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STRONGLY ALGEBRAIC LATTICES AND CONDITIONS OF MINIMAL MAPPING PRESERVING INFS**

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Abstract

The author gives some characterizations of strongly algebraic lattices, and proves that the category of strongly algebraic lattices is complete and cocomplete. Finally, this paper gives the complete conditions under which the minimal mapping $\beta : L \to 2^L$ on a completely distributive lattice L preserves finite infs and arbitrary infs.

Keywords Strongly algebraic lattice, Category of strongly algebraic lattices, Completeness and cocompleteness, Minimal mapping.
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Algebraic lattices were first invented in the forties by G. Birkhoff and O. Frink[1] and L. Nachbin[2]. Now, in universal algebra, algebraic lattices have become familiar objects as lattice of congruences and lattice of subalgebras of an algebra (see [3]). In [4], L. Geissinger and W. Graves have studied strongly algebraic lattices and category of strongly algebraic lattices. In this paper, we give some characterizations of strongly algebraic lattices and prove that the category of strongly algebraic lattices is complete and cocomplete. In [5], Wang Guojun has introduced an important mapping β: $L \to 2^L$—the minimal mapping on a completely distributive lattice L. It is easy to see that β preserves arbitrary sups, but in general, β does not preserve infs. In the last part of this paper, we give a complete answer to the following question: Under what conditions does the minimal mapping β: $L \to 2^L$ on a completely distributive lattice L preserve finite infs and arbitrary infs?

For the definitions and notations used here, see [3, 6–9]. In general, if the proof of a result is easy, we omit it.

§1. Preliminaries

Definition 1.1[5]. *Let L be a complete lattice and $x \in L$. $\phi \neq A \subseteq L$ is called a minimal set of x if $\bigvee A = x$ and, for each $\phi \neq B \subseteq L$ with $\bigvee B \geq x$ and each $a \in A$, there exists a $b \in B$ such that $a \leq b$; $\phi \neq C \subseteq L$ is called a maximal set of x if C is a minimal set of x in L^{0p}. If the minimal sets (resp., maximal sets) of x exist, let $\beta(x)$ (resp., $\alpha(x)$) denote the union of all minimal (resp., maximal) sets of x.*

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Theorem 1.1[5]. *Let L be a complete lattice. Then the following statements are equivalent:*

(1) *L is completely distributive;*

(2) *For each $x \in L$, $\beta(x)$ is a minimal set of x;*

(3) *For each $x \in L$, $\alpha(x)$ is a maximal set of x.*

Definition 1.2[10]. *Let L be a complete lattice. We define two binary relations $\triangleleft$ and $\lneqq$ on L as follows : $x \triangleleft y$ iff for each $\phi \neq D \subseteq L$ with $y \leq \bigvee D$ there exists a $d \in D$ such that $x \leq d$; $x \lneqq y$ iff $y \triangleleft x$ in L^{0p}. Let $\downarrow\!\!\!\!\bigtriangledown x = \{y \in L : y \triangleleft x\}$.*

Let L be completely distributive. Then for each $x \in L, \downarrow\!\!\!\!\bigtriangledown x = \beta(x)$ and $\alpha(x) = \{y \in L : x \lneqq y\}$[10]. Define β: $L \to 2^L$, $x \mapsto \beta(x)$; and α: $L \to 2^L$. β and α are called the minimal mapping and the maximal mapping on L respectively[5].

From Theorem 1.1, we can directly obtain Raney's intrinsic characterization of complete distributivity:

Theorem 1.2(Raney)[10,11]. *Complete lattice L is completely distributive iff whenever $u \not\leq v$ in L there exist elements p and q in L such that $u \not\leq p$, $v \not\geq q$ and, for each $x \in L$, either $x \leq p$ or $x \geq q$.*

Definition 1.3. *Let L be a complete lattice. An equivalence relation $R \subseteq L \times L$ is called a congruence of L iff R is a complete sublattice of $L \times L$.*

For the definitions and notations of induced mappings in complete lattices, see [8, 9]. It is easy to get the following (cf. [8, 9]):

Lemma 1.1. *Let L be complete. Then we have*

(1) *If $c : L \to L$ is a closure operator preserving arbitrary infs, then $c^{\wedge_1} : L \to L$ is a kernel operator preserving arbitrary sups, and for $x, y \in L$, $c(x) = c(y)$ iff $c^{\wedge_1}(x) = c^{\wedge_1}(y)$.*

(2) *If $k : L \to L$ is a kernel operator preserving arbitrary sups, then $k^{\vee_1} : L \to L$ is a closure operator preserving arbitrary infs, and for $x, y \in L$, $k(x) = k(y)$ iff $k^{\vee_1}(x) = k^{\vee_1}(y)$.*

Proposition 1.1. *Let L be complete and Δ the diagonal of $L \times L$. For an equivalence relation $R \subseteq L \times L$, the following conditions are equivalent:*

(1) *R is a congruence;*

(2) *There is a closure operator $c : L \to L$ preserving arbitrary infs such that $R = (c \times c)^{-1}(\Delta)$;*

(3) *There is a kernel operator $k : L \to L$ preserving arbitrary sups such that $R = (k \times k)^{-1}(\Delta)$.*

Proof. (1) implies (2): For each $x \in L$, let $\overline{x} = \{y \in L : yRx\}$ and define $c : L \to L$, $x \mapsto \bigvee \overline{x} = x_R$. It is easy to prove that c is a closure operator and preserves arbitrary infs, and $R = (c \times c)^{-1}(\Delta)$.

(2) implies (3): Let $k = c^{\wedge_1}$: $L \to L$. By Lemma 1.1, k is a kernel operator preserving arbitrary sups, and $R = (c \times c)^{-1}(\Delta) = (k \times k)^{-1}(\Delta)$.

(3) implies (1): Since k: $L \to L$ is a kernel operator preserving arbitrary sups, $R = (k \times k)^{-1}(\Delta)$ is closed with respect to sups of arbitrary sets. If $k(x_i) = k(y_i)$ for each $i \in I$, then by Proposition 1.5 of [10], $k(\bigwedge_{i} {}_L x_i) = \bigwedge_{i} {}_{K(L)} k(x_i) = \bigwedge_{i} {}_{K(L)} k(y_i) = k(\bigwedge_{i} {}_L y_i)$, it follows that R is closed with respect to infs of arbitrary sets. Hence, R is a congruence.

Corollary 1.1. *Let L be complete and Δ the diagonal of $L \times L$. Then we have*

(1) $c \mapsto (c\times c)^{-1}(\Delta)$ *associates with a closure operator its kernel congruence and induces an isomorphism from the lattice of all closure operators of* L *which preserves arbitrary infs* (*under the pointwise order*) *onto the lattice of all congruences of* L (*under the inclusion order*).

(2) $k \mapsto (k\times k)^{-1}(\Delta)$ *associates with a kernel operator its kernel congruence and induces a dual isomorphism from the lattice of all kernel operator of* L *which preserves arbitrary sups* (*under the pointwise order*) *onto the lattice of all congruences of* L.

Let L be complete and R a congruence of L, and $L/R = \{\overline{x} : x \in L\}$ the quotient set. We define an order on L/R as follows: $\overline{x} \le \overline{y}$ iff $(x \vee y, y) \in R$. Under this order, L/R is called a quotient of L.

Lemma 1.2. (1) $\overline{x} \le \overline{y}$ *iff* $x_R = \bigvee \overline{x} \le y_R = \bigvee \overline{y}$;

(2) $\bigvee\limits_i \overline{x}_i = \overline{\bigvee\limits_i x_i}$, $\bigwedge\limits_i \overline{x}_i = \overline{\bigwedge\limits_i x_i}$.

Corollary 1.2. *Let* L *be completely distributive. Then* L/R *is completely distributive.*

§2. Strongly Algebraic Lattices and Their Characterizations

Definition 2.1. *Let* P *be a partially ordered set.* $x \in P$ *is called coprime, if* $x \le a$ *or* $x \le b$ *whenever* $a \vee b$ *exists in* P *and* $x \le a \vee b$. *Prime elements can be defined dually. Let* PRI (P) *and* COPRI (P) *denote the sets of prime elements and of coprime elements respectively.*

Definition 2.2.[4] *Let* L *be complete.* $x \in L$ *is called strongly compact* (*resp., strongly cocompact*) *iff* $x \lhd x$ (*resp.,* $x \lneqq x$) *in* L. *Let* $K_c(L)$ *and* $\overline{K}_c(L)$ *denote the sets of strongly compact elements and of strongly cocompact elements respectively.* L *is called strongly algebraic iff, for each* $x \in L$, $x = \bigvee\{y \in L : y \in K_c(L)$ *and* $y \le x\}$.

Lemma 2.1.[12] L *is strongly algebraic iff* L *is isomorphic to a complete ring of sets.*

Corollary 2.1. *Let* L *be strongly algebraic. If* $L_0 \subseteq L$ *is a sub-complete lattice of* L, *then* L_0 *is strongly algebraic* (*relative to the induced order*).

Corollary 2.2. *Let* L *be strongly algebraic. Then we have*

(1) *If* $c : L \to L$ *is a closure operator preserving arbitrary sups, then* $c(L)$ *is strongly algebraic* (*relative to the induced order*).

(2) *If* $k : L \to L$ *is a kernel operator preserving arbitrary infs, then* $k(L)$ *is strongly algebraic* (*relative to the induced order*).

For a partially ordered set P, $x \in P$, and $A \subseteq P$, let $\downarrow x = \{y \in P : y \le x, \uparrow x = \{y \in P : x \le y\}$, $\downarrow A = \bigcup\limits_{x\in A} \downarrow x$ and $\uparrow A = \bigcup\limits_{x\in A} \uparrow x$. A is called a semi–ideal (resp., a semi–filter) of P iff $A =\downarrow A$ (resp., $A =\uparrow A$). Let Sid (P) (resp., Sfil (P)) denote the complete lattice of semi–ideals (resp., semi–filters) of P. $M \in$ Sid (P) is called an ideal of P iff for x, $y \in P$, if $x \vee y$ exists in P, then $x \vee y \in M$; filters of P can be defined dually. Let Id (p) (resp., Fil (P)) denote the complete lattice of ideals (resp., filters) of P.

Obviously, Sid (P) and Sfil (P) are complete sublattices of 2^P, so both of them are strongly algebraic, $K_c(\text{Sid } (P)) = \{\downarrow x : x \in P\}$, $K_c(\text{Sfil } (P)) = \{\uparrow x : x \in P\}$, and $P \cong K_c(\text{ Sid } (P)) \cong \overline{K}_c(\text{Sfil } (P))$. For a strongly algebraic lattice L, it is easy to see that $L \cong \text{Sid } (K_c(L))$ and $L^{0p} \cong \text{Sfil } (\overline{K}_c(L))$. We can easily prove that Id (P) and Fil (P) are algebraic. For an algebraic lattice L, it is well–known that $L \cong \mathrm{Id}\ (K(L))$, where $K(L)$ denotes the set of compact elements of L[6].

Lemma 2.2. *Let S be a sup–semilattice. Then we have*

(1) $K(\mathrm{Id}\ (S)) = \{\downarrow x : x \in S\}$, *and* $S \cong K(\mathrm{Id}\ (S))$;

(2) $K_c(\mathrm{Id}\ (S)) = \{\downarrow x : x \in \mathrm{PRI}\ (S)\}$.

Definition 2.1. *Let S be a sup–semilattice. $S_0 \subseteq S$ is said to be a finite sup–generator of S iff, for each $x \in S$, there is a finite set $A \subseteq S_0$ such that $x = \bigvee A$.*

It is not difficult to get the following

Theorem 2.1. *For a sup–semilattice S, the following two statements are equivalent:*

(1) $\mathrm{Id}\ (S)$ *is strongly algebraic;*

(2) $\mathrm{COPRI}\ (S)$ *is a finite sup–generator of S.*

Corollary 2.3. *For a complete lattice L, the following statements are equivalent:*

(1) *L is strongly algebraic;*

(2) *There is a partially ordered set P such that $L \cong \mathrm{Sid}\ (P)$;*

(3) *There is a sup–semilattice S such that* $\mathrm{COPRI}\ (S)$ *is a finite sup–generator of S and $L \cong \mathrm{Id}\ (S)$.*

It follows from Lemma 2.1 that the dual of a strongly algebraic lattice is still one, so by Theorem I–4.23 of [6] we have

Proposition 2.1. *The following conditions are equivalent:*

(1) *L is strongly algebraic;*

(2) *L is algebraic and L^{0p} is a Heyting algebra;*

(3) *L^{0p} is algebraic and L is a Heyting algebra;*

(4) *L is distributive and both L and L^{0p} are algebraic.*

Corollary 2.4. *Let L be a finite lattice. Then L is strongly algebraic iff L is distributive.*

By Corollary 2.2 and Proposition 1.5 of [10], we get the following

Theorem 2.2. *For a complete lattice L, the following statements are equivalent:*

(1) *L is strongly algebraic;*

(2) *For some set X, L is isomorphic to the image of some closure operator $c : 2^X \to 2^X$ which preserves arbitrary sups;*

(3) *For some set X, L is isomorphic to the image of some kernel operator $k : 2^X \to 2^X$ which preserves arbitrary infs.*

By Theorem 2.2, Lemma 2 of [13], Proposition 1.5 of [10] and the proof of Theorem I–4.16 of [6], we get the following

Theorem 2.3. *For a complete lattice L, the following statements are equivalent:*

(1) *L is complete distributive;*

(2) *There is a strongly algebraic lattice $\overline{L}$ and a complete lattice homomorphism $f : \overline{L} \to L$ such that $L = f(\overline{L})$;*

(3) *There is a set X and a projection operator $p : 2^X \to 2^X$ preserving arbitrary sups such that $L \cong p(2^X)$;*

(4) *There is a set X and a projection operator $q : 2^X \to 2^X$ preserving arbitrary infs such that $L \cong q(2^X)$.*

From Theorem 2.3, it is note worthy that the class of strongly algebraic lattices is not closed under the formation of complete lattice homomorphic images.

Theorem 2.4. *For a complete lattice L, the following conditions are equivalent:*

(1) *L is strongly algebraic;*

(2) *Whenever $u \not\leq v$ in L, there exist p, $q \in L$ such that $p \in \overline{K}_c(L)$, $q \in K_c(L)$, $u \not\leq p$, $v \not\geq q$ and, for each $x \in L$, either $x \leq p$ or $x \geq q$;*

(3) *Whenever $u \not\leq v$ in L, there exist p, $\overline{q} \in L$ such that $\overline{q} \in K_c(L)$, $u \not\leq p$, $v \not\geq \overline{q}$ and, for each $x \in L$, either $x \leq p$ or $x \geq \overline{q}$;*

(4) *Whenever $u \not\leq v$ in L, there exist $\overline{p}$, $q \in L$ such that $\overline{p} \in \overline{K}_c(L)$, $u \not\leq \overline{p}$, $v \not\geq q$ and, for each $x \in L$, either $x \leq \overline{p}$ or $x \geq q$.*

Proof. (1) implies (2): Since strongly algebraic lattices are completely distributive, by Theorem 1.2, there exist p_1, $q_1 \in L$ such that $u \not\leq p_1$, $v \not\geq q_1$ and, for each $x \in L$, either $x \leq p_1$ or $x \geq q_1$. Since L and L^{0p} are strongly algebraic, there are $p \in \overline{K}_c(L)$ and $q \in K_c(L)$ such that $p_1 \leq p$, $q_1 \geq q$, $u \not\leq p$ and $v \not\geq q$. It is easy to see that p and q satisfy the conditions of (2).

(2) implies (3): Trivial.

(3) implies (1): For $u \in L$, let $v = \bigvee\{y \in L : y \in K_c(L) \text{ and } y \leq u\}$. If $u \neq v$, then there exist $p \in L$ and $\overline{q} \in K_c(L)$ such that $u \not\leq p$, $v \not\geq \overline{q}$ and, for each $x \in L$, either $x \leq p$ or $x \geq \overline{q}$. It follows that $u \geq \overline{q}$; hence, $\overline{q} \leq v$, which is contradictory to $v \not\geq \overline{q}$. Whence $u = v$; therefore, L is strongly algebraic.

The equivalence of (1) and (2) can be proved similarly.

Theorem 2.5. *For two strongly algebraic lattices L_1 and L_2 and a complete lattice homomorphism $h : L_1 \to L_2$, let $E_h = \{(x, y) \in L_1 \times L_1 : h(x) = h(y)\}$. Then E_h is a congruence of L_1, and the quotien L_1/E_h is strongly algebraic.*

Proof. Obviously, E_h is a congruence of L_1. By Corollary 2.1, $h(L_1)$ is strongly algebraic (relative to the induced order). Define $\overline{h} : L_1/E_h \to h(L_1), \overline{x} \mapsto h(x)$. It is not difficult to prove that $\overline{h}$ is an isomorphism; hence L_1/E_h is strongly algebraic.

§3. Completeness and Cocompleteness of the Category of Strongly Algebraic Lattices

Taking strongly algebraic lattices for objects and complete lattice homomorphisms for morphisms, we get the category of strongly algebraic lattices[4]. Let SAL denote this category. Obviously, SAL is a full subcategory of Lat*—the category of completely distributive lattices[9]. In [9], we have proved that Lat* has product and coproduct for every family of objects. In this section, we prove that SAL is complete and cocomplete.

It is not difficult to get the following

Theorem 3.1. *$\{\prod\limits_i L_i, p_i \mid i \in I\}$ is the product of $\{L_i : i \in I\}$ in* SAL, *where $p_i : \prod\limits_i L_i \to L_i$ is the natural projection.*

Definition 3.1.[9,14] *For a family of complete lattices $\{L_\lambda : \lambda \in \Lambda\}$, we define a lattice $L = \bigotimes\limits_\lambda L_\lambda$ as follows:*

The elements of L are the subsets A of $\prod\limits_\lambda L_\lambda$ which satisfy:

(1) Let $Z = \{\{a_\lambda\} \in \prod_\lambda L_\lambda : (\exists \lambda_0 \in \Lambda)\ (a_{\lambda_0} = 0)\}$, then $Z \subseteq A$;

(2) $A = \downarrow A$;

(3) If $B_\lambda \subseteq L_\lambda$ and $\prod_\lambda B_\lambda \subseteq A$, then $\{b_\lambda\} \in A$, where $b_\lambda = \sup B_\lambda$. The order on L is the ordering of set inclusion.

Lemma 3.1. *Let $\{L_\lambda : \lambda \in \Lambda\}$ be a family of completely distributive lattices. Then in $\bigotimes_\lambda L_\lambda$, $\bigwedge_i A_i = \bigcap_i A_i$, and $\bigvee_i A_i = \{\{a_\lambda\} \in \prod_\lambda L_\lambda \mid \exists B_\lambda \subseteq L_\lambda$ such that $\prod_\lambda B_\lambda \in \bigcup_i A_i$ and $a_\lambda = \sup B_\lambda$ for each $\lambda \in \Lambda\}$, where "$\bigcap$" and "$\bigcup$" denote the intersection and union of sets.*

Lemma 3.2. *Let L_λ be completely distributive for each $\lambda \in \Lambda$. Then we have*

(1) For $A, B \in \bigotimes_\lambda L_\lambda$, $A \lhd B$ iff there is $\{z_\lambda\} \in B$ such that $A \lhd \downarrow \{b_\lambda\} \cup Z$.

(2) For $\{x_\lambda\}, \{y_\lambda\} \in \prod_\lambda L_\lambda$, $\downarrow \{x_\lambda\} \cup Z \lhd \downarrow \{y_\lambda\} \cup Z$ iff $x_\lambda \lhd y_\lambda$ for each $\lambda \in \Lambda$.

Proof. (1) By Theorem 6 of [9], $\bigotimes_\lambda L_\lambda$ is completely distributive, so $B = \bigvee_{\{b_\lambda\} \in B} \downarrow \{b_\lambda\} \cup Z = \bigvee_{\{b_\lambda\} \in B} \bigvee \overset{\downarrow}{\triangledown} (\downarrow \{b_\lambda\} \cup Z)$. Therefore, if $A \lhd B$, then there is $\{b_\lambda\} \in B$ such that $A \lhd \downarrow \{b_\lambda\} \cup Z$. Conversely, if $A \lhd \downarrow \{b_\lambda\} \cup Z$ for some $\{b_\lambda\} \in B$, then it follows obviously that $A \lhd B$.

(2) $\downarrow \{y_\lambda\} \cup Z = \bigvee_{\{u_\lambda\} \in \prod_\lambda \overset{\downarrow}{\triangledown} y_\lambda} \downarrow \{u_\lambda\} \cup Z$, by $\downarrow \{x_\lambda\} \cup Z \lhd \downarrow \{y_\lambda\} \cup Z$, there is $\{u_\lambda\} \in \prod_\lambda \overset{\downarrow}{\triangledown} y_\lambda$ such that $\downarrow \{x_\lambda\} \cup Z \leq \downarrow \{u_\lambda\} \cup Z$. It follows that $x_\lambda \lhd y_\lambda$ for each $\lambda \in \Lambda$. Conversely, for every family $\{A_i : i \in I\} \subseteq \bigotimes_\lambda L_\lambda$ such that $\downarrow \{y_\lambda\} \cup Z = \bigvee_i A_i$, by Lemma 3.1, there exists $B_\lambda \subseteq L_\lambda$ $(\lambda \in \Lambda)$ such that $\prod_\lambda B_\lambda \subseteq \bigcup_i A_i$ and $y_\lambda = \sup B_\lambda$ $(\lambda \in \Lambda)$. Since $x_\lambda \lhd y_\lambda$ for each $\lambda \in \Lambda$, there is $b_\lambda \in B_\lambda$ with $x_\lambda \leq b_\lambda$ $(\lambda \in \Lambda)$. There exists $i \in I$ such that $\{b_\lambda\} \in A_i$; therefore, $\downarrow \{x_\lambda\} \cup Z \leq A_i$. Hence, $\downarrow \{x_\lambda\} \cup Z \lhd \downarrow \{y_\lambda\} \cup Z$.

Corollary 3.1. *Let $\{L_\lambda : \lambda \in \Lambda\}$ be a family of completely distributive lattices. Then $A \in K_c(\bigotimes_\lambda L_\lambda)$ iff $A = \downarrow \{x_\lambda\} \cup Z$ such that $x_\lambda \in K_c(L_\lambda)$ for each $\lambda \in \Lambda$.*

Theorem 3.2. *$\bigotimes_\lambda L_\lambda$ is strongly algebraic iff L_λ is strongly algebraic for each $\lambda \in \Lambda$.*

Proof. Define $q_{\lambda_0} : L_{\lambda_0} \to \bigotimes_\lambda L_\lambda$, $x \mapsto \downarrow \{x_\lambda\} \cup Z$, $x_\lambda = \begin{cases} 1, & \lambda \neq \lambda_0 \\ x, & \lambda = \lambda_0. \end{cases}$
Let $\overline{L}_{\lambda_0} = \{q_{\lambda_0}(x) : x \in L_{\lambda_0}\}$. It is easy to see that $\overline{L}_{\lambda_0}$is a complete sublattice of $\bigotimes_\lambda L_\lambda$ and L_{λ_0} is isomorphic to $\overline{L}_{\lambda_0}$. If $\bigotimes_\lambda L_\lambda$ is strongly algebraic, then by Corollary 2.1, $\overline{L}_{\lambda_0}$ is strongly algebraic; whence, L_{λ_0} is strongly algebraic $(\lambda_0 \in \Lambda)$. Conversely, for every $A \in \bigotimes_\lambda L_\lambda$, $A = \bigvee_{\{x_\lambda\} \in A} \downarrow \{x_\lambda\} \cup Z$. Let $B_\lambda = \{y_\lambda \in L_\lambda : y_\lambda \in K_c(L_\lambda)$ and $y_\lambda \leq x_\lambda\}$. If L_λ is strongly algebraic for each $\lambda \in \Lambda$, then $\downarrow \{x_\lambda\} \cup Z = \bigvee \{\downarrow \{b_\lambda\} \cup Z : \{b_\lambda\} \in \prod_\lambda B_\lambda\}$. By Corollary 3.1, $A = \bigvee \{C \in \bigotimes_\lambda L_\lambda : C \in K_c(\bigotimes_\lambda L_\lambda)$ and $C \leq A\}$. So $\bigotimes_\lambda L_\lambda$ is strongly algebraic.

By Theorem 3.2 and Theorem 6 of [9], we get the following

Theorem 3.3. *$\{\bigotimes_\lambda L_\lambda : q_\lambda \mid \lambda \in \Lambda\}$ is the coproduct of $\{L_\lambda : \lambda \in \Lambda\}$ in SAL, where $q_{\lambda_0} : L_{\lambda_0} \to \bigotimes_\lambda L_\lambda$, $x \mapsto \downarrow \{x_\lambda\} \cup Z$, $x_\lambda = \begin{cases} 1, & \lambda \neq \lambda_0 \\ x, & \lambda = \lambda_0 \end{cases}$ $(\lambda_0 \in \Lambda)$.*

Lemma 3.3. *For a pair morphisms* $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ *in SAL, let* $L_0 = \{x \in L_1 : f(x) = g(x)\}$. *Then* $L_0 \xrightarrow{e} L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ *is the equivalizer of* $\langle f, g\rangle$, *where* e *is the injection of* L_0 *into* L_1.

Proof. *The proof is standard and is left as an exercise.*

Lemma 3.4. *For a pair morphisms* $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ *in SAL, let* $\{E_i : i \in I\}$ *denote the set of congrunces of* L_2 *such that* $E_i \supseteq M(f, g) = \{(f(x), g(x)) : x \in L_1\}$ *and the quotien* L_2/E_i *is strongly algebraic* $(i \in I)$. *Let* $E = \bigcap_i E_i$. *Then* $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2 \xrightarrow{q} L_2/E$ *is the coequalizer of* $\langle f, g\rangle$, *where* $q : L_2 \to L_2/E$ *is the natural projection.*

Proof. First, we prove that L_2/E is strongly algebraic. Since L_2/E_i is strongly algebraic for each $i \in I$, $\prod_i L_2/E_i$ is strongly algebraic. Let $q_i : L_2 \to L_2/E_i$ be the projection $(i \in I)$ and define k: $L_2 \to \prod_i L_2/E_i$, $x \mapsto \{q_i(x)\}$. It is easy to see that $E = \bigcap_i E_i = E_k = \{(x, y) \in L_2 \times L_2 : k(x) = k(y)\}$. By Theorem 2.5, L_2/E is strongly algebraic. For every morphism $h : L_2 \to L$ in SAL such that $h \circ f = h \circ g$, then there exists $i \in I$ such that $E_i = E_h = \{(x, y) \in L_2 \times L_2 : h(x) = h(y)\}$. Define $\overline{h} : L_2/E \to L$, $\overline{x} \mapsto h(x)$. We can easily prove that $\overline{h}$ is the unique morphism such that $h = \overline{h} \cdot q$. Whence, $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2 \xrightarrow{q} L_2/E_i$ is the coequalizer of $\langle f, g\rangle$.

By Theorem 3.1, Lemma 3.3, and Theorem V–1 of [7], we get the following

Theorem 3.4. *SAL is complete.*

By Theorem 3.2 and Lemma 3.4, we have dually the following

Theorem 3.5. *SAL is cocomplete.*

§4. Conditions of Minimal Mapping Preserving Infs

Definition 4.1. *Let* L *be a complete lattice. The relation* $\lhd$ *on* L *is called multiplicative* (*resp., completely multiplicative*) *iff* $a \lhd x$ *and* $b \lhd y$ *imply* $a \wedge b \lhd x \wedge y$ *for all* $a, b, x, y \in L$ (*resp., for all* $\{x_j : j \in J\} \subseteq L$ *and* $\{y_j : j \in J\} \subseteq L$, $x_j \lhd y_j$ *for each* $j \in J$ *implies* $\bigwedge_j x_j \lhd \bigwedge_j y_j$); *the relation* $\lessgtr$ *is called co-multiplicative* (*resp., completely co-multiplicative*) *iff for all* $a, b, x, y \in L$, $a \lessgtr x$ *and* $b \lessgtr y$ *imply* $a \vee b \lessgtr x \vee y$ (*resp., for all* $\{x_j : j \in J\} \subseteq L$ *and* $\{y_j : j \in J\} \subseteq L$, $x_j \lessgtr y_j$ *for each* $j \in J$ *implies* $\bigvee_j x_j \lessgtr \bigvee_j y_j$).

Proposition 4.1. *Let* L *be strongly algebraic. Then the following statements are equivalent:*

(1) $K_c(L)$ *is closed under finite infs* (*resp., under arbitrary infs*);

(2) $\lhd$ *is multiplicative* (*resp., completely multiplicative*).

Dually, we have

Proposition 4.2. *Let* L *be strongly algebraic. Then the following statements are equivalent:*

(1) $\overline{K}_c(L)$ *is closed under finite sups* (*resp., under arbitrary sups*);

(2) $\lessgtr$ *is co–multiplicative* (*resp., completely co–multiplicative*).

Lemma 4.1.[10] *For a complete lattice* L *and* $I \in \mathrm{Sid}\ (L)$, *the following conditions are equivalent:*

(1) $I \in \mathrm{PRI}(\mathrm{Sid}\ (L))$ (*resp.,* $I \in \overline{K}_c(\mathrm{Sid}(L))$);

(2) $L\backslash I$ *is a filter* (*resp., a principal filter, namely,* $I = L\backslash \uparrow a$ *for some* $a \in L$)*;*

(3) *The relation* $x \wedge y \in I$ *always implies* $x \in I$ *or* $y \in I$ (*resp., the relation* $\bigwedge_j x_j \in I$ *always implies* $x_j \in I$ *for some* $j \in J$)*.*

Dually, we have

Lemma 4.2. *For a complete lattice* L *and* $F \in \mathrm{Sfil}(L)$*, the following conditions are equivalent:*

(1) $F \in \mathrm{PRI}(\mathrm{Sfil}(L))$ (*resp.,* $F \in \overline{K}_c(\mathrm{Sfil}(L))$)*;*

(2) $L\backslash F$*is an ideal* (*resp., a principal ideal, namely,* $F = L\backslash \downarrow a$ *for some* $a \in L$)*;*

(3) *The relation* $x \vee y \in F$ *always implies* $x \in F$ *or* $y \in F$ (*resp., the relation* $\bigvee_j x_j \in F$ *always implies* $x_j \in F$ *for some* $j \in J$)*.*

Definition 4.2. *Let* L *be a complete lattice and* $p \in L$*.* p *is called quasiprime* (*resp., strongly quasicocompact*) *iff* $p = \sup P$ *for some* $P \in \mathrm{PRI}(\mathrm{Sid}(L))$ (*resp.,* $P \in \overline{K}_c(\mathrm{Sid}(L))$)*;* $q \in L$ *is called quasicoprime* (*resp., strongly quasicompact*) *iff* $q = \inf Q$ *for some* $Q \in \mathrm{PRI}(\mathrm{Sfil}(L))$ (*resp.,* $Q \in \overline{K}_c(\mathrm{Sfil}\ (L))$*. Let* $\mathrm{QPRI}(L)$, $\mathrm{QCOPRI}(L), QK_c(L)$ *and* $Q\overline{K}_c(L)$ *denote the sets of quasiprime elements, of quasicoprime elements, of strongly quasicompact elements and of strongly quasicocompact elements, respectively.*

$p \in\mathrm{PRI}(L)$ (resp., $p \in \overline{K}_c(L)$) gives rise to $\downarrow p \in \mathrm{PRI}(\mathrm{sid}(L))$ (resp., $\downarrow p \in \overline{K}_c(\mathrm{Sid}(L))$, so $\mathrm{PRI}(L) \subseteq \mathrm{QPRI}(L)$ and $\overline{K}_c((L) \subseteq Q\overline{K}_c(L)$. Dually, we have $\mathrm{COPRI}(L) \subseteq \mathrm{QCOPRI}(L)$ and $K_c(L) \subseteq QK_c(L)$.

Theorem 4.1.[10] *Let* L *be completely distributive and* $p \in L$*. Then the following conditions are equivalent:*

(1) $p \in \mathrm{QPRI}(L)$ (*resp.,* $p \in Q\overline{K}_c(L)$)*;*

(2) *The relation* $x \wedge y \lhd p$ (*resp.,* $\bigwedge_j x_j \lhd p$) *always implies* $x \leq p$ *or* $y \leq p$ (*resp.,* $x_j \leq p$ *for some* $j \in J$);

(3) *The filter generated by* $L\backslash \downarrow p$ (*resp., the principal filter* $\uparrow a$*,* $a = \bigwedge L\backslash \downarrow p$) *does not meet*$\underset{\triangledown}{\downarrow} p$*.*

Dually, we have

Theorem 4.2. *Let* L *be completely distributive and* $q \in L$*. Then the following conditions are equivalent:*

(1) $q \in \mathrm{QCOPRI}(L)$ (*resp.,* $q \in QK_c(L)$)*;*

(2) *The relation* $q \lessgtr x \vee y$ (*resp.,* $q \lessgtr \bigvee_j x_j$) *always implies* $q \leq x$ *or* $q \leq y$ (*resp.,* $q \leq x_j$ *for some* $j \in J$);

(3) *The ideal generated by* $L\backslash \uparrow q$ (*resp., the principal ideal* $\downarrow a$*,* $a = \bigvee L\backslash \uparrow q$) *does not meet* $\{z \in L : q \lessgtr z\}$*.*

Theorem 4.3.[10] *Let* L *be completely distributive. Then the following conditions are equivalent:*

(1) *The relation* $\lhd$ *is multiplicative* (*resp., completely multiplicative*);

(2) $\mathrm{PRI}(L) = \mathrm{QPRI}(L)$ (*resp.,* $\overline{K}_c(L) = QK_c(L)$)*.*

Dually, we have

Theorem 4.4. *Let* L *be completely distributive. Then the following conditions are equivalent:*

(1) *The relation* $\lessgtr$ *is co-multiplicative* (*resp., completely comultiplicative*);

(2) COPRI(L) =QCOPRI(L) (*resp.,* $K_c(L) = QK_c(L)$).

By Proposition 4.1 and Theorem 4.3, we get the following two results:

Theorem 4.5. *Let L be completely distributive. Then the following statements are equivalent:*

(1) *The minimal mapping* $\beta : L \to 2^L$ *preserves finite infs;*

(2) *The relation* $\lhd$ *is multiplicative;*

(3) PRI(L) =QPRI(L).

Theorem 4.6. *Let L be completely distributive. Then the following statements are equivalent:*

(1) *The minimal mapping* $\beta : L \to 2^L$ *preserves arbitrary infs;*

(2) *The relation* $\lhd$ *is completely multiplicative;*

(3) *L is strongly algebraic and* $K_c(L)$ *is closed under arbitrary infs;*

(4) *L is strongly algebraic and* $\overline{K}_c(L) = Q\overline{K}_c(L)$.

Dually, we have the following two results:

Theorem 4.7. *Let L be completely distributive. Then the following conditions are equivalent:*

(1) *The maximal mapping* $\alpha : L \to 2^L$ *is a finite* $\wedge - \vee$ *mapping;*

(2) *The relation* $\lessgtr$ *is co–multiplicative;*

(3) COPRI(L) =QCOPRI(L).

Theorem 4.8. *Let L be completely distributive. Then the following conditions are equivalent:*

(1) *The maximal mapping* $\alpha : L \to 2^L$ *is an arbitrary* $\vee - \wedge$ *mapping;*

(2) *The relation* $\lessgtr$ *is completely co–multiplicative;*

(3) *L is strongly algebraic and* $\overline{K}_c(L)$ *is closed under arbitrary sups;*

(4) *L is strongly algebraic and* $K_c(L) = QK_c(L)$.

From Theorems 4.6 and 4.8, it follows that Theorems 2.7, 2.9, 3.12 and 3.13 in [15] are not correct.

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