangent boundle over M^n and its fibre at $p\in M$, respectively. Then, we consider the function $f : UM \longrightarrow R$ defined by

$$f(u) = \|\sigma(u,u)\|^2 = h(u,u)^2 + k(u,u)^2, \quad \forall u \in UM_p.$$

We may obtain

Lemma. *Let M^n $(n\geq 2)$ be a compact Kaehler hypersurface immersed in $CP^{n+1}(1)$ which is not totally geodesic. Then there exist some $p \in M$ and some vector $v \in UM_p$ such that*

$$\|A_\xi u\|^2(p) = h(A_\xi u, u) \geq \frac{1}{4} \tag{3.1}$$

for any unit normal vector $\xi\in T_p^\perp M$ and any unit vector $u\in P^\perp(\{v, Jv\})$, where $P^\perp(\{v, Jv\})$ denotes the orthogonal complement space of the holomorphic plane spanned by v and Jv in UM_p.

Proof. Since M^n is compact, the function f attains its maximum at some vector $v\in UM_p^n$ for some $p\in M$. Fixed $v\in UM$, for any vector $u\in M_p^n$, let $\gamma_u(t)$ be the geodesic in M^n determined by the initial conditions $\gamma_u(0) = p$, $\gamma_u'(0) = u$. Parallel translation of v along $\gamma_u(t)$ yields vector field $V_u(t)$. Let $f_u(t) = f(V_u(t))$. By similar computations as in [6], we obtain

$$0 = \frac{d}{dt} f_u(0) = 2h(v,v)[(\nabla h)(u,v,v)] + 2k(v,v)[(\nabla k)(u,v,v)], \tag{3.2}$$

$$0 = \frac{d}{dt} f_{Ju}(0) = 2h(v,v)[(\nabla h)(Ju,v,v)] + 2k(v,v)[(\nabla k)(Ju,v,v)]. \tag{3.3}$$

Now we suppose that $u\in UM_p$ satisfies the condition that $g(u,v) = g(Ju,v) = 0$. Then[6]

$$h(u,v)h(v,v) + k(u,v)k(v,v) = 0, \tag{3.4}$$

$$h(u,v)k(v,v) - k(u,v)h(v,v) = 0, \tag{3.5}$$

$$\frac{d^2}{dt^2} f_u(0) =2[(\nabla h)(u,v,v)]^2 + 2h(v,v)[(\nabla^2 h)z(u,u,v,v)] \\ + 2[(\nabla^2 k)(u,v,v)]^2 + 2k(v,v)[(\nabla^2 k)(u,u,v,v)],$$

$$\frac{d^2}{dt^2} f_{Ju}(0) =2[(\nabla h)(Ju,v,v)]^2 + 2h(v,v)[(\nabla^2 k)(Ju,Ju,v,v)] \\ + 2[(\nabla k)(Ju,v,v)]^2 + 2k(v,v)[(\nabla^2 k)(Ju,Ju,v,v)],$$

$$\frac{d^2}{dt^2} f_u(0) + \frac{d^2}{dt^2} f_{Ju}(0) \leq 0. \tag{3.6}$$

Using (2.15), (2.16), (2.17) and (2.12), we easily obtain

$$\begin{aligned}
&(\nabla^2 h)(Ju, Ju, v, v)\\
=&-(\nabla^2 h)(u,u,v,v) + [\nabla_{Ju}(k(v,v)) + \nabla_u(h(v,v))]s(Ju)\\
&+[\nabla_u(k(v,v)) - \nabla_{Ju}(h(v,v))]s(u) + k(v,v)[Ju(s(Jw) + u(s(u))]\\
&+h(v,v)[u(s(Jw) - Ju(s(w))] - h(R)(Ju,u)Jv,v) - h(Jv, R(Ju,u)v).
\end{aligned}$$

Similarly, using (2.15), (2.16), (2.17) and (2.13), we have

$$\begin{aligned}
&(\nabla^2 k)(Ju, Ju, v, v)\\
=&-(\nabla^2 k)(u,u,v,v) - [\nabla_u(h(v,v)) + \nabla_{Ju}(k(u,u))s(u)\\
&+[-\nabla_{Ju}(h(v,v)) + \nabla_u(k(v,v))]s(Ju) + k(v,v)[-Ju(s(u)) + u(s(Ju))]\\
&-h(v,v)[Ju(s(Ju)) + u(s(u))] - k(R(Ju,u)Jv,v) - k(Jv, R(Ju,u)v).
\end{aligned}$$

Substituting these into (3.6) and using (3.2), (3.3), (2.15) and (2.16), we have

$$\begin{aligned}
&h(v,v)[(\nabla^2 h)(Ju,Ju,v,v) + (\nabla^2 h)(u,u,v,v)]\\
&+k(v,v)[(\nabla^2 k)(Ju,Ju,v,v) + (\nabla^2 k)(u,u,v,v)]\\
=&-(h(v,v)^2 + k(v,v)^2)(s(Ju)^2 + s(u)^2)\\
&+(h(v,v)^2 + k(v,v)^2)(u(s(Ju)) - Ju(s(u)))\\
&-h(v,v)[h(R(Ju,u)Jv,v) + h(Jv,R(Ju,u)v)]\\
&+k(v,v)[k(R(Ju,u)Jv,v) + k(v,R(Ju,u)v)].
\end{aligned}$$

Noticing the Gauss equation and that $g(u,v) = g(Ju,v) = 0$, one can see that

$$\begin{aligned}
h(R(Ju,u)Jv,v) =&h(Jv,R(Ju,u)v)\\
=&\frac{1}{2}h(v,v) + 2h(v,v)h(A_\xi u,v) - 2k(u,v)k(A_\xi u,v),\\
k(R(Ju,u)Jv,v) =&k(Jv,R(Ju,u)v)\\
=&-\frac{1}{2}k(v,v) + 2k(u,v)k(A_\xi u,v) + 2h(u,v)k(A_\xi u,v).
\end{aligned}$$

Then (3.6) is equivalent to

$$\begin{aligned}
&(h^2+k^2)[1-(s(Ju))^2 - (s(u))^2 + u(s(Ju)) - (Ju)(s(u))]\\
&-4[h(u,v)h(v,v) + k(v,v)k(u,v)]h(A_\xi u,v)\\
&+4[h(v,v)k(u,v) - k(v,v)k(u,v)]k(A_\xi u,v)\\
\le&0.
\end{aligned}$$

Substituting (3.4), (3.5) and (2.11) into the above, we finally obtain

$$(h(v,v)^2 + k(v,v)^2)[\frac{1}{2} - 2h(A_\xi u,u) - (s(u))^2 - (s(Ju))^2 - s([Ju,u])\le 0. \tag{3.7}$$

In the neighborhood $U(p)$ of p, $\sigma(X,Y) = h(X,Y)\xi + k(X,Y)J\xi$, where σ is a unit normal vector field, which can be obtained by parallelly translating the unit normal vector ξ_p at p in the normal bundle along the geodesics on M^n starting from the point p. So, we have $\nabla^{\perp}_{\gamma'}\xi\big|_p = 0$, where $\gamma(t)$ is any geodesic on M^n through p. Then it is easily seen from (2.4) that

$$s(u) = s(Ju) = s([Ju,u]) = 0$$

at the point p. Thus it follows from (3.7) that

$$[h^2(v,v)+k^2(v,v)][\frac{1}{2}-2h(A_\xi u,u)]\le 0 \tag{3.8}$$

at p. Since M^n is not totally geodesic, $h^2(v,v)+k^2(v,v)\neq 0$. At the point p we have from (3.8)

$$h(A_\xi u,u)\ge\frac{1}{4},\quad \forall u\in P^\perp(\{v,Jv\}).$$

The Lemma is proved.

§4. Proof of Theorem

At first, by Proposition 6.12 in [3], we note that M^n is compact under the hypothesis as in theorem.

We assume that the function f defined as in §3 attains its maximum at $v\in UM_p$ for some $p\in M$. All computations below will be restricted at the point p. From (3.4) and (3.5) we have

$$h(v,v)g(A_\xi u,v)+k(v,v)g(JA_\xi u,v)=0,$$
$$-k(v,v)g(A_\xi u,v)+h(v,v)g(JA_\xi u,v)=0$$

for any $u\in M_p$ such that $g(u,v)=g(u,Jv)=0$.

Now suppose that M^n would be not totally geodesic so that $h^2(v,v)+k^2(v,v)\neq 0$. Then,

$$g(A_\xi u,v)=g(JA_\xi u,v)=0,$$

i.e.,

$$h(u,v)=k(u,v)=0.$$

Therefore,

$$A_\xi v=h(v,v)v,\quad JA_\xi v=k(v,v)v,$$

$$\left.\begin{aligned}A_{\sigma(v,v)}u&=h(v,v)A_\xi u+k(v,v)JA_\xi u,\\ A_{\sigma(v,v)}Ju&=k(v,v)A_\xi u-h(v,v)JA_\xi u.\end{aligned}\right\} \tag{4.1}$$

If we take an eigenvector $u\in P^\perp(\{v,Jv\})$ of $A_{\sigma(v,v)}$, then

$$A_{\sigma(v,v)}u=[h(v,v)h(u,u)+k(v,v)k(u,u)]u.$$

It is clear that $A_{\sigma(v,v)}u\neq 0$. From the Gauss equation we have

$$\begin{aligned}A_{\sigma(v,v)}u&=\frac{1}{2}(R(u,v)v-R(u,Jv)Jv)\\ &=\frac{1}{2}(K(u,v)-K(u,Jv)).\end{aligned} \tag{4.2}$$

By similar arguments, we obtain

$$A_{\sigma(v,v)}Ju=\frac{1}{2}(K(Ju,v)-K(Ju,Jv))Ju. \tag{4.3}$$

From (4.1), (4.2) and (4.3) we can find

$$\begin{aligned} A_\xi u =& \frac{1}{2(h^2+k^2)}[h(v,v)(K(u,v)-K(u,Jv))u \\ &+ k(v,v)(K(Ju,v)-K(Ju,Jv))Ju], \\ h(A_\xi u,u) =& \frac{1}{2(h^2+k^2)}[h(v,v)h(u,u)(K(u,v)-K(u,Jv)) \\ &- k(v,v)k(u,u)(K(Ju,v)-K(Ju,Jv))], \end{aligned} \tag{4.4}$$

where

$$\left.\begin{aligned} K(u,v) = K(Ju,Jv) = \frac{1}{4} + h(u,u)h(v,v) + k(u,u)k(v,v) > 0, \\ K(u,Jv) = K(Ju,v) = \frac{1}{4} - h(u,u)h(v,v) - k(u,u)k(v,v) > 0. \end{aligned}\right\} \tag{4.5}$$

Then, (4.4) yields

$$\begin{aligned} h(A_\xi u,u) =& \frac{1}{h^2+k^2}(h(u,u)h(v,v)+k(u,u)k(v,v))^2 \\ \le& \frac{1}{\|\sigma(v,v)\|^2}|h(u,u)h(v,v)+k(u,u)k(v,v)| \cdot \|\sigma(u,u)\| \cdot \|\sigma(v,v)\| \\ \le& |h(u,u)h(v,v)+k(u,u)k(v,v)| < \frac{1}{4}, \end{aligned}$$

where the first inequality is from the Schwarz inequality, the second is due to

$$\|\sigma(u,u)\|^2 \ge \|\sigma(u,u)\|^2,$$

and the third is from (4.5). This contradicts the Lemma. Hence, M^n must be totally geodesic in CP^{n+1}. Our theorem is proved.

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