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ON THE FEIGENBAUM'S FUNCTIONAL EQUATION $f^P(\lambda x)=\lambda f(x)$**

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Abstract

The author considers the Feigenbaum's functional equation $f^P(\lambda x) = \lambda f(x)$ for each $p \geq 2$. The existence of even unimodal C^1 solutions to this equation is discussed and a feasible method to construct such solutions is given.

Keywords Functional equation, Continuous Single–Valley solution,
Even unimodal C^1 solution.
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§1. Introduction

Recently the research for the Feigenbaum phenomenon has been attached importance to by mathematicians, theoretical physicists and theoretical biologists, etc. The following functional equation was exactly posed by Feigenbaum[5] himself first for explaining this phenomenon:

$$\begin{cases} f(x) = -\dfrac{1}{\lambda} f^2(-\lambda x), \\ f(0) = 1, \qquad -1 \leq f(x) \leq 1, \end{cases} \tag{1.1}$$

where $\lambda \in (0,1)$ is to be determined, $x \in [-1,1]$.

A key problem is whether the Equation (1.1) has any solution, in particular, any even unimodal C^1 solution. For this purpose we may consider under a broader sense the equation:

$$\begin{cases} f(x) = \dfrac{1}{\lambda} f^p(\lambda x), \\ f(0) = 1, \qquad -1 \leq f(x) \leq 1, \end{cases} \tag{1.2}$$

where $\lambda \in (0,1)$ is to be determined, $x \in [-1,1]$, $p \geq 2$ is an integer, f^p the p-fold iteration of f.

It is easy to see that (1.1) is a special case of (1.2). When $p = 2$, the existence of even unimodal C^1 solutions to (1.2) was proved by many authors (see [1], [3], [6], [8]). When $p = 3$, a method to construct the even C^1 solutions of (1.2) was pointed out in [2] essentially. For p large enough, it was shown in [4] that (1.2) has a solution similar to the quadratic function $f(x) = 1 - 2x^2$.

In this paper, we will not only pose the conditions that (1.2) has even unimodal C^i solutions for any $p \geq 2$ and each $i = 0$, 1, but also contribute a feasible method to construct

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these solutions. The main results will be given in Theorem 2.1, Theorem 3.1 and Theorem 4.2.

For simplifying the problem, we consider the following equation:

$$\begin{cases} f(x) = \dfrac{1}{\lambda} f^p(\lambda x), \\ f(0) = 1, \qquad 0 \le f(x) \le 1, \end{cases} \tag{1.3}$$

where $\lambda \in (0,1)$ is to be determined , $x \in [0,1]$.

The connection between (1.2) and (1.3) will be given in Theorem 4.1.

If f is a solution of (1.3), then it is easy to check

$$f(x) = \lambda^{-n} f^{p^n}(\lambda^n x) \tag{1.4}$$

for all $n \ge 0$ and each $x \in [0,1]$.

§2. Continuous Single–Valley Solutions

Definition 2.1. *We call f a continuous single–valley solution of* (1.3), *if* (1) $f : [0,1] \to [0,1]$ *is continuous*, (2) $f(0) = 1$, (3) $f(\alpha) = 0$ *for some* $\alpha \in (0,1)$ *such that f is strictly decreasing on* $[0,\alpha]$ *and strictly increasing on* $[\alpha, 1]$.

In Lemma 2.1–Lemma 2.7, f is always supposed to be a continuous single–valley solution of (1.3) with $f(\alpha) = 0$, where $\alpha \in (0,1)$.

Lemma 2.1. $f^{p^n}(0) = \lambda^n \to 0$ *as* $n \to \infty$. *And* 0 *is recurrent but not periodic.*

Proof. It follows immediately by taking $x = 0$ in (1.4).

Lemma 2.2. *f has a unique fixed point e in* $[0,1]$, *and* $0 < e < \alpha$.

Proof. Obviously, f has only one fixed point in $(0,\alpha)$. If f has another fixed point q, then by Lemma 2.1 and the fact that $f(\alpha) = 0$, $q \in (\alpha, 1)$. Since f is strictly increasing on $[\alpha, 1]$, it follows that $q = f(q) < f(1)$. By induction, $q = f^m(q) < f^m(1)$ for all $m > 0$. In particular,

$$q = f^{p^n-1}(q) < f^{p^n-1}(1) = f^{p^n-1}(f(0)) = f^{p^n}(0).$$

This contradicts the fact that $f^{p^n}(0) \to 0$ as $n \to \infty$.

Lemma 2.3. *Let* $x \in [0,\lambda]$ *and* $0 \le i \le p-1$. *Then* $f^i(x) = \alpha$ *iff* $x = \lambda\alpha$ *and* $i = p-1$.

Proof. Suppose $f^i(x) = \alpha$ for some $x \in [0,\lambda]$ and $0 \le i \le p-1$. First we can know from (1.3) that $\lambda\alpha$ is the only local extremum point of f^p in $(0,\lambda)$. Secondly, since $f(\alpha) = 0$ is not periodic, it follows that $x \ne 0$ and α is not periodic. Noting that

$$f^{p+1}(\alpha) = f^p(f(\alpha)) = f^p(0) = \lambda,$$

we must have $x \ne \lambda$. By aperiodicity of α, we know that if $j \ne i$ then $f^j(x) \ne \alpha$. Thus x is an extremum point of f^p in $(0,\lambda)$. By uniqueness, $x = \lambda\alpha$ and $i = p-1$. Conversely let $x = \lambda\alpha$ and $i = p-1$. We must show that $f^{p-1}(\lambda\alpha) = \alpha$. Take $x = \alpha$ in (1.3). We have $f^p(\lambda\alpha) = 0$, i. e., $f(f^{p-1}(\lambda\alpha)) = 0$. Hence $f^{p-1}(\lambda\alpha) = \alpha$.

Lemma 2.4. *For each* $i = 1, 2, \cdots, p-1$,

(1) $f^i(\lambda\alpha) > \lambda$,

(2) $f^i(x) > \lambda, \forall x \in [0, \lambda\alpha]$,

(3) $f^i(x) > \lambda\alpha, \forall x \in (\lambda\alpha, \lambda]$.

Proof. (1) If $f^i(\lambda\alpha) = x \le \lambda$ for some i with $1 \le i \le p-1$, then

$$f^{p-1-i}(x) = f^{p-1-i}f^i(\lambda\alpha) = f^{p-1}(\lambda\alpha) = \alpha.$$

This contradicts Lemma 2.3.

(2) By Lemma 2.3, $f^i : [0, \lambda\alpha] \to f^i([0, \lambda\alpha])$ is a homeomorphism. It suffices from conclusion (1) to show that $f^i(0) > \lambda$. If $f^j(0) = x \le \lambda$ for some j with $1 \le j \le p-1$, then

$$f^{p-j}(x) = f^{p-j}f^j(0) = f^p(0) = \lambda.$$

Furthermore

$$f^j(\lambda) = f^j f^{p-j}(x) = f^p(x) = \lambda f(\frac{x}{\lambda}) \le \lambda.$$

Since $f^j|_{[0,\lambda]}$ is also a homeomorphism, we have $f^j(\lambda\alpha) \le \lambda$. This contradicts conclusion (1). The result then follows.

(3) If $f^j(x) = y \le \lambda\alpha$ for some j with $1 \le j \le p-1$ and $x \in (\lambda\alpha, \lambda]$, then

$$f^{p-j}(y) = f^{p-j}f^j(x) = f^p(x) \le \lambda.$$

This contradicts conclusion (2).

Lemma 2.5. *For each $l = 1, 2, \cdots, p-1$, f has no periodic point of period l on $[0, \lambda]$.*

Proof. Assume for contradiction that the conclusion fails, i. e., there were $x \in [0, \lambda]$ and $1 \le l \le p-1$ such that x were a periodic point of f with period l. By Lemma 2.4, (2), $x \in (\lambda\alpha, \lambda]$. Let

$$y = \min\{\, x, f(x), \cdots, f^l(x)\,\}.$$

Then by Lemma 2.4, (3), $y \in (\lambda\alpha, \lambda]$. Since $\frac{y}{\lambda} \in [\alpha, 1)$ and $f(\alpha) = 0 < \alpha$, it follows from Lemma 2.2 that $f(\frac{y}{\lambda}) < \frac{y}{\lambda}$. Hence

$$f^p(y) = \lambda f(\frac{y}{\lambda}) < \lambda\frac{y}{\lambda} = y.$$

This contradicts the property of y.

Lemma 2.6. *Let $J = [0, \lambda]$, $J_0 = f(J)$, and $J_i = f^i(J_0)$. Then*

(1) *for each $i = 0, 1, \cdots, p-2$, $f^i|_{J_0} : J_0 \to J_i$ is a homeomorphism.*

(2) *$J_0, J_1, \cdots, J_{p-2} \subset (\lambda, 1]$ are pairwise disjoint.*

Proof. By Lemma 2.3, $f^{i+1}|_J$ is injective for $0 \le i \le p-2$, so is $f^i|_{J_0}$. Thus (1) holds from the continuity of f^i. To prove (2), it suffices to show $J_i \cap J = \emptyset$ for $0 \le i \le p-2$. We claim that $f^{i+1}(\lambda) > \lambda$. If otherwise, $f^l(\lambda) \le \lambda$ for some l with $1 \le l \le p-1$. Then we know from Lemma 2.4, (1) that there exists a fixed point of f in $[\lambda\alpha, \lambda]$. This contradicts Lemma 2.5. So the claim holds. Now we continue proving the lemma. Noting that $f^{i+1}|_J : J \to J_i$ is also a homeomorphism, we get from Lemma 2.4, (2) that $J_i \cap J = \emptyset$.

Lemma 2.7. *The equation $f^{p-1}(x) = \lambda x$ has only one solution $x = 1$ in $(f(\lambda\alpha), 1]$.*

Proof. Recall (1.3). Clearly $x = 1$ is a solution of the equation $f^{p-1}(x) = \lambda x$. Suppose $x = x_0$ is an arbitrary solution of this equation. Since $f([0, \lambda\alpha]) \supset (f(\lambda\alpha), 1]$, it follows that $f(y_0) = x_o$ for some $y_0 \in [0, \lambda\alpha]$. So $f^{p-1}(f(y_0)) = \lambda x_0$. Furthermore,

$$\lambda f(\frac{y_0}{\lambda}) = f^{p-1}(f(y_0)) = \lambda x_0,$$

i.e., $f(\frac{y_0}{\lambda}) = x_0$. Noting that $\frac{y_0}{\lambda} \in [0, \alpha]$ and f is strictly decreasing on $[0, \alpha]$, we have $y_0 = \frac{y_0}{\lambda}$. In other words,

$$y_0(1 - \frac{1}{\lambda}) = 0.$$

Since $\lambda \neq 1$, the only possible case is $y_0 = 0$. Hence

$$x_0 = f(0) = 1.$$

Theorem 2.1 *Let f_0 be a continuous function on $[\lambda, 1]$, where $0 < \lambda < 1$. If*

(1) *there exists some $\alpha \in (\lambda, 1)$ such that $f_0(\alpha) = 0$ and f_0 is strictly decreasing on $[\lambda, \alpha]$ and strictly increasing on $[\alpha, 1]$;*

(2) $f_0^{p-1}(1) = \lambda$, $f_0^p(\lambda) = \lambda f_0(1)$;

(3) *denote $[f_0(\lambda), 1]$ by J_0 and $f^i(J_0)$ by J_i,*

then

(a) *$J_0, J_1, \cdots, J_{p-2} \subset (\lambda, 1]$ are pairwise disjoint,*

(b) *$f^i|_{J_0} : J_0 \to J_i$ is a homeomorphism for each $i = 0, 1, \cdots, p-2$,*

(c) *α is in the interior of J_{p-2};*

(4) *the equation $f_0^{p-1}(x) = \lambda x$ has only one solution $x = 1$ on $(\alpha_0, 1]$, where $\alpha_0 \in J_0$ with $f_0^{p-1}(\alpha_0) = 0$,*

then the equation (1.3) *has exactly one single-valley continuous solution f with $f|_{[\lambda,1]} = f_0$. Conversely, if f_0 is the restriction on $[\lambda, 1]$ of a single-valley continuous solution to* (1.3), *then* (1)–(4) *must hold.*

Proof. Suppose that f_0 is the restriction on $[\lambda, 1]$ of some single-valley continuous solution to (1.3). Then it is easy to prove that f_0 satisfies (1)–(4). Indeed, (1) follows from Definition 2.1 and Lemma 2.4; (2) can be concluded directly from (1.3); (3) is a direct conclusion of Lemmas 2.3 and 2.6; And Lemma 2.7 implies (4).

Conversely, suppose that f_0 satisfies the conditions (1)–(4). Set

$$g_+ = f_0^{p-1}|_{[\alpha_0,1]},\ g_- = f_0^{p-1}|_{[f_0(\lambda),\alpha_0]}.$$

Then it is easy to see that

$$g_+ : [\alpha_0, 1] \to g_+([\alpha_0, 1]) \text{ and}$$

$$g_- : [f_0(\lambda), \alpha_0] \to g_-([f_0(\lambda), \alpha_0])$$

are both homeomorphisms and g_+ is strictly increasing and g_- strictly decreasing. Set

$$I_0 = [\lambda, 1],\ I_k = [\lambda^{k+1}, \lambda^k],\ \forall k \geq 1.$$

Then f_0 is well-defined on I_0. For $x \in I_1$, we set

$$f_1(x) = \begin{cases} g_+^{-1}(\lambda f_0(\frac{x}{\lambda})),\ x \in [\lambda^2, \lambda\alpha], \\ g_-^{-1}(f_0(\frac{x}{\lambda})),\ x \in [\lambda\alpha, \lambda]. \end{cases} \tag{2.1}$$

And then for each $k \geq 1$, we define inductively

$$f_{k+1}(x) = g_+^{-1}(\lambda f_k(\frac{x}{\lambda})),\ x \in I_{k+1}. \tag{2.2}$$

Finally, let

$$f(x) = \begin{cases} 1,\ x = 0, \\ f_k(x),\ x \in I_k. \end{cases} \tag{2.3}$$

We prove that f is exactly what we need.

(1) f is well-defined.

To see this it suffices to show that f_k and f_{k+1} coincide at $I_k \cap I_{k-1} = \{\lambda^k\}$. We use the induction. For $k = 1$,

$$f_1(\lambda) = g_-^{-1}(\lambda f_0(\frac{\lambda}{\lambda})) = g_-^{-1}(\lambda f_0(1)) = f_0(\lambda).$$

The last equality holds because

$$g_-(f_0(\lambda)) = f_0^{p-1} f_0(\lambda) = f_0^p(\lambda) = \lambda f_0(1).$$

Suppose that for $k = n$, $f_n(\lambda^n) = f_{n-1}(\lambda^n)$ has been proved. For $k = n+1$, we have from (2.2)

$$f_{n+1}(\lambda^{n+1}) = g_+^{-1}(\lambda f_n(\frac{\lambda^{n+1}}{\lambda})) = g_+^{-1}(\lambda f_n(\lambda^n))$$

$$= g_+^{-1}(\lambda f_{n-1}(\lambda^n)) = g_+^{-1}(\lambda f_{n-1}(\frac{\lambda^{n+1}}{\lambda})) = f_n(\lambda^{n+1}).$$

The induction is complete.

(2) f is continuous.

Since one can see easily that f is continuous on each I_k, it suffices to show that f is continuous at $x = 0$. By induction, we can see that f is strictly decreasing on $(0, \alpha]$. Therefore $\{f_k(\lambda^k \alpha)\}_{k=2}^{\infty}$ is a strictly increasing sequence on $[\alpha, 1]$. Let

$$\lim_{k \to \infty} f_k(\lambda^k \alpha) = \beta.$$

Then $\beta \in [\alpha, 1]$. By (2.2), $g_+(f_k(\lambda^k \alpha)) = \lambda f_{k-1}(\lambda^{k-1}\alpha)$. Also, it may be written as $f_0^{p-1}(f_k(\lambda^k \alpha)) = \lambda f_{k-1}(\lambda^{k-1}\alpha)$.

Letting $k \to \infty$, we obtain $f_0^{p-1}(\beta) = \lambda\beta$. By condition (4) in Theorem 2.1, $\beta = 1 = f(0)$. This proves that f is continuous at $x = 0$.

(3) f is the unique single-valley solution of equation (1.3) determined by f_0.

From the definition of f and (1.3), this can be concluded by induction.

Thus the proof of Theorem 2.1 is complete.

Remark 2.1. When $p = 2$, the condition (3) in Theorem 2.1 implies $\lambda < f_0(\lambda) < \alpha < 1$, which is identical with the results in [7] and [8]. When $p = 3$, this condition implies

$$\lambda < f_0^2(\lambda) < \alpha < f_0(1) < f_0(\lambda) < 1,$$

which is just the same as the results in [2].

§3. Piecewise Smooth Single–Valley Solutions

Definition 3.1. *A continuous single–valley solution f of* (1.3) *is said to be piecewise C^1, if f is continuously differentiable on each interval where it is monotone.*

Restricting the initial function f_0 by additional condition, we can obtain the piecewise C^1 single–valley solutions of (1.3) which are related to the even unimodal C^1 solutions of (1.2).

Theorem 3.1. *Let $0 < \lambda < 1$, $\alpha \in (\lambda, 1)$ and let f_0 be continuous on $[\lambda, 1]$, and be C^1 on each of $[\lambda, \alpha]$ and $[\alpha, 1]$. If*

(1) $f_0(\alpha) = 0$;

(2) $f_0^{p-1}(1)=\lambda$, $f_0^p(\lambda)=\lambda f_0(1)$;

(3) *Denote* $[f_0(\lambda),1]$ *by* J_0 *and* $f^i(J_0)$ *by* J_i, *then*

(a) $J_0,\ J_1,\ \cdots,\ J_{p-2}\subset(\lambda,1]$ *are pairwise disjoint*,

(b) $f_0^i|_{J_0}:J_0\to J_i$ *is a diffeomorphism for each* $i=0,\ 1,\ \cdots,\ p-2$,

(c) α *is in the interior of* J_{p-2};

(4) *The equation* $f_0^{p-1}(x)=\lambda x$ *has only one solution* $x=1$ *on* $(\alpha_0,1]$, *where* $\alpha_0\in J_0$ *with* $f_0^{p-1}(\alpha_0)=0$;

(5) $f_0'(x)>0$ *for each* $x\in[\alpha,1]$ *and* $f_0'(x)<0$ *for each* $x\in[\lambda,\alpha]$; $f_0'(\alpha+0)=-f_0'(\alpha-0)$;

$$f_0'(1)=f_0'(\lambda)\prod_{i=1}^{p-1}f_0'(f_0^i(\lambda));\quad \frac{df_0^{p-1}}{dx}(1)>1,$$

then there exists an unique piecewise C^1 *single–valley solution* f *of equation* (1.3) *satisfying* (1) $f|_{[\lambda,1]}=f_0$, (2) $f'(0)=0$, (3) $f'(x)<0$ *for each* $x\in(0,\lambda]$.

Proof. By Theorem 2.1, we may assume that f is the unique continuous single–valley solution of (1.3) determined by f_0. By induction, it is easy to check that f is C^1 and has negative derivative on each of $[\lambda^2,\lambda\alpha]$, $[\lambda\alpha,\lambda]$ and I_2, I_3, $\cdots$. To complete the proof of the theorem, we shall first prove that f is continuously differentiable at $x=\lambda\alpha$ and $f'(\lambda\alpha)<0$.

Differentiating the equation $f(x)=\dfrac{1}{\lambda}f^p(\lambda x)$ with respect to x, we have

$$f'(x)=\prod_{i=0}^{p-1}f'(f^i(\lambda x)). \tag{3.1}$$

It can be written as

$$f'(x)=f'(f^{p-1}(\lambda x))f'(\lambda x)\prod_{i=1}^{p-2}f'(f^i(\lambda x)). \tag{3.2}$$

Letting $x\to\alpha+0$ and $x\to\alpha-0$ respectively, we have

$$f'(\alpha+0)=f'(\alpha\pm0)f'(\lambda\alpha+0)\prod_{i=1}^{p-2}f'(f^i(\lambda\alpha)), \tag{3.3}$$

and

$$f'(\alpha-0)=f'(\alpha\mp0)f'(\lambda\alpha-0)\prod_{i=1}^{p-2}f'(f^i(\lambda\alpha)), \tag{3.4}$$

where $f'(\alpha\pm0)=-f'(\alpha\mp0)$.

Comparing (3.3) with (3.4), we can know that f is continuously differentiable at $x=\lambda\alpha$. Furthermore from

$$|f'(\lambda\alpha)|=|f'(\lambda\alpha+0)|=|f'(\lambda\alpha-0)|=\Big|\prod_{i=1}^{p-2}f'(f^i(\lambda\alpha))\Big|^{-1}\neq0,$$

$$f'(\lambda\alpha)=f'(\lambda\alpha+0)<0.$$

Secondly, we prove that f is continuously differentiable at $x=\lambda^k$ and $f'(\lambda^k)<0$ for each $k=1,\ 2,\ \cdots$. Letting $x\to1-0$ for (3.1), we obtain

$$f'(1)=f'(\lambda-0)\prod_{i=1}^{p-1}f_0'(f_0^i(\lambda)).$$

With reference to condition (5) of Theorem 3.1, we know immediately that f is continuously differentiable at $x = \lambda$ and $f'(\lambda) = f_0'(\lambda) < 0$. Now suppose that f is continuously differentiable at $x = \lambda^n$ and $f'(\lambda^n) < 0$. For (3.1), letting $x \to \lambda^n + 0$ and $x \to \lambda^n - 0$ respectively, we obtain

$$f'(\lambda^{n+1} + 0) = f'(\lambda^n)\Big[\prod_{i=1}^{p-1} f_0'(f_0^i(\lambda^{n+1}))\Big]^{-1} = f'(\lambda^{n+1} - 0).$$

This shows that f is continuously differentiable at $x = \lambda^{n+1}$. Since

$$f'(\lambda^n)\Big[\prod_{i=1}^{p-1} f_0'(f_0^i(\lambda^{n+1}))\Big]^{-1} \neq 0,$$

it follows that $f'(\lambda^{n+1}) = f'(\lambda^{n+1} + 0) < 0$. Thus by induction we have proved that f is continuously differentiable at $x = \lambda^k$ and $f'(\lambda^k) < 0$ for each $k = 1, 2, \cdots$.

Finally, we rewrite (3.1) as $f'(x) = f'(\lambda x)\dfrac{df^{p-1}}{dx}(f(\lambda x))$. Taking absolute value in both sides of this equality, we have

$$|f'(\lambda x)| = \Big|\frac{df^{p-1}}{dx}(f(\lambda x))\Big|^{-1}|f'(x)|. \tag{3.5}$$

By condition (5), there are $0 < r < 1$ and $x_0 > 0$ such that if $x < x_0$ then

$$\Big|\frac{df^{p-1}}{dx}(f(\lambda x))\Big|^{-1} \le r.$$

From (3.5), for $x < x_0$

$$|f'(\lambda x)| \le r|f'(x)|. \tag{3.6}$$

Set $K = \max\{|f'(x)| : \lambda x_0 \le x \le x_0\}$. It is clear that for each $x < \lambda x_0$ there are some $n = n(x)$ and some $\overline{x} \in [\lambda x_0, x_0]$ such that $x = \lambda^n \overline{x}$. Using (3.6) repeatedly, we get

$$|f'(x)| = |f'(\lambda^n \overline{x})| \le r|f'(\lambda^{n-1}\overline{x})| \le \cdots \le r^n|f'(\overline{x})| \le r^n K.$$

Therefore $\lim\limits_{x\to 0} f'(x) = \lim\limits_{n\to\infty} r^n K = 0$. This implies that f is continuously differentiable at $x = 0$ and $f'(0) = 0$. The proof of Theorem 3.1 is finished.

§4. Even Unimodal C^1 Solutions

Definition 4.1. *Let f be a continuous map of $[-1, 1]$ into itself. We call f an even unimodal solution, if (1) $f(0) = 1$; (2) for each $x \in [-1, 1]$, $f(x) = f(-x)$; (3) f is strictly decreasing for $x > 0$. If, in addition, f is C^k ($k \ge 0$), then f is said to be an even unimodal C^k solution.*

The proofs of the following Lemma 4.1 and Theorem 4.1 are simple, they are omitted here.

Lemma 4.1. *If f is an even unimodal C^0 solution of equation (1.2), then $f(1) < 0$. Therefore $f(\alpha) = 0$ for some $\alpha \in (0, 1)$.*

Theorem 4.1. *For fixed p, there are following relations between the solutions of (1.2) and (1.3):*

(1) *If $g(x)$ is an even unimodal C^1 (or C^0) solution of equation (1.2) relative to λ, then $f(x) = |g(x)|$ ($x \in [0, 1]$) is a piecewise single–valley C^1 (C^0, respectively) solution of equation (1.3) relative to $|\lambda|$.*

(2) *If* $f(x)$ *is a piecewise single–valley* C^1 *solution of equation* (1.3) *relative to* λ, *satisfying* $f'(0) = 0$ *and* $f'(\alpha + 0) = -f'(\alpha - 0)$, *then* $g(x) = \mathrm{Sgn}\ (\alpha - |x|)f(|x|)$ *is an even unimodal* C^1 *solution of equation* (1.2) *relative to* $\mathrm{Sgn}\ (\alpha - f^{p-2}(1))\lambda$.

As a direct conclusion of Theorems 2.1 and 3.1, we give

Theorem 4.2. *Let* $0 < |\lambda| < 1$, *and* f_0 *be a* C^1 *function on* $[-1, |\lambda|] \cup [|\lambda|, 1]$. *If*

(1) $f_0(x) = f_0(-x)$ *and* $f_0(\alpha) = 0$ *for some* $\alpha \in (|\lambda|, 1)$;

(2) $f_0^{p-1}(1) = \lambda$, $f_0^p(\lambda) = \lambda f_0(1)$;

(3) *Denote* $[f_0(\lambda), 1]$ *by* J_0 *and* $f_0^i(J_0)$ *by* J_i, *then*

(a) $J_0, J_1, \cdots, J_{p-2} \subset [-1, -|\lambda|) \cup (|\lambda|, 1]$ *are pairwise disjoint;*

(b) $f_0^i|_{J_0} : J_0 :\to J_i$ *is a diffeomorphism for each* $i = 0, 1, \cdots, p-2$,

(c) α *is in the interior of* J_{p-2};

(4) *The equation* $f_0^{p-1}(x) = \lambda x$ *has only one solution* $x = 1$, *where* $\alpha_0 \in J_0$ *with* $f_0^{P-1}(\alpha_0) = 0$;

(5) $f_0'(1) = f_0'(\lambda)\prod_{i=1}^{p-1} f_0'(f_0^i(\lambda))$, $\dfrac{df_0^{p-1}}{dx}(1) < -1$ *and* $f_0'(x) < 0$ *for* $|\lambda| \le x \le 1$,

then equation (1.2) *has only one even unimodal* C^1 *solution* f *with* $f|_{[-1,-|\lambda|]\cup[|\lambda|,1]} = f_0$.

Remark 4.1. For making the solution smoother, it suffices to restrict the initial function f_0 further. We can see from the proof of Theorem 2.1 that Theorems 2.1, 3.1, 4.2 not only reveal the existence of some kind of solutions, but also give a feasible method to construct such solutions. An interesting problem is if there exist initial functions relative to some one–parameter families (for example, $f(x) = 1 - \mu x^2$, etc.). We shall discuss this problem in another paper.

In addition, the condition (5) of Theorem 4.2 is not necessary to an even unimodal C^1 solution f of equation (1.2), but we can conclude from Theorems 4.1 and 3.1 that f must satisfy the first four conditions. Hence we can check easily that for $p = 2$ or 3 equation (1.2) may have some unimodal C^0 solutions only if $\lambda < 0$.

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