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SOURCE–TYPE SOLUTIONS OF A QUASILINEAR DEGENERATE PARABOLIC EQUATION WITH ABSORPTION

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Abstract

The existence and nonexistence of non–trivial solutions for the Cauchy problem of the form

$$u_t = \text{div } (|\nabla u|^{p-2}\nabla u) - u^q,$$
$$u(x,0) = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}$$

are studied.

Keywords Quasilinear degenerate parabolic equation, Cauchy problem, Non–trivial solution.

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§1. Introduction

In this paper we consider the Cauchy problem

$$u_t = \text{div } (|\nabla u|^{p-2}\nabla u) - u^q \text{ in } S_T = \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = 0 \qquad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.2}$$

where $p > \dfrac{2N}{N+1}$, $q > 0$.

Equation (1.1) is a prototype of a certain class of degenerate equations and appears to be relevant to the theory of non-Netonian fluids[1]. The case when the initial datum is a measure is also a model for physical phenomena. The goal of this paper is to give necessary and sufficient conditions which guarantee that (1.1) (1.2) has a non–trivial solution. For the case when $p = 2$, it was shown in [2] that if $0 < q < 1 + \dfrac{2}{N}$, the problem (1.1) (1.2) has a solution satisfying the initial condition

$$u(x,0) = \delta(x), \tag{1.3}$$

where $\delta(x)$ denotes the Dirac mass centered at the origin, and that if $q \geq 1 + \dfrac{2}{N}$, the problem (1.1), (1.3) has no solution. In addition, it was shown in [3] and [4] that if $p \geq 2$ and $p - 1 < q < p - 1 + \dfrac{p}{N}$, the equation (1.1) has a very singular solution, i.e., a solution

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w with the following properties:

$$w \in C(\overline{S}_T \setminus \{(0,0)\}), \tag{1.4}$$

$$w(x,0) = 0 \text{ if } x \in I\!R^N \setminus \{0\}, \tag{1.5}$$

$$\lim_{t\to 0^+} \int_{|x|<r} w(x,t)dx = +\infty, \text{ for every } r > 0. \tag{1.6}$$

Our results are as follows.

Theorem 1.1. *Suppose that* $p > \dfrac{2N}{N+1}$ *and* $0 < q < p - 1 + \dfrac{P}{N}$. *Then* (1.1) (1.3) *has a solution.*

Theorem 1.2. *Suppose that either* $p < \dfrac{2N}{N+1}$, $\quad q \ge 0$ *or* $p \ge \dfrac{2N}{N+1}$ *and* $q > p-1+\dfrac{p}{N}$. *Then* (1.1) (1.3) *has no solution.*

Theorem 1.3. *Suppose that* $p > \dfrac{2N}{N+1}$ *and* $\max\{1, p-1\} < q < p - 1 + \dfrac{p}{N}$. *Then* (1.1) (1.3) *has a very singular solution.*

Since the equation (1.1) is a quasilinear degenerate equation when $p \ne 2$, the proofs of Theorem 1.1 and Theorem 1.2 are different from [2]. In [3] and [4], the existence of very singular solution is obtained by using O.D.E. method by means of the symmetric property of the equation. In the proof of Theorem 1.3, we adopt a P.D.E. method such as Imbedding Theorem and Moser iteration, which seems a more natural and more powerful approach. Moreover, Theorem 1.3 permits $\dfrac{2N}{1+N} < p < 2$. Thus we generalize the results in [4].

§2. Proof of Theorem 1.1

Definition 2.1. *A solution u of* (1.1) (1.3) *is a nonnegative function defined in S_T such that*

1. $u \in C(0,T; L^1(I\!R^N)) \cap L^\infty(I\!R^N \times (\tau, T)) \cap C(\overline{S}_T \setminus \{(0,0)\})$ *for every* $\tau \in (0,T)$,
2. $|\nabla u| \in L^p_{\text{Loc}}(S_T)$,
3. $\int_{I\!R^N} u\eta(x,\tau)dx\big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{I\!R^N} (u\eta_t - |\nabla u|^{p-2}\nabla u\cdot\nabla\eta - u^q\eta)dxdt = 0$ *for every* $\eta \in C^2(S_T)$ *which vanishes for large* $|x|$, *and* $0 < t_1 < t_2$,
4. $\lim\limits_{t\to 0^+} \int_{I\!R^N} u(x,t)\psi(x)dx = \psi(0)$ *for every* $\psi \in C_0^\infty(I\!R^N)$.

First, we discuss the following Dirichlet problem.

$$u_t = \text{div}\left((|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla u\right) - u^q \text{ in } B_{\varepsilon^{-1}} \times (0,T), \tag{2.1}$$

$$u(x,t) = 0 \quad \text{on} \quad |x| = \varepsilon^{-1}, \tag{2.2}$$

$$u(x,0) = k^N h(kx) \quad \text{on} \quad B_{\varepsilon^{-1}}, \tag{2.3}$$

where $h(x) \in C_0^\infty(B_{\varepsilon^{-1}})$, $h(x) \ge 0$, $\int_{I\!R^N} h(x)dx = 1$, $k > 0$, B_R denotes the N-ball of radius R with centre 0.

It is well known that (2.1)–(2.3) has a nonnegative classical solution u_k^ε [5]. We have also the following estimates.

By means of approximate process, in the following context, we can assume $u_k^\varepsilon > 0$. Moreover, without loss of generality, we use C to denote the constants independent of k, ε, although they may change from line to line in the same proof.

Lemma 2.1. *The solution* u_k^ε *of* (2.1)–(2.3) *satisfies*

$$\int_{B_{\varepsilon^{-1}}} u_k^\varepsilon(x,t)dx + \int_0^t \int_{B_{\varepsilon^{-1}}} (u_k^\varepsilon)^q dxds \le 1.$$

Proof. For simplicity of notation we drop the superscript and the subscript. Multiplying (2.1) by $\dfrac{u}{u+r}$ $(r>0)$ and integrating over $B_{\varepsilon^{-1}} \times (0,t)$, we have

$$\begin{aligned}&\int_{B_{\varepsilon^{-1}}} \frac{u^2(x,t)}{u(x,t)+r}dx + \int_0^t \int_{B_{\varepsilon^{-1}}} \frac{u^{q+1}}{u+r}dxd\tau \\ =&\int_{B_{\varepsilon^{-1}}} \frac{k^{2N}h^2(kx)}{k^N h(kx)+r}dx - \int_0^t \int_{B_{\varepsilon^{-1}}} \frac{r(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2}{(u+r)^2}dxd\tau \\ &+\int_0^t \int_{B_{\varepsilon^{-1}}} \frac{ru_t u}{(u+r)^2}dxd\tau. \end{aligned} \tag{2.4}$$

Letting $r \to 0^+$ in (2.4), we obtain

$$\int_{B_{\varepsilon^{-1}}} u(x,t)dx + \int_0^1 \int_{B_{\varepsilon^{-1}}} u^q dxds \le \int_{I\!R^N} k^N h(kx)dx = 1.$$

Lemma 2.2. *Let* $p>1$. *Then the solution* u_k^ε *of* (2.1)–(2.3) *satisfies*

$$\int_0^T \int_{B_{\varepsilon^{-1}}} \frac{(u_k^\varepsilon)^{\alpha-1}}{(1+(u_k^\varepsilon)^\alpha)^2}|\nabla u_k^\varepsilon|^p dxdt < C(\alpha), \tag{2.5}$$

$$\int_0^T \int_{B_k} (u_k^\varepsilon)^{p-1+\frac{p}{N}-\alpha} dxdt \le C(\alpha). \tag{2.6}$$

where $2R<\varepsilon^{-1}$, $0<\alpha<p-1$.

Proof. We multiply (2.1) by $u^\alpha(1+u^a)^{-1}$ and integrate it to obtain

$$\begin{aligned}&\int_{B_{\varepsilon^{-1}}} \int_0^{u(x,T)} \frac{s^\alpha}{1+s^\alpha}dsdx + \alpha\int_0^T \int_{B_{\varepsilon^{-1}}} \frac{u^{\alpha-1}}{(1+u^\alpha)^2}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2 dxdt \\ \le&\int_{B_{\varepsilon^{-1}}} \int_0^{k^N h(kx)} \frac{s^\alpha}{1+s^\alpha}dsdx.\end{aligned}$$

This implies (2.5).

Let

$$v=(u-M)^+ + M, \qquad w = v^{\frac{p-1-\alpha}{p}}.$$

Then by Sobolev's imbedding inequality ([5], p. 62), we get

$$\Big(\int_{I\!R^N} \xi^p w^r dx\Big)^{\frac{1}{r}} \le C\Big(\int_{I\!R^N} |\nabla(\xi w)|^p dx\Big)^{\frac{\theta}{p}} \Big(\int_{B_{2R}} w^{\frac{p}{p-1-\alpha}}dx\Big)^{(1-\theta)\frac{p-1-a}{p}},$$

where

$$\theta = \Big(\frac{p-1-\alpha}{p} - \frac{1}{r}\Big)\Big(\frac{1}{N} - \frac{1}{p} + \frac{p-1-\alpha}{p}\Big)^{-1}.$$

From (2.7), we have

$$\int\int_{S_T} \xi^p w^r dxdt \le C\int_0^T \Big(\int_{I\!R^N} |\nabla(\xi w)|^p dx\Big)^{\frac{\theta r}{p}} dt \cdot \sup_{t\in(0,T)} \Big(\int_{B_{2R}} w^{\frac{p}{p-1-\alpha}}dx\Big)^{(1-\theta)r\frac{p-1-a}{p}}.$$

For

$$r=\frac{p(p-1+\frac{p}{N}-\alpha)}{p-1-\alpha},$$

we obtain

$$\int\int_{S_T}\xi^p w^r dxdt\le C\int\int_{S_T}|\nabla(\xi w)|^p dxdt\sup_{t\in(0,T)}\Big(\int_{B_{2R}}w^{\frac{p}{p-1-\alpha}}dx\Big)^{(r-p)\frac{p-a-1}{p}}.$$

Hence, by Lemma 2.1 and (2.5), we have

$$\int\int_{S_T}\xi^p u^{p-1+\frac{p}{N}-\alpha}dxdt\le C(\alpha)\Big\{1+\int\int_{S_T}|\nabla\xi|^p u^{p-1-\alpha}dxdt\Big\}. \tag{2.8}$$

We take $\xi=\psi^b$, where $\psi\in C_0^\infty(B_{2R})$, $0\le\psi\le 1$, $\psi=1$ if $x\in B_{I\!R}$, and $b=\dfrac{(p-1+\frac{p}{N}-\alpha)N}{p}$ in (2.8). We obtain

$$\int\int_{S_T}\psi^{bp}u^{p-1+\frac{p}{N}-\alpha}dxdt\le C(\alpha)\Big(1+(\int\int_{S_T}\psi^{bp}u^{p-1+\frac{p}{N}\alpha}dxdt)^{\frac{p-1-\alpha}{p-1+\frac{p}{N}-\alpha}}\Big).$$

This implies (2.6).

Denote

$$Q_R=B_R(x_0)\times(t_0-R^p,t_0).$$

Lemma 2.3. *Let* $p>\dfrac{2N}{N+1}$, $0<q<p-1+\dfrac{p}{N}$. *Then* u_k^ε *satisfies*

$$\int\int_{Q_R}(u_k^\varepsilon)^m dxdt\le C(R),\qquad\forall m>0, \tag{2.9}$$

$$\sup_{Q_R}u_k^\varepsilon\le C(r)\Big\{\int\int_{Q_{2R}}(u_k^\varepsilon)^{\frac{2N}{p}-N+r}dxdt\Big\}^{\frac{1}{r}},\ \text{if } p\le 2, \tag{2.10}$$

$$\sup_{Q_R}u_k^\varepsilon\le C(r)\Big\{\int\int_{Q_{2R}}(u_k^\varepsilon)^{p-2+r}dxdt\Big\}^{\frac{1}{r}},\ \text{if } p>2, \tag{2.11}$$

$$\int\int_{Q_R}|\nabla u_k^\varepsilon|^p dxdt\le C(R), \tag{2.12}$$

where $0<r<p/N+1$, $2R<\varepsilon^{-1}$, $t_0-2R^r>0$.

Proof. Let $\xi(x,t)$ be a cutoff function in Q_{2R}, $0\le\xi\le 1$. We multiply (2.1) by $\xi^p u^{2\gamma-1}$ with $\gamma>\dfrac{1}{2}$ to obtain

$$\begin{aligned}&\frac{1}{2\gamma}\int_{B_{2R}}\xi^p u^{2\gamma}(x,t)dx+(2\gamma-1)\int_0^t\int_{B_{2R}}\xi^p u^{2\gamma-2}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2dxds\\ \le&p\int_0^t\int_{B_{2R}}\xi^{p-1}|\nabla\xi|u^{2\gamma-1}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|dxds+\frac{p}{2\gamma}\int_0^t\int_{B_{2R}}\xi^{p-1}|\xi_t|u^{2\gamma}dxds.\end{aligned}$$

Note that if $1<p<2$,

$$(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2\ge(|\nabla u|^2+\varepsilon)^{\frac{p}{2}}-\varepsilon^{\frac{p}{2}},$$
$$(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|\le(|\nabla u|^2+\varepsilon)^{\frac{p-1}{2}};$$

if $p\ge 2$,

$$(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2\ge|\nabla u|^p,$$
$$(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|\le C(|\nabla u|^{p-1}+1).$$

Using Young's inequality, we can obtain if $1 < p < 2$

$$\begin{aligned}&\sup_{t_0-2R^p<t<t_0}\int_{B_{2R}}\Big(\xi u^{\frac{2\gamma-2+p}{p}}\Big)^{\frac{2\gamma p}{2\gamma-2+p}}dx+\int\int_{Q_{2R}}\Big|\nabla\Big(\xi u^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt\\ \le& C\int\int_{Q_{2R}}|\nabla\xi|^p u^{2\gamma-2+p}dxdt+C\gamma^{p-2}\int\int_{Q_{2R}}|\xi_t|u^{2\gamma}dxdt\\ &+C\gamma^p\varepsilon^{\frac{p}{2}}\int\int_{Q_{2R}}\xi^p u^{2\gamma-2}dxdt; \qquad (2.13)\end{aligned}$$

if $p \ge 2$

$$\begin{aligned}&\sup_{t_0-2R^p<t<t_0}\int_{B_{2R}}\Big(\xi^{\frac{2\gamma-2+p}{2\gamma}}u^{\frac{2\gamma-2+p}{p}}\Big)^{\frac{2\gamma p}{2\gamma-2+p}}dx+\int\int_{Q_{2n}}\Big|\nabla\Big(\xi^{\frac{2\gamma-2+p}{2\gamma}}u^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt\\ \le& C\int\int_{Q_{2R}}|\nabla\xi|^p u^{2\gamma-2+p}dxdt+C\gamma^{p-2}\int\int_{Q_{2R}}|\xi_t|u^{2\gamma}dxdt\\ &+C\gamma^{p-1}\int\int_{Q_{2R}}\xi^{p-1}|\nabla\xi|u^{2\gamma-1}dxdt. \qquad (2.14)\end{aligned}$$

First, we discuss the case when $p < 2$. Let

$$v=\xi u^{\frac{2\gamma-2+p}{p}}.$$

Using the imbedding inequality, we have

$$\begin{aligned}\int\int_{Q_{2R}}v^r dxdt \le& C\Big(\sup_{t_0-(2R)^p<t<t_0}\int_{B_{2R}}v^{\frac{2\gamma p}{2\gamma-2+p}}dx\Big)^{\frac{2\gamma-2+p}{2\gamma p}(1-\delta)r}\times\\ &\times\int_{t_0-(2R)^p}^{t_0}\Big(\int_{B_{2R}}|\nabla v|^p dx\Big)^{\frac{\delta r}{p}}dt,\end{aligned}$$

where

$$\delta=\Big(\frac{2\gamma-2+p}{2\gamma p}-\frac{1}{r}\Big)\Big(\frac{1}{N}-\frac{1}{p}+\frac{2\gamma-2+p}{2\gamma p}\Big)^{-1}.$$

Hence choosing

$$r=p\Big(1+\frac{2\gamma p}{N(2\gamma-2+p)}\Big),$$

we obtain

$$\begin{aligned}&\int\int_{Q_{2R}}\xi^r u^{2\gamma-2+p+\frac{2\gamma p}{N}}dxdt\\ \le& C_1\Big(\sup_{t_0-(2R)^p<t<t_0}\int_{B_{2R}}\xi^{\frac{2\gamma p}{2\gamma-2+p}}u^{2\gamma}dx\Big)^{\frac{p}{N}}\cdot\int\int_{Q_{2R}}\Big|\nabla\Big(\xi u^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt\\ \le& C_2\Big\{\sup_{t_0-(2R)^p<t<t_0}\int_{B_{2R}}\xi^{\frac{2\gamma p}{2\gamma-2+p}}u^{2\gamma}dx+\int\int_{Q_{2R}}\Big|\nabla\Big(\xi u^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt\Big\}^{1+\frac{p}{N}}\\ \le& C_3\Big(\int\int_{Q_{2R}}|\nabla\xi|^p u^{2\gamma-2+p}dxdt+\gamma^{p-2}\int\int_{Q_{2R}}|\xi_t|u^{2\gamma}dxdt\\ &+\gamma^p\int\int_{Q_{2R}}\xi^p u^{2\gamma-2}dxdt\Big)^{1+\frac{p}{N}}. \qquad (2.15)\end{aligned}$$

Taking $1 < 2\gamma < p-1+\dfrac{p}{N}$ in (2.15), by Lemma 2.2, we obtain $u \in L^{2\gamma-2+p+\frac{2\gamma p}{N}}(Q_R)$. Since

$$2\gamma-2+p+\frac{2\gamma p}{N}>2\gamma \quad \text{if } p>\frac{2N}{N+1},$$

we can repeat above process to obtain (2.9).

For $\tau \in \Big[\frac{1}{2},1\Big)$, set

$$R_t = 2R\Big(\tau + \frac{1-\tau}{2^l}\Big), \qquad l = 0,1,2,\cdots$$

and let $\xi_t(x,t)$ be a cutoff function in Q_{R_l}, with $\xi_l = 1$ in $Q_{R_{l+1}}$. Denote $1+\dfrac{p}{N}$ by K and choose y such that $2\gamma = \dfrac{2N}{p} - N + K^l$. Let $u_1 = \max\{,u\}$. Then we get from (2.14)

$$\int\int_{Q_{R_{l+1}}} u^{\frac{2N}{p}-N+K^{l+1}}dxdt \le \int\int_{Q_{R_{l+1}}} u_1^{\frac{2N}{p}-N+K^{l+1}}dxdt$$
$$\le \Big\{\frac{C^l}{((1-\tau)R)^p}\int\int_{Q_{R_l}} u_1^{\frac{2N}{p}-N+K^l}dxdt\Big\}^k.$$

Hence the standard Moser's iteration yields

$$\sup_{Q_{2R}} u_1 \le C\Big\{\frac{1}{((1-\tau)R)^{p+N}}\int\int_{Q_{2R}} u_1^{\frac{2N}{p}-N+K}dxdt\Big\}^{\frac{1}{K}}. \tag{2.16}$$

It follows from (2.16) that

$$\sup_{Q_{2R}} u_1 \le \big(\sup_{Q_{2R}} u_1\big)^{\frac{K-r}{K}}\Big\{\frac{1}{((1-\tau)R)^{N+p}}\int\int_{Q_{2R}} u_1^{\frac{2N}{p}-N+r}dxdt\Big\}^{\frac{1}{K}}.$$

Applying Holder's inequality, we get

$$\sup_{Q_{2R}} u_1 \le \frac{1}{2}\sup_{Q_{2R}} u_1 + C(r)\Big\{\frac{1}{((1-\tau)R)^{N+p}}\int\int_{Q_{2R}} u_1^{\frac{2N}{p}-N+r}dxdt\Big\}^{\frac{1}{r}}.$$

Hence by [6, p. 161, Lemma 3.1], we obtain

$$\sup_{Q_{2R}} u_1 \le C(r)\Big\{\frac{1}{R^{N+p}}\int\int_{Q_{2R}} u_1^{\frac{2N}{p}-N+r}dxdt\Big\}^{\frac{1}{r}} \text{ for } \tau \in \Big[\frac{1}{2},1\Big).$$

If $p \ge 2$, we can obtain from (2.14)

$$\int\int_{Q_{R_{l+1}}} u_1^{p-2+K^{l+1}}dxdt \le \Big\{CC_1^l\int\int_{Q_{k_l}} u_1^{p-2+k^l}dxdt\Big\}^K.$$

Hence, we can repeat the same argument as $p<2$ to obtain (2.11).

Combining (2.9), (2.10), (2.11), (2.13) and (2.14) we can obtain (2.12), and Lemma 2.3 is proved.

Lemma 2.4. *For all ball* $\overline{B_R(x_0)} \subset I\!R^N\setminus\{0\}$ *and* $\forall m>0$, $0<r<1+p/N$, *the solution* u_k^ε *of* (2.1)–(2.3) *satisfies*

$$\int_0^T\int_{B_R}(u_k^\varepsilon)^m dxdt \le C(R), \tag{2.17}$$

$$\sup_{B_R(x_0)\times(0,T)} u_k^\varepsilon \le C(r)\Big\{\int_0^T\int_{B_R(x_0)}(u_k^\varepsilon)^{\frac{2N}{p}-N+r}dxdt\Big\}^{\frac{1}{r}} \ \textit{for}\ p\le 2, \tag{2.18}$$

$$\sup_{B_R(x_0)\times(0,T)} u_k^\varepsilon \le C(r)\Big\{\int_0^T\int_{B_R(x_0)}(u_k^\varepsilon)^{p-2+r}dxdt\Big\}^{\frac{1}{r}} \ \textit{for}\ p\ge 2, \tag{2.19}$$

$$\int_0^T\int_{B_R(x_0)}|\nabla u_k^\varepsilon|^p dxdt \le C. \tag{2.20}$$

Proof. Let $\xi(x) \in C_0^2(I\!R^N)$, $0 \le \xi \le 1$, supp $\xi \subset I\!R^N \setminus \{0\}$, $\xi = 1$ on $B_R(x_0)$. Multiplying (2.1) by $\xi^p u^{2\gamma-1}$ with $\gamma > \dfrac{1}{2}$ and integrating, we have

$$\int_{I\!R^N} \frac{u^{2\gamma}(x,t)}{2\gamma} dx + (2\gamma-1) \int_0^t \int_{I\!R^N} \xi^p u^{2\gamma-2} (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u|^2 dxds$$
$$= -p \int_0^t \int_{I\!R^N} \xi^{p-1} u^{2\gamma-1} (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u \cdot \nabla \xi dxds + \frac{1}{2\gamma} \int_{I\!R^N} \xi^p (k^N h(kx))^{2\gamma} dx.$$

Since supp $\xi \subset I\!R^N \setminus \{0\}$, $h \in C_0^\infty(I\!R^N)$, we have

$$k^N h(kx) \le C_0 \qquad x \in \text{ supp } \xi.$$

Hence, using Young's inequality, as in Lemma 2.3, we get that if $1 < p < 2$

$$\int_{I\!R^N} \xi^p u^{2\gamma}(x,t) dx + \gamma^2 \int_0^t \int_{I\!R^N} \xi^p u^{2\gamma-2} |\nabla u|^p dxds$$
$$\le C\Big\{\gamma^{2-p} \int_0^T \int_{I\!R^N} |\nabla \xi|^p u^{2\gamma-2+p} dxdt + \gamma^2 \int_0^T \int_{I\!R^N} \xi^p u^{2\gamma-2} dxdt\Big\}$$
$$+ \int_{I\!R^N} \xi^p C_0^{2\gamma} dx;$$

if $p \ge 2$

$$\int_{I\!R^N} \xi^p u^{2\gamma}(x,t) dx + \gamma^2 \int_0^t \int_{I\!R^N} \xi^p u^{2\gamma-2} |\nabla u|^p dxds$$
$$\le C\Big\{\gamma^{2-p} \int_0^T \int_{I\!R^N} |\nabla \xi|^p u^{2\gamma-2+p} dxdt + \gamma \int_0^T \int_{I\!R^N} \xi^{p-1} |\nabla \xi| u^{2\gamma-1} dxdt\Big\}$$
$$+ C \int_{I\!R^N} \xi^p C_0^{2\gamma} dx.$$

Let $w = \max\{C_0, u\}$. Then, we have

$$\sup_{t\in(0,T)} \int_{I\!R^N} \xi^p w^{2\gamma} dx + \int_0^T \int_{I\!R^N} \Big|\nabla \Big(\xi w^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt$$
$$\le C\gamma^p \sup |\nabla \xi|^p \int_0^T \int_{I\!R^N \cap \text{supp } \xi} w^{2\gamma} dxdt, \qquad \text{if } 1 < p < 2;$$

$$\sup_{t\in(0,T)} \int_{I\!R^N} \xi^p w^{2\gamma} dx + \int_0^T \int_{I\!R^N} \Big|\nabla \Big(\xi w^{\frac{2\gamma-2+p}{p}}\Big)\Big|^p dxdt$$
$$\le C\gamma^{p-1} \sup |\nabla \xi|^p \int_0^T \int_{I\!R^N \cap \text{supp } \xi} w^{2\gamma-2+p} dxdt, \qquad \text{if } p > 2.$$

Therefore (2.17), (2.18), (2.19), (2.20) can be obtained by using a similar argument as in the proof of Lemma 2.3.

Lemma 2.5. *For every compact set $K \subset \overline{S_T} \setminus \{(0,0)\}$ the solution u_k^ε of* (2.1)–(2.3) *satisfies*

$$\sup_k |\nabla u_k^\varepsilon| \le C(K). \tag{2.21}$$

Proof. Differentiating (2.1) with respect to x_1 and setting $v = |\nabla u|^2$, we get

$$\frac{\partial}{\partial t} u_{x_1} = \text{div } \Big((v+\varepsilon)^{\frac{p-2}{2}} \nabla u_{x_1} + \frac{\partial}{\partial x_1} (v+\varepsilon)^{\frac{p-2}{2}} \nabla u\Big) - q u^{q-1} u_{x_1}. \tag{2.22}$$

Let $\overline{B_{2R}(x_0)} \subset I\!R^N \setminus \{0\}$, $\xi(x) \in C_0^1(B_{2R}(x_0))$, $0 \le \xi \le 1$, $\xi = 1$ on $x \in B_R(x_0)$. Multiply (2.21) by $\xi^2(v+\varepsilon)^\alpha u_{x_j}$ $\Big(\alpha \ge \min\Big\{0, \dfrac{p-2}{2}\Big\}\Big)$ and integrate over $I\!R^N \times (0,t)$. After some computation (cf. [7]), we get

$$\begin{aligned}
&\frac{1}{2(\alpha+1)}\int_{I\!R^N}\xi^2(v(x,t)+\varepsilon)^{\alpha+1}dx + \frac{p+2\alpha-2}{4}\int_0^t\int_{I\!R^N}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-4}{2}}|\nabla v|^2 dxds\\
&+\int_0^t\int_{I\!R^N}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-2}{2}}u_{x_ix_j}u_{x_ix_j}dxds + q\int_0^t\int_{I\!R^N}u^{q-1}(v+\varepsilon)^\alpha dxds\\
&+\frac{\alpha(p-2)}{2}\int_0^t\int_{I\!R^N}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-6}{2}}(\nabla u\cdot\nabla v)^2 dxds\\
\le& C\int_0^t\int_{I\!R^N}\xi|\nabla\xi|(v+\varepsilon)^{\frac{p+2\alpha-2}{2}}|\nabla v|dxds\\
&+\frac{1}{2(\alpha+1)}\int_{I\!R^N}\xi^2(k^{2N}|\nabla h(kx)|^2+\varepsilon)^{\alpha+1}dx. \qquad (2.23)
\end{aligned}$$

Note that

$$(v+\varepsilon)^{\frac{p+2\alpha-2}{2}}u_{x_ix_j}u_{x_ix_j} \ge (v+\varepsilon)^{\frac{p+2\alpha-4}{2}}v^2u_{x_ix_j}u_{x_ix_j} \ge \frac14 v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2.$$

and

$$k^{2N}|\nabla h(kx)|^2+\varepsilon \le M \qquad \text{on } \overline{B_R(x_0)},$$

where M is a constant independent of k, ε. We obtain from (2.23)

$$\begin{aligned}
&\frac{1}{2(\alpha+1)}\sup_{0<t<T}\int_{I\!R^N}\xi^2(v+\varepsilon)^{\alpha+1}dx + \frac{p+2\alpha-1}{4}\int\int_{S_T}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-4}{2}}|\nabla v|^2dxdt\\
&+\frac{\alpha(p-2)}{2}\int\int_{S_T}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-6}{2}}(\nabla u\cdot\nabla v)^2dxdt\\
\le& C\Big(\int\int_{S_T}\xi|\nabla\xi|(v+\varepsilon)^{\frac{p+2\alpha-2}{2}}|\nabla v|dxdt + \frac{1}{2(\alpha+1)}M^{\alpha+1}\Big).
\end{aligned}$$

If $p \ge 2$, we apply Cauchy inequality to obtain

$$\begin{aligned}
&\frac{1}{2(\alpha+1)}\sup_{0<t<T}\int_{I\!R^N}\xi^2(v+\varepsilon)^{\alpha+1}dx + \frac{p+2\alpha-1}{8}\int\int_{S_T}\xi^2(v+\varepsilon)^{\frac{p+2\alpha-4}{2}}|\nabla v|^2dxdt\\
\le& \frac{C}{p+2\alpha-1}\int\int_{S_T}|\nabla\xi|^2(v+\varepsilon)^{\frac{p+2\alpha}{2}}dxdt + \frac{CM^{\alpha+1}}{2(\alpha+1)}. \qquad (2.24)
\end{aligned}$$

Letting $u_M = \max\{v+\varepsilon, M\}$, we obtain from (2.24)

$$\begin{aligned}
&\frac{1}{2(\alpha+1)}\sup_{0<t<T}\int_{B_{2R}(x_0)}\xi^2v_M^{\alpha+1}dx + \frac{p+2\alpha-1}{8}\int_0^T\int_{B_{2R}(x_0)}\xi^2v_M^{\frac{p+2\alpha-4}{2}}|\nabla v_M|^2dxdt\\
\le& \frac{C}{p+2\alpha-1}\int_0^T\int_{B_{2R}(x_0)}v_M^{\frac{p+2\alpha}{2}}|\nabla\xi|^2dxdt + \frac{C}{2(\alpha+1)}\int_0^T\int_{B_{2R}(x_0)}\xi v_M^{\alpha+1}dxdt.
\end{aligned}$$

Hence, as in [8], we can obtain

$$\max_{B_R(x_0)\times(0,T)}|\nabla u|^2 \le C\max\{1, \int_t^T\int_{B_{2R}(x_0)}|\nabla u|^p dxdt\}. \qquad (2.25)$$

In analogous fashion, we can prove that for $\tau \in (0,T)$

$$\max_{B_R(x_0)\times(0,T)}|\nabla u|^2 \le C\max C\max\{1, \int_t^T\int_{B_{2R}(0)}|\nabla u|^p dxdt\}. \qquad (2.26)$$

Combining (2.25), (2.26), Lemmas 2.3 and 2.4, (2.21) is proved.

As in [7–8] we can extend (2.21) to case $\frac{2N}{N+1} < p < 2$.

By Lemma 2.5, as in [8], we can prove

Lemma 2.6. *For every compact set $K \subset S_T$, ∇u_k^ε are uniformly Hölder continuous.*

Proof of Theorem 1.1. By Lemma 2.2–Lemma 2.6, the solutions u_k^ε are uniformly bounded and Hölder continuous for k, ε on any compact set $K \subset \overline{S}_T \setminus \{(0,0)\}$. Thus there exist a subsequence $\{u_{k_j}^{\varepsilon_j}\}$, and function $u \in C(\overline{S}_T \setminus \{(0,0)\}$ such that for any compact set $K \subset \overline{S}_T \setminus \{(0,0)\}$,

$$u_{k_j}^{\varepsilon_j} \to u \ \text{ as } j \to \infty \ \text{ in } C(K).$$

We now prove that u is a solution of (1.1) (1.3). It is easy to verify that u has the properties 1 and 2 in Definition 2.1. We need only to verify property 3.

Denote $u_{k_j}^{\varepsilon_j}$ by u_j. Using (2.1), we can obtain

$$\begin{aligned}
&\Big| \int_{I\!R^N} u_j(x,t)\psi(x)dx - \int_{I\!R^N} k^N h(kx)\psi(x)dx \Big| \\
\le & A\int_0^t \int_{B_R} u_j^q dxds + B\int_0^t \int_{B_R} (|\nabla u_j|^2+\varepsilon)^{\frac{p-2}{2}} |\nabla u_j| dxds,
\end{aligned} \tag{2.27}$$

where $\psi \in C_0^\infty(B_R(0))$, $A = |\psi|_{L^\infty}$, $B = |\nabla\psi|_{L^\infty}$.

By Lemma 2.2, we have

$$\begin{aligned}
&\int_0^t \int_{B_R} u_j^q dxds \le C\Big\{ \int_0^t \int_{B_R} (u_j)^{p-1+\frac{p}{N}-\alpha} dxds \Big\}^{\frac{q}{p-1+\frac{p}{N}-\alpha}} t^{\frac{p-1+\frac{p}{N}-a-q}{p-1+\frac{p}{N}-\alpha}} \\
\le & C_1 t^{\frac{p-1+\frac{p}{N}-\alpha-q}{p-1+\frac{p}{N}-\alpha}}, \\
&\int_0^t \int_{B_R} (|\nabla u_j|^2+\varepsilon)^{\frac{p-1}{2}} dxds \\
\le & C\Big\{ t + \Big[\int_0^t \int_{B_R} \frac{u_j^{\alpha-1}}{(1+u_j^\alpha)^2} |\nabla u_j|^p dxds \Big]^{\frac{p-1}{p}} \times \\
& \times \Big[\int_0^t \int_{B_R} (1+u_j^\alpha)^{2(p-1)} u_j^{(1-\alpha)(p-1)} dxds \Big]^{\frac{1}{p}} \Big\} \\
\le & C_1 \Big\{ t + \Big(\int_0^t \int_{B_R} u_j^{(1-\alpha)(p-1)} dxds \Big)^{\frac{1}{p}} + \Big(\int_0^t \int_{B_R} u_j^{(p-1)(\alpha+1)} dxdt \Big)^{\frac{1}{p}} \Big\} \\
\le & C_2 \Big\{ t + \Big(\int_0^t \int_{B_R} u_j^{p-1+\frac{p}{N}-\alpha} dxds \Big)^{\frac{(1-\alpha)(p-1)}{p(p-1+\frac{p}{N}-\alpha)}} t^{\frac{\frac{p}{N}+\alpha p-2\alpha}{p(p-1+\frac{p}{N}-\alpha)}} \\
& + \Big(\int_0^t \int_{B_R} u_j^{p-1+\frac{p}{N}-\alpha} dxds \Big)^{\frac{(1-\alpha)(p-1)}{p(p-1+\frac{p}{N}-\alpha)}} t^{\frac{\frac{p}{N}-\alpha p}{p(p-1+\frac{p}{N}-\alpha)}} \\
\le & C_3 t^{\frac{\frac{1}{N}-\alpha}{p-1+\frac{p}{N}-\alpha}},
\end{aligned}$$

where $\alpha < \min\Big\{ \frac{1}{N}, p-1+\frac{p}{N}-q \Big\}$.

Thus

$$\Big| \int_{I\!R^N} u_j(x,t)\psi(x)dx - \int_{I\!R^N} k^N h(kx)\psi(x)dx \Big| \le C(t),$$

where $C(t)$ is a constant independent of ε, k and $\lim_{t\to 0} C(t) = 0$. Letting $j \to \infty$, we obtain

$$\Big| \int_{I\!R^N} u(x,t)\psi(x)dx - \psi(0)\Big| \le C(t),$$

which completes the proof of Theorem 1.1.

§3. Proof of Theorem 1.2

The proof of Theorem 1.2 proceeds via two lemmas.

Lemma 3.1. *Suppose that the hypotheses of Theorem* 1.2 *are fulfilled. Then the solution of* (1.1) (1.3) *satisfies*

(1) *for any* $R > 0$

$$\int_0^T \int_{B_R} u^{q_1} dxdt < \infty,$$

and

(2)

$$\int_0^T \int_{B_R} \frac{u^{\alpha-1}}{(1+u^\alpha)^2} |\nabla u|^p dxdt \le C,$$

where $\alpha > 0$, $q_1 = \max\{1, q\}$.

Proof. (1) From Definition 2.1 and $u \in C(\overline{B_R} \times [\varepsilon, T])$, $|\nabla u| \in L^p(B_R \times (\varepsilon, T))$, we deduce that for any $\psi(x) \in C_0^\infty(I\!R^N)$ and $\varepsilon \in (0, T)$

$$\begin{aligned}
&\int_{I\!R^N} \int_0^{u(x,T)} \frac{s^\alpha}{1+s^\alpha} ds\psi^p(x)dx + \int_\varepsilon^T \int_{I\!R^N} u^q \frac{u^\alpha}{1+u^\alpha} \psi^p(x)dxdt \\
&\quad + \int_\varepsilon^T \int_{I\!R^N} \frac{\alpha u^{\alpha-1}}{(1+u^\alpha)^2} |\nabla u|^p \psi^p dxdt \\
&= -(p-1) \int_\varepsilon^T \int_{I\!R^N} \frac{u^\alpha}{1+u^\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla\psi \psi^{p-1} dxdt \\
&\quad + \int_{I\!R^N} \int_0^{u(x,\varepsilon)} \frac{s^\alpha}{1+s^\alpha} ds\psi^p(x)dx. \qquad (3.1)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^T \int_{I\!R^N} \frac{u^\alpha}{1+u^\alpha} |\nabla u|^{p-1} |\nabla\psi| \psi^{p-1} dxdt \\
\le& \eta \int_0^T \int_{I\!R^N} \frac{u^{\alpha-1}}{(1+u^\alpha)^2} |\nabla u|^p \psi^p dxdt + C(\eta) \Big\{ \int_0^T \int_{I\!R^N} u^{p-1+\alpha} |\nabla\psi|^p dxdt \\
&+ \int_0^T \int_{I\!R^N} u^{(1+\alpha)(p-1)} |\nabla\psi|^p dxdt \Big\} \qquad (3.2)
\end{aligned}$$

and

$$\int_{I\!R^N} \int_0^{u(x,\varepsilon)} \frac{s^\alpha}{1+s^\alpha} ds\psi^p(x)dx \le \int_{I\!R^N} u(x,\varepsilon)\psi^p dx \le C. \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3) and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\sup_{0<t<T} \int_{I\!R^N} u_1 \psi^p dx + \int_0^T \int_{I\!R^N} \frac{u^{\alpha-1}}{(1+u^\alpha)^2} |\nabla u|^p \psi^p dxdt + \int_0^T \int_{I\!R^N} \psi^p u_1^q dxdt \\
\le& C\Big\{1 + \int_0^T \int_{I\!R^N} u_1^{(1+\alpha)(p-1)} |\nabla\psi|^p dxdt\Big\}, \qquad (3.4)
\end{aligned}$$

where $u_1 = \max\{u, 1\}$. This implies

$$\int_0^T \int_{I\!R^N} u_1^{q_1} \psi^p dxdt \le C\Big\{1 + \int_0^T \int_{I\!R^N} u_1^{(1+\alpha)(p-1)} |\nabla\psi|^p dxdt\Big\}.$$

We choose $\alpha < \dfrac{q_1 - p + 1}{p-1}$. Using a similar argument as in the proof of Lemma 2.2, we obtain

$$\int_0^T \int_{I\!R^N} u^{q_1} dxdt \le C. \tag{3.5}$$

Hence, (3.4) and (3.5) imply conclusion (2).

Lemma 3.2. *Suppose that the hypotheses of Theorem* 1.2 *are fulfilled. Then the solution of* (1.1) (1.3) *satisfies*

$$\int\int_{S_T} \{u\xi_t - |\nabla u|^{p-2}\nabla u \cdot \nabla\xi - u^q \xi\} dxdt = 0,$$

for any $\xi \in C_0^\infty\{I\!R^N \times (-T, T)\}$.

Proof. First, we consider the case when $p > \dfrac{2N}{N+1}$, $q > p - 1 + \dfrac{p}{N}$.

Let

$$\psi_k(x,t) = \eta_k(|x|^2 + t^{\frac{2}{\gamma N}})\xi(x,t), \tag{3.6}$$

where $\xi(x,t) \in C_0^\infty(I\!R^N \times (-T,T))$, $\gamma = p - 2 + \dfrac{p}{N}$ and $\eta \in C^\infty(R)$ with the properties: $\eta(s) = 1$ if $s \ge 2$, $\eta(s) = 0$ if $s < 1$ and $\eta_k(s) = \eta(ks)$.

Since, for every $k > 0$, $\psi_k(x,t)$ vanishes in a neighbourhood of the origin (0,0), it follows from Definition 2.1 that u satisfies

$$\int_0^T \int_{I\!R^N} (u\psi_{kt} - |\nabla u|^{p-2}\nabla u \cdot \nabla\psi_k - u^q \psi_k) dxdt = 0.$$

Therefore, it is sufficient to prove that as $k \to \infty$

$$\int_0^T \int_{I\!R^N} u\eta_{kt}\xi dxdt \to 0, \qquad \int_0^T \int_{I\!R^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\eta_k \xi dxdt \to 0. \tag{3.7}$$

Set

$$D_k = \Big\{(x,t) : t > 0, k^{-1} < |x|^2 + t^{\frac{2}{\gamma N}} < 2k^{-1}\Big\}.$$

Then

$$\Big|\int_0^T \int_{I\!R^N} u\eta_{kt}\xi dxdt\Big| \le Ck^{\frac{\gamma N}{2}} \int\int_{D_k} u dxdt,$$

$$\Big|\int_0^T \int_{I\!R^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\eta_k \xi dxdt\Big| \le Ck^{\frac{1}{2}} \int\int_{D_k} |\nabla u|^{p-1} dxdt.$$

Note that

$$|D_k| = \text{measure of } D_k = Ck^{-\frac{p+Np-N}{2}}.$$

Let $q = p - 1 + \dfrac{p}{N} + \varepsilon_0$, $\alpha < \dfrac{\varepsilon_0}{p-1+\frac{p}{N}}$. Using Hölder inequality and Lemma 3.1, we obtain

$$\begin{aligned} k^{\frac{\gamma N}{2}} \int\int_{D_k} u dxdt &\le Ck^{\frac{\gamma N}{2}} \Big\{\int\int_{D_k} u^{p-1+\frac{p}{N}} dxdt\Big\}^{\frac{1}{p-1+\frac{p}{N}}} |D_k|^{\frac{\frac{p}{N}+p-2}{p-1+\frac{p}{N}}} \\ &\le C\Big\{\int\int_{D_k} u^{p-1+\frac{p}{N}} dxdt\Big\}^{\frac{1}{\frac{p}{N}+p-1}}, \end{aligned}$$

$$
\begin{aligned}
&k^{\frac{1}{2}}\int\int_{D_k}|\nabla u|^{p-1}dxdt\\
\leq&k^{\frac{1}{2}}\Big\{\int\int_{D_k}\frac{u^{\alpha-1}}{(1+u^\alpha)^2}|\nabla u|^p dxdt\Big\}^{\frac{p-1}{p}}\Big\{\int\int_{D_k}(1+u^\alpha)^{2(p-1)}u^{(1-a)(p-1)}dxdt\Big\}^{\frac{1}{p}}\\
\leq&Ck^{\frac{1}{2}}\Big\{\int\int_{D_k}u_1^{(p-1)(1+\alpha)}dxdt\Big\}^{\frac{1}{p}}\\
\leq&C_1\Big(\int\int_{D_k}u_1^q dxdt\Big)^{\frac{(1+\alpha)(p-1)}{pq}}k^{-\{\frac{(p-1)(\varepsilon_0 N-\alpha(p+Np-N))}{2pq}\}}\\
\leq&C_2k^{-\{(p-1)\frac{(\varepsilon_0 N-\alpha(p+Np-N))}{2pq}\}},
\end{aligned}
$$

where $u_1=\max\{1,u\}$.
Since

$$\int\int_{D_k}u^q dxdt \quad \text{as} \quad k\to\infty,$$

(3.7) is proved.

We now consider the case when $p\leq\dfrac{2N}{N+1}$. Let

$$\psi_k(x,t)=\eta_k(|x|^2)\xi(x,t)$$

where η_k and ξ are the functions in (3.6). Clearly, it suffices to verify

$$\int\int_{S_T}|\nabla u|^{p-2}\nabla u\cdot\nabla\eta_k\xi dxdt\to 0 \quad \text{as} \quad k\to\infty. \tag{3.8}$$

Set

$$D_k=\{x: k^{-1}<|x|^2<2k^{-1}\}.$$

We have

$$|D_k|\leq Ck^{-\frac{N}{2}}.$$

For $p-2+\dfrac{p}{N}<0$, we choose $0<\alpha<\dfrac{2-p-\frac{p}{N}}{p-1}$. Hence using Hölder's inequality and Lemma 3.1, we obtain

$$
\begin{aligned}
&k^{\frac{1}{2}}\int\int_{D_k}|\nabla u|^{p-1}dxdt\\
\leq&k^{\frac{1}{2}}\Big\{\int\int_{D_k}\frac{u^{\alpha-1}}{(1+u^\alpha)^2}|\nabla u|^p dxdt\Big\}^{\frac{p-1}{p}}\Big\{\int\int_{D_k}(1+u^\alpha)^{2(p-1)}u^{(p-1)(1-\alpha)}dxdt\Big\}^{\frac{1}{p}}\\
\leq&Ck^{\frac{1}{2}}\Big\{\int\int_{D_k}u_1^{(p-1)(1+\alpha)}dxdt\Big\}^{\frac{1}{p}}\\
\leq&C_1\Big(\int\int_{D_k}u_1 dxdt\Big)^{\frac{(p-1)(1+\alpha)}{p}}k^{\frac{1}{2}-\frac{N}{2p}+\frac{N}{2p}(p-1)(1+\alpha)}.
\end{aligned}
$$

Noting that

$$\frac{1}{2}-\frac{N}{2p}+\frac{N}{2p}(p-1)(1+\alpha)<0,$$

we get (3.8).

If $2-p-\dfrac{p}{N}=0$, we have $q>p-1+\dfrac{p}{N}=1$. Hence using Hölder's inequality and

Lemma 3.1, we have

$$k^{\frac{1}{2}}\int\int_{D_k}|\nabla u|^{p-1}dxdt\le Ck^{\frac{1}{2}}\Big\{\int\int_{D_k}u_1^{(p-1)(1-\alpha)}dxdt\Big\}^{\frac{1}{p}}$$

$$\le C_1\Big\{\int\int_{D_k}u_1^q dxdt\Big\}^{\frac{(p-1)(1+\alpha)}{pq}}k^{\frac{1}{2}-\frac{Nq-N(p-1)(1+\alpha)}{2pq}},$$

where $\alpha<q-1$. Noting that

$$\frac{1}{2}-\frac{Nq-N(p-1)(1+\alpha)}{2pq}<0,$$

we obtain (3.8). Thus Lemma 3.2 is proved.

Proof of Theorem 1.2. Suppose to the contrary that (1.1) (1.3) has a solution. Then by Lemma 3.2, we have

$$\int_0^T\int_{I\!R^N}(u\xi_t-|\nabla u|^{p-2}\nabla u\cdot\nabla\xi-u^q\xi)dxdt=0\ \text{ for any }\xi\in C_0^\infty(I\!R^N\times(-T,T)).\quad(3.9)$$

Let $j(s)\in C_0^\infty(R)$, $j\ge 0$, $j(s)=0$ if $|s|>1$ and $\int_R j(s)ds=1$. For $h>0$, we define $j_h(s)=h^{-1}j\Big(\frac{s}{h}\Big)$ and

$$\eta_h(t)=1-\int_{-\infty}^{t-\tau-2h}j_h(s)ds,$$

where $\tau\in(0,T)$ is a fixed number. Clearly, $\eta_h\in C^\infty(R)$, $\eta_h(t)=1$ if $t<\tau+h$, $0\le\eta_h\le 1$ and $\lim\limits_{h\to 0}\eta_h(t)=0$ if $t>\tau$.

For $X\in C_0^\infty(R^N)$, we set $\xi(x,t)=X(x)\eta_h(t)$ in (3.9) to obtain

$$-\int_0^T\int_{I\!R^N}j_h(t-\tau-2h)uXdxdt-\int_0^T\int_{I\!R^N}\{|\nabla u|^{p-2}\nabla u\cdot\nabla X\eta_h-u^q\eta_h X\}dxdt=0.$$

If we now let $h\to 0^+$, we obtain

$$\int_{I\!R^N}u(x,\tau)X(x)dx=-\int_0^\tau\int_{I\!R^N}\{|\nabla u|^{p-2}\nabla u\cdot\nabla X-u^qX\}dxds,$$

and this implies

$$\lim_{\tau\to 0}\int_{I\!R^N}u(x,\tau)X(x)dx=0$$

for every $X\in C_0^\infty(R^N)$. This contradicts (1.3).

§4. Proof of Theorem 1.3

We discuss the solution of equation (2.1) (2.2) with initial data

$$u(x,0)=k^{N+1}h(kx),\quad(4.1)$$

where $h(x)\in C_0^\infty(I\!R^N)$, $h(x)\ge 0$, $\int_{I\!R^N}h(x)dx=1$.

By [5], (2.1) (2.2) (4.1) has a nonnegative classical solution. We now prove the following estimates. As in section 2 we use letter C to denote the constants independent of k, ε.

Lemma 4.1. *Let* $q>\max\{1,p-1\}$. *Then the solution* u_k^ε *of problem* (2.1) (2.2) (4.1) *satisfies*

$$\int_{t_1}^T\int_{B_R(0)}(u_k^\varepsilon)^r dxdt<C(r,t_1)\quad\textit{for every } r,R>0,t_1\in(0,T).\quad(4.2)$$

Proof. Let $\xi(x,t)\in C_0^\infty(S_{T_1})$, $T_1>T$, $0\le\xi\le 1$, $\xi=1$ on $B_R(0)\times(t_1,T)$. Multiplying (2.1) by $u^{2\gamma-1}\xi^p$ $(\gamma>1)$, and integrating over S_T, we have

$$\begin{aligned}&\int_{I\!R^N}\frac{u^{2\gamma}(x,T)}{2\gamma}\xi^p dx+\int\int_{S_T}u^{2\gamma-1+q}\xi^p dxdt\\&+(2\gamma-1)\int\int_{S_T}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2u^{2\gamma-2}\xi^p dxdt\\=&-p\int\int_{S_T}u^{2\gamma-1}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}\xi^{p-1}\nabla u\cdot\nabla\xi dxdt\\&+\frac{p}{2\gamma}\int\int_{S_T}u^{2\gamma}\xi^{p-1}\xi_t dxdt.\end{aligned}\tag{4.3}$$

As in Lemma 2.3, using Holder's inequality, we have

$$\begin{aligned}&\int\int_{S_T}u^{q+2\gamma-1}\xi^p dxdt\\\le&C\Big\{\int\int_{S_T}u^{2\gamma-2+p}|\nabla\xi|^p dxdt+\int\int_{S_T}u^{2\gamma}\xi^{p-1}|\xi_t|dxdt\\&+\int\int_{S_T}u^{2\gamma-2}\xi^p dxdt\Big\}\quad\text{if } 1<p\le 2,\\&\int\int_{S_T}u^{q+2\gamma-1}\xi^p dxdt\\\le&C\Big\{\int\int_{S_T}u^{2\gamma-2+p}|\nabla\xi|^p dxdt+\int\int_{S_T}u^{2\gamma-1}\xi^{p-1}|\nabla\xi|dxdt\\&+\int\int_{S_T}u^{2\gamma}\xi^{p-1}dxdt\Big\}\quad\text{if } p\ge 2.\end{aligned}$$

Note that

$$q+2\gamma-1>\max\{2\gamma+p-2,2\gamma\}.$$

As in Lemma 2.2, using Young's inequality we obtain (4.2).

Lemma 4.2. *Let* $\overline{B_R(x_0)}\subset I\!R^N\setminus\{0\}$. *Then the solution of problem* (2.1) (2.2) (4.1) *satisfies*

$$\int_0^T\int_{B_R(x_0)}(u_k^\varepsilon)^r dxdt\le C(r)\quad\textit{for every } r>0.$$

Proof. Let $\xi\in C_0^\infty(I\!R^N)$, $0\le\xi\le1$, $\xi(x)=1$ if $x\in B_R(x_0)$, supp $\xi\subset I\!R^N\setminus\{0\}$. We multiply (2.1) by $u^{2\gamma-1}\xi^p$ and integrate it to obtain

$$\begin{aligned}&\frac{1}{2\gamma}\int_{I\!R^N}u^{2\gamma}(x,T)\xi^p dx+\int_0^T\int_{I\!R^N}u^{2\gamma-1+q}\xi^p dxdt\\&+(2\gamma-1)\int\int_{S_T}u^{2\gamma-2}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|^2\xi^p dxdt\\\le&p\int_0^T\int_{I\!R^N}u^{2\gamma-1}(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u|\xi^{p-1}\|\nabla\xi|dxdt\\&+\frac{1}{2\gamma}\int_{I\!R^N}\xi^p(k^{N+1}h(kx))^{2\gamma}dx.\end{aligned}\tag{4.4}$$

Note that

$$k^{N+1}h(kx)\le C_0\quad\text{on supp }\xi.$$

Hence, using an analogous argument with Lemma 4.1, we can prove Lemma 4.2.

Lemma 4.3. *For every $r > 0$, the solution u_k^ε of* (2.1) (2.2) (4.1) *satisfies*

$$\sup_{Q_R} u_k^\varepsilon \le C(r)\Big\{\int\int_{Q_{2R}} (u_k^\varepsilon)^{\frac{2N}{p}-N+r} dxdt\Big\}^{\frac{1}{r}} \quad \textit{if } p<2,$$

$$\sup_{Q_R} u_k^\varepsilon \le C(r)\Big\{\int\int_{Q_{2R}} (u_k^\varepsilon)^{p-2+r} dxdt\Big\}^{\frac{1}{r}} \quad \textit{if } p\ge 2,$$

$$\int\int_{Q_R} |\nabla u_k^\varepsilon|^p dxdt \le C(R),$$

where $2R \le \varepsilon^{-1}$, $t_0 - (2R)^p > 0$.

Lemma 4.4. *For every $\overline{B_R(x_0)} \subset I\!R^N \setminus \{0\}$ and $r > 0$, the solution u_k^ε of* (2.1) (2.2) (4.1) *satisfies*

$$\sup_{B_R(x_0)\times(0,T)} u_k^\varepsilon \le C(r)\Big\{\int_0^T\int_{B_R(x_0)} (u_k^\varepsilon)^{\frac{2N}{p}-N+r} dxdt\Big\}^{\frac{1}{r}} \quad \textit{for } p<2,$$

$$\sup_{B_R(x_0)\times(0,T)} u_k^\varepsilon \le C(r)\Big\{\int_0^T\int_{B_R(x_0)} (u_k^\varepsilon)^{p-2+r} dxdt\Big\}^{\frac{1}{r}} \quad \textit{for } p\ge 2,$$

$$\int_0^T\int_{B_R(x_0)} |\nabla u|^p dxdt \le C(R).$$

Lemma 4.5. *For every compact set $K \in \overline{S_T} \setminus \{(0,0)\}$, the solution u_k^ε satisfies*

$$\sup_K |\nabla u_k^\varepsilon| \le C(K).$$

Lemma 4.6. *For every compact set $K \subset S_T$, ∇u_k^ε are uniformly Hölder cotinuous.*

The proofs of Lemma 4.3–Lemma 4.6 are similar to the proofs of Lemma 2.3–Lemma 2.6, respectively.

Proof of Theorem 1.3. By Lemma 4.1–Lemma 4.5, the solutions u_k^ε are uniformly Hölder continuous on every compact set $K \subset \overline{S_T} \setminus \{(0,0)\}$. Thus, there exists a subsequence $\{u_{k_j}^{\varepsilon_j}\}$ of $\{u_k^\varepsilon\}$ and a function $U \in (\overline{S_T} \setminus \{(0,0)\}) \cap L^p_{\text{loc}}[0,T; W^{1,p}_{\text{loc}}(I\!R^N)]$ such that for every compact set $K \subset \overline{S_T} \setminus \{(0,0)\}$

$$u_{k_j}^{\varepsilon_j} \to U \quad \text{as } j\to\infty \quad \text{in } C(K)$$

and $U(x,0) = 0$ if $x \ne 0$.

Clearly U satisfies (1.1) in the sense of distributions. It remains to prove

$$\lim_{t\to 0^+}\int_{B_R} U(x,t)dx = \infty \quad \text{for every } R>0.$$

Let u_{Lk}^ε be the solution of (2.1) (2.2) with initial data

$$u_{Lk}(x,0) = Lk^N h(kx). \tag{4.5}$$

Clearly, if k is large enough,

$$Lk^N h(kx) \le k^{N+1} h(kx).$$

Hence, if k is large enough, we have

$$u_k^\varepsilon(x,t) \ge u_{Lk}^\varepsilon(x,t), \tag{4.6}$$

Letting $k = k_j$, $\varepsilon = \varepsilon_j$, and $j \to \infty$ in (4.6), we obtain

$$U(x,t) \geq u_L.$$

Hence, by the proof of Theorem 1.1, we have

$$\lim_{t\to 0^+} \int_{B_R} U(x,t)dx \geq \lim_{t\to 0^+} \int_{B_R} u_L(x,t)dx = L.$$

Letting $L \to \infty$, we obtain

$$\lim_{t\to 0^+} \int_{B_R} U(x,t)dx = \infty,$$

and Theorem 1.3 is proved.

References

[1] Alikakos, N. D. & Evans, L. C., Continuity of the gradient for weak solutions of a degenerate parabolic equation, *J. Math. Pures et Appl.,* **62** (1983), 253-268.

[2] Brezis, H. & Friedman, A., Nonlinear parabolic equations involving measures as initial conditions, *J. Math. Pures et Appl.,* **62** (1983), 73-97.

[3] Brezis, H., Peletier, L. A. & Terman, D., A very singular solution of the heat equation with absorption, *Arch. Rational Anal.,* **95** (1986), 185-207.

[4] Peletier, L. A. & Wang Junyu, A very singular solution of a quasilinear degenerate diffusion with absorption, *Trans. Amer. Math. Soc.,* **307:2** (1988), 813-826.

[5] Ladyzenskaja, O. A., Solonnikov, V. A. & Uralceva, N. N., Linear and quasilinear equations of parabolic type, *Trans. of Math. Mono.,* Providence (1968).

[6] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems, Princeton Univ. Press, 1983.

[7] Dibenedetto, E. & Friedman, A., Regularity of solutions of nonlinear degenerate parabolic systems, *J. Reine Angew. Math.,* **349** (1984), 83-128.

[8] Chen Yazhe, Hölder continuity of the gradient of the solutions of certain degenerate parabolic equations, *Chin. Annals of Math.,* **8B:3** (1987), 343-356.