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A PICARD TYPE THEOREM AND BLOCH LAW

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Abstract

A picard type theorem is proved, and a counterexample is given to show that the Bloch Law is not true generally.

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§1. Introduction

In 1959, W. K. Hayman[1] proved the following theorem: Let f be a transcendental meromorphic function in the plane. If n is an integer not less than 5 and a is a finite nonzero complex number, then $f' - af^n$ assumes every finite complex value infinitely often. According to Bloch Law, which is a well-known heuristic principle in the theory of functions asserting that a family of holomorphic (meromorphic) functions which have a property P in common in a domain D is a normal family in D if P cannot be possessed by non-constant entire (meromorphic) functions in the plane, the criterion for normality which corresponds to the above theorem was recently proved by J. K. Langley[2] and Li Xianjin[3] respectively. The further results on this respect were investigated by E. Mues[4] and Pang Xuecheng[5]. In this paper, we shall show that Bloch Law is not true generally by proving a Picard type theorem and giving a counterexample.

§2. Statement of Results

Theorem 2.1. a) *Let f be a transcendental entire function. If $a \neq 0$ is a finite complex number and $n \geq 2$ is an integer, then $f + af'^n$ assumes all finite complex numbers infinitely often.*

b) *Let f be a transcendental meromorphic function. If $a \neq 0$ is a finite complex number and $n \geq 3$ is an integer, then $f + af'^n$ assumes all finite complex numbers infinitely often.*

Example. Let $F = \{f_m = mz\}, z \in D$, where D is a unit disc. Then $f_m + af'^n_m \neq 0$, $z \in D$, but F is not normal on D.

The above example shows that Bloch Law is not true. However, if we give an additional condition, then we have the following

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Theorem 2.2. *Let F be a family of meromorphic functions in a domain D, $f \neq b$ and $f + af'^n \neq b$ for every $f \in F$, where $n \geq 2$ is an integer and $a \neq 0, b$ are two finite complex numbers. Then F is normal.*

By the above example, the condition $f \neq b$ is necessary in a sense.

§3. Some Lemmas and the Proof of Theorems

In the following, we will use the usual Nevanlinna theory, e.g., [6], for notations and results. In particular $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set of values r of finite linear measure.

Lemma 3.1.[7] *Let f be a transcendental meromorphic function and take any $K > 1$. Then there exists a set M_K of upper logarithmic density $\overline{\log\operatorname{des}} M_K \leq \delta_K < 1$ such that*

$$\lim_{M_K \ni r \to \infty} \frac{T(r, f)}{T(r, f')} \leq 3eK.$$

Lemma 3.1 will play an important rôle in the proof of our theorems.

Lemma 3.2. *Let f be a transcendental meromorphic function. If $a \neq 0, b$ are two finite complex numbers and $n \geq 2$ is an integer, then we have*

$$(n-1)T(r, f') \leq 4\overline{N}(r, f) + 9N\Big(r, \frac{1}{f + a(f')^n - b}\Big) + S(r, f).$$

Proof. Without loss of generality, we may assume $a = 1$ and $b = 0$.

Set

$$g = f + f'^n \quad \text{and} \quad \varphi = \frac{g'}{g}. \tag{3.1}$$

Then it is obvious that $\varphi \not\equiv 0$ (otherwise f must be a constant, a polynomial of degree 2 or an algebraic function).

By elementary Nevanlinna theory and by (3.1), we deduce that $T(r, g) \leq 0(T(r, f))$, so $m(r, \varphi) = S(r, f)$.

From (3.1) we have

$$f' + nf'^{n-1}f'' = \varphi(f + f'^n). \tag{3.2}$$

We rewrite (3.2) in the form

$$f'^n(n\frac{f''}{f'} - \varphi) = \varphi f - f' \tag{3.3}$$

and denote $\psi = n\frac{f''}{f'} - \varphi$, so

$$\psi f'^n = \varphi f - f'. \tag{3.4}$$

Then $\psi \not\equiv 0$; otherwise by integrating we may obtain $f + (1 - C)f'^n = 0$, but it is impossible by the same reasons above.

By differentiation of (3.4), then

$$\psi' f'^n + n\psi f'^{n-1} f'' = \varphi' f + \varphi f' - f''. \tag{3.5}$$

Now we eliminate f between (3.4) and (3.5); we arrive at

$$f'^n P = Q \tag{3.6}$$

where

$$P=\varphi\psi'+n\varphi\psi\frac{f''}{f'}-\varphi'\psi, \tag{3.7}$$

$$Q=f'\varphi'+f'\varphi^2-\varphi f''. \tag{3.8}$$

If $P\equiv 0$, then $Q\equiv 0$. By (3.8), we can easily have $f'^{n-2}f''=C$, but this is impossible since f is transcendental.

Thus from now on we may suppose $P\not\equiv 0$. From (3.6) and (3.8), we claim that the poles of f cannot be the poles of P for $n\geq 2$, so by (3.7) the poles of P may be caused by the zero points of f' and $f+f'^n$. Hence

$$N(r,P)\leq 2\overline{N}\Big(r,\frac{1}{f'}\Big)+3\overline{N}\Big(r,\frac{1}{f+f'^n}\Big). \tag{3.9}$$

By (3.8), we have

$$N(r,Q)\leq N(r,f')+2\overline{N}(r,f)+2\overline{N}\Big(r,\frac{1}{f+f'^n}\Big). \tag{3.10}$$

It is easy to see that the following is true:

$$m(r,P)=S(r,f) \tag{3.11}$$

and

$$m(r,Q)\leq m(r,f')+S(r,f), \tag{3.12}$$

by the logrithm lemma.

Hence, by (3.6), (3.9), (3.10), (3.12), we have

$$\begin{aligned} nT(r,f')&\leq T(r,Q)+T(r,P)+O(1)\\ &\leq T(r,f')+2\overline{N}(r,f)+2\overline{N}\Big(r,\frac{1}{f'}\Big)+5\overline{N}\Big(r,\frac{1}{f+f'^n}\Big)+S(r,f). \end{aligned}$$

By (3.2) we have

$$\begin{aligned} N\Big(r,\frac{1}{f'}\Big)&\leq N\Big(r,\frac{1}{\varphi}\Big)+N\Big(r,\frac{1}{f+f'^n}\Big)\\ &\leq N(r,\varphi)+N\Big(r,\frac{1}{f+f'^n}\Big)+S(r,f)\\ &\leq \overline{N}(r,f)+2N\Big(r,\frac{1}{f+f'^n}\Big)+S(r,f). \end{aligned}$$

Thus we immediately have the assertion by above two inequalities.

Lemma 3.3. *Let f be a transcendental meromorphic function, $n\geq 2$ be an integer. If $N_{1)}(r,\frac{1}{f})=S(r,f)$, then $f+f'^n$ assumes zero value infinitely.*

Proof. By FFT and SFT, we have

$$\begin{aligned}
m\Big(r, \frac{1}{f'^n}\Big) \le & m\Big(r, \frac{f}{f'^n}\Big) + m\Big(r, \frac{1}{f}\Big) \\
\le & m\Big(r, \frac{f}{f'^n} + 1\Big) + m\Big(r, \frac{f'}{ff'}\Big) + O(1) \\
\le & m\Big(r, \frac{f + f'^n}{f'^n}\Big) + m\Big(r, \frac{1}{f'}\Big) + S(r, f) \\
= & T\Big(r, \frac{f'^n}{f + f'^n}\Big) - N\Big(r, \frac{f + f'^n}{f'^n}\Big) + m\Big(r, \frac{1}{f'}\Big) + S(r, f) \\
\le & \overline{N}\Big(r, \frac{f'^n}{f + f'^n}\Big) + \overline{N}\Big(r, \frac{f + f'^n}{f'^n}\Big) + \overline{N}\Big(r, \frac{1}{\frac{f'^n}{f+f'^n} - 1}\Big) \\
& - N\Big(r, \frac{f + f'^n}{f'^n}\Big) + m\Big(r, \frac{1}{f'}\Big) + S(r, f). \qquad (3.13)
\end{aligned}$$

By a simple observation, we have

$$\begin{aligned}
\overline{N}\Big(r, \frac{f'^n}{f + f'^n}\Big) \le & \overline{N}\Big(r, \frac{1}{f + f'^n}\Big), \\
\overline{N}\Big(r, \frac{f + f'^n}{f'^n}\Big) \le & \overline{N}\Big(r \frac{1}{f'}\Big), \\
\overline{N}\Big(r, \frac{f + f'^n}{f}\Big) \le & \overline{N}(r, f) + N_{1)}\Big(r, \frac{1}{f}\Big), \\
N\Big(r, \frac{f + f'^n}{f'^n}\Big) \ge & N\Big(r, \frac{1}{f'^n}\Big) - N\Big(r, \frac{1}{f + f'^n}\Big). \qquad (3.14)
\end{aligned}$$

Hence by (3.13), (3.14) and FFT, we have

$$\begin{aligned}
nT(r, f') \le & \overline{N}\Big(r, \frac{1}{f'}\Big) + \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f + f'^n}\Big) \\
& + N\Big(r, \frac{1}{f + f'^n}\Big) + N_{1)}\Big(r, \frac{1}{f}\Big) + m\Big(r, \frac{1}{f'}\Big) + S(r, f) \\
\le & T(r, f') + \overline{N}(r, f) + 2N\Big(r, \frac{1}{f + f'^n}\Big) + S(r, f). \qquad (3.15)
\end{aligned}$$

If $f + f'^n$ assumes zero value finitely, then $N(r, \frac{1}{f+f'^n}) = S(r, f)$. By (3.15), then we have

$$(n - 1 - \frac{1}{2})T(r, f') \le S(r, f). \qquad (3.16)$$

But by Lemma 3.1, (3.16) is impossible and we have the assertion.

Proof of Theorem 2.1. If $f + af'^n$ assumes some finite complex number b finitely, then $N(r, \frac{1}{f+af'^n-b}) = S(r, f)$. For the sake of convenience, we assume $a = 1$ and $b = 0$. If f is an entire function, then by Lemma 3.2 we see that $(n - 1)T(r, f') = S(r, f)$. But it is impossible by Lemma 3.1. If $n > 3$ and f is a transcendental meromorphic function, then, from Lemma 3.2, we have

$$(n - 1)T(r, f') \le 4\overline{N}(r, f) + S(r, f) \le 2N(r, f') + S(r, f), \qquad (3.17)$$

so $(n - 3)T(r, f') \le S(r, f)$. But it is also impossible by Lemma 3.1. Now it remains to prove the case when $n = 3$ and f is a transcendental meromorphic function. By (3.17) we easily have $m(r, f') = S(r, f)$.

Rewriting (3.3) in the form

$$f'(nf'^{n-2}f'' - \varphi f'^{n-1}) = \varphi \cdot f - f' \tag{3.18}$$

and denoting $H = nf'^{n-2}f'' - \varphi f'^{n-1}$, from (3.18) we see that the poles of f cannot be the poles of H. Hence $N(r, H) = N(r, \frac{1}{f+f'^n}) = S(r, f)$, and

$$\begin{aligned} T(r,H) &= m(r,H) + N(r,H) \\ &\le m(r, f'^{n-1}) + m\Big(r, n\frac{f''}{f'} - \varphi\Big) + N(r,H) = S(r,f). \end{aligned}$$

By rewritting (3.18) in the form $f'(1+H) = \varphi f$, we see that the simple zero points of f must be the zero points of $H+1$, since such points cannot be the poles of φ.

Hence

$$N_{1)}\Big(r,, \frac{1}{f}\Big) \le N\Big(r, \frac{1}{H+1}\Big) \le T(r,H) + O(1) = S(r,f)$$

and we have the assertion by Lemma 3.3.

Corollary 3.1. *Let f be a meromorphic function, $n \ge 3$ be an integer and $a \ne 0$ be a finite complex number. If $f + af'^n \ne b$ for some finite complex number, then f must be a constant.*

Corollary 3.2. *Let f be a meromorphic function, $a \ne 0$ be a finite complex number. If f satisfies the following conditions*

i) $f + af'^2 \ne b$,

ii) $N_{1)}(r, \frac{1}{f-b}) = S(r, f)$,

then f must be a constant.

Corollary 3.3. *Let f be an entire function and $a \ne 0$ be a finite complex number. If $f + af'^2 \ne b$ for some finite complex number b, then f is either a constant or a polynomial of degree 2.*

The Corollaries 3.1, 3.2, 3.3 are obvious according to the proof of above lemmas.

To prove Theorem 2.2 we need the following

Lemma 3.4.[8] *Let $F = \{f\}$ be a family of meromorphic functions defined on unit disc D. If F is not normal on D and $f \ne 0$ for all $f \in F$, then for every given real number $k (k < 1)$ there exist*

(1) *a real number $r, 0 < r < 1$,*

(2) *complex numbers $z_n, |z_n| < r$,*

(3) *functions $f_n \in F, n = 1, 2, \cdots$,*

(4) *positive numbers ρ_n, which satisfy*

$$\lim_{n\to\infty} \rho_n = 0 \quad and \quad \lim_{n\to+\infty} \frac{r - |z_n|}{\rho_n} = +\infty$$

such that $\rho_n^k f_n(z_n + \rho_n\zeta) \longrightarrow g(\zeta)$, spherically on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function on $\mathbb{C}$.

Proof of Theorem 2.2. Without loss of generality, we may assume that $a = 1, b = 0$ and D is unit disc. If F is not normal on D, then for $k = \frac{n}{1-n} < 1$ there exist r, z_m, f_m, ρ_m by Lemma 3.4 such that $g_m(\zeta) = \rho_m^k f_m(z_m + \rho_m\zeta)$ is convergent to $g(\zeta)$ uniformaly on compact subsets of $\mathbb{C}$ where $g(\zeta)$ is a non-constant meromorphic function.

Therefore, $g_m + (g'_m)^n$ is also convergent to $g + g'^n$ uniformly on compact subsets of $\mathbb{C}$. On the other hand

$$g_m + g'^n_m = \rho^k_m f_m + \rho^{(k+1)n} f'^n_m = \rho^k_m (f_m + f'^n_m) \neq 0,$$

so either $g+g'^n$ has no zero points or $g+g'^n$ is identical zero by Hurwitz theorem. But by our Corollaries 3.1 and 3.2 these two cases cannot takeplace if g is a non-constant meromorphic function and we have the assertion.

Remark. It remains open whether or not Theorem 2.1 is true if $n = 2$ and f is meromorphic.

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