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BAHADUR ASYMPTOTIC EFFICIENCY IN A SEMIPARAMETRIC REGRESSION MODEL

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Abstract

The authors give MLE $\hat{\theta}_{1ML}$ of θ_1 in the model $Y = \theta_1 + g(T) + \varepsilon$, then consider Bahadur asymptotic efficiency of $\hat{\theta}_{1ML}$, where T and ε are independent, g is unknown, $\varepsilon \sim \varphi(\cdot)$ is known with mean 0 and variance σ^2.

Keywords Semiparametric regression model, Bahadur asymptotic efficiency, Maximum likelihood estimation.

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§1. Introduction and Statement of Preliminary

Consider the semiparametric regression model given by

$$Y = \theta_1 + g(T) + \varepsilon.$$

Here g is an unknown Hölder continuous function of known order p in R^1, θ_1 is an unknown parameter to be estimated, T and ε are independent, $\varepsilon \sim \varphi(\cdot)$ is known with mean 0 and variance σ^2, T ranges over a nondegenerate compact 1-dimensional interval C, without loss of generality $C = [0, 1]$.

Let Y and T be random variables such that T ranges over $[0, 1]$ and Y is real valued.

Let $\{T_i,\ Y_i,\ i = 1, \cdots, n\}$ denote a sample of size n from the

$$Y_i = \theta_1 + g(T_i) + \varepsilon_i,$$

where the errors ε_i are assumed to be independent and identical, T_i and ε_i are independent. Denote

$$\begin{aligned}
&\mathbf{Y} = (Y_1, \cdots, Y_n)^\tau,\\
&\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^\tau,\\
&\mathbf{1}^* = (1, \cdots, 1)^\tau,\\
&\theta_1^* = \mathbf{1}^*\theta_1,\\
&\mathbf{g}(\mathbf{T}) = (g(T_1), \cdots, g(T_n))^\tau,\\
&\mathcal{F} \text{ is an indentity matrix,} \quad R_n = 1^{*\tau}(\mathcal{F} - P)1^*.
\end{aligned}$$

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Condition 1. The distribution of T is absolutely continuous and its density is bounded away from 0 on $[0,1]$.

Condition 2. Let r and M denote nonnegative real constants, $0 < r \le 1$, m is a nonnegative integer such that

$$|g^{(m)}(t') - g^{(m)}(t)| \le M|t'-t|^r, \quad \text{for } 0 \le t, t' \le 1, \qquad p = m + r.$$

Condition 3. $\psi(\cdot) = \frac{\varphi'}{\varphi}(\cdot)$, $\lim\limits_{\delta\to 0}\int \sup\limits_{|h|\le\delta} |\psi'(y+h) - \psi'(y)|\varphi(y)\,dy = 0$, $I(\varphi) < \infty$.

Condition 4. $\varphi(x) > 0, x \in R^1$, is twice differentiable and $\lim\limits_{|x|\to\infty}\varphi(x) = \lim\limits_{|x|\to\infty}\varphi'(x) = 0$.

First we describe a piecewise polynomial estimator of g given in [2] (1988), which has been investigated by some authors. Given a positive M_n, the estimator has the form of a piecewise polynomial of degree m based on M_n intervals of length $\frac{1}{M_n}$, where the $(m+1)M_n$ coefficients are chosen by the method of least squares on the basis of the data $(T_1, Y_1), \cdots, (T_n, Y_n)$, $1 \le i \le n$. Let $I_{n\nu} = [\frac{\nu-1}{M_n}, \frac{\nu}{M_n})$ for $1 \le \nu < M_n$ and $I_{nM_n} = [1 - \frac{1}{M_n}, 1]$. Let $\chi_{n\nu}$ denote the indicator function for the interval $I_{n\nu}$, so that $\chi_{n\nu}$=1 or 0 according to $t \in I_{n\nu}$ or $t \notin I_{n\nu}$. Consider the piecewise polynomial estimators of g of degree m given by

$$\hat{g}_n(t) = \sum_{\nu=1}^{M_n} \chi_{n\nu}(t)\hat{P}_{nm\nu}(t),$$

where $\{\hat{P}_{nm\nu}(t)\}$ are polynomials of degree m chosen to minimize the residual sum of squares

$$\sum_{i=1}^{n}(Y_i - \theta_1 - \hat{g}_n(T_i))^2.$$

Denote

$$\hat{P}_{nm\nu}(x) = a_{0\nu} + a_{1\nu}x + \cdots + a_{m\nu}x^m,$$

$$Z = \begin{pmatrix} \psi_{n1}(T_1) & \dots & \psi_{n1}(T_1)T_1^m & \dots & \psi_{nM_n}(T_1) & \dots & \psi_{nM_n}(T_1)T_1^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{n1}(T_n) & \dots & \psi_{n1}(T_n)T_n^m & \dots & \psi_{nM_n}(T_n) & \dots & \psi_{nM_n}(T_n)T_n^m \end{pmatrix}_{n\times mM_n}.$$

$$\alpha = (a_{01}, \cdots, a_{m1}, a_{02}, \cdots, a_{m2}, \cdots, a_{0M_n}, \cdots, a_{mM_n})^{\tau}_{(m+1)M_n},$$

$$\begin{pmatrix} \hat{g}_n(T_1) \\ \vdots \\ \hat{g}_n(T_n) \end{pmatrix} = \begin{pmatrix} \sum\limits_{\nu=1}^{M_n} \chi_{n\nu}(T_1)\hat{P}_{nm\nu}(T_1) \\ \vdots \\ \sum\limits_{\nu=1}^{M_n} \chi_{n\nu}(T_n)\hat{P}_{nm\nu}(T_n) \end{pmatrix} = Z\alpha.$$

We know from [7](1986) that

$$\rho^*_\theta = \rho_\theta, \quad I_* = I(\varphi) = \int \frac{{\varphi'}^2}{\varphi}d\nu.$$

We adopt $\hat{g}(t)$ given in [2] (1988) as an estimator of $g(t)$, and then get MLE of θ_1. Denote

$$\xi_n = R_n^{-1}\mathbf{1}^{*\tau}(\mathcal{F} - P)\mathbf{g}(T),$$

$$\eta_n = R_n^{-1}\mathbf{1}^{*\tau}(\mathcal{F}-P).$$

Since

$$|g^{(m)}(t') - g^{(m)}(t)| \le M|t'-t|^r \quad \text{for } 0 \le t, t' \le 1,$$

according to arguments of [2] (1988), $\mathcal{F}-P$ is an idempotent matrix,

$$(\mathcal{F}-P)\mathbf{g}(T) = (\mathcal{F}-P)(\mathbf{g}(T) - Z\alpha),$$

and there is a constant $B_2 > 0$ such that

$$|\psi_{n\nu}(t)(\hat{P}_{mn\nu} - g(t)| \le B_2 M_n^{-p} \quad \text{for all } \nu \text{ and } t \in C.$$

$$g(T_i) - \hat{g}_n(T_i) = g(T_i) - \sum_{\nu=1}^{M_n} \psi_{n\nu}(T_i)\hat{P}_{mn\nu}(T_i) = \sum_{\nu=1}^{M_n} \psi_{n\nu}(T_i)[g(T_i) - \hat{P}_{mn\nu}(T_i)].$$

So

$$|g(T_i) - \hat{g}_n(T_i)| \le B_2 M_n^{-p}. \tag{1.1}$$

Lemma 1.1. $\sqrt{n}\xi_n \to 0.$

Proof. From (1.1) we know $\big(\mathbf{g}(T) - Z\alpha\big)^\tau \big(\mathbf{g}(T) - Z\alpha\big) \le B_2^2 n M_n^{-2p}$.

$$\begin{aligned}
&\left|\sqrt{n}R_n^{-1}\mathbf{1}^{*\tau}(\mathcal{F}-P)\mathbf{g}(T)\right| \\
=&\left|\sqrt{n}R_n^{-1}\mathbf{1}^{*\tau}(\mathcal{F}-P)\big(\mathbf{g}(T) - Z\alpha\big)\right| \\
\le&\left|\sqrt{n}R_n^{-1}\sqrt{\mathbf{1}^{*\tau}(\mathcal{F}-P)\mathbf{1}^*}\sqrt{\big(\mathbf{g}(T) - Z\alpha\big)^\tau\big(\mathbf{g}(T) - Z\alpha\big)}\right| \\
\le& B_2\sqrt{n}R_n{}^{-\frac{1}{2}}(\sqrt{n}M_n^{-p}) \to 0.
\end{aligned}$$

§2. On Bahadur Asymptotic Efficiency of $\hat{\theta}_{1ML}$

Since Bahadur posed his asymptotically efficient concept of consistent estimate, many authors have discussed this efficiency in parametric models. Now we consider Bahadur asymptotic efficiency of $\hat{\theta}_{1ML}$.

Definition 2.1. *Estimators $\{\tilde{h}_n(Y_1, \cdots, Y_n)\}$ of θ_1 are called locally uniformly consistent estimators, if for every $\theta_{10} \in R^1$ there exists a $\delta > 0$ such that for each $\zeta > 0$*

$$\lim_{n\to\infty} \sup_{|\theta_1-\theta_{10}|<\delta} P_{\theta_{10}}\{|\tilde{h}_n - \theta_{10}| > \zeta\} = 0.$$

Definition 2.2. *Consistent estimators $\{\tilde{h}_n\}$ of θ_1 are called Bahadur asymptotically efficient, if for each $\theta_{10} \in R^1$*

$$\limsup_{\zeta\to 0}\limsup_{n\to\infty} \frac{1}{n\zeta^2} I^{-1}(\varphi) \log P_{\theta_{10}}\{|\tilde{h}_n - \theta_{10}| > \zeta\} \le -\frac{1}{2}.$$

Theorem 2.1. *Suppose conditions* 1-4 *hold, $\{\tilde{h}_n\}$ are locally uniformly consistent estimators. Then for each $\theta_{10} \in R^1$*

$$\liminf_{\zeta\to 0}\liminf_{n\to\infty} \frac{1}{n\zeta^2} I^{-1}(\varphi) \log P_{\theta_{10}}\{|\tilde{h}_n - \theta_{10}| > \zeta\} \ge -\frac{1}{2}.$$

Proof. For each $\zeta > 0$, set $\theta_{1n} = \theta_{10} + \zeta$. Consider test hypothesis $H_0 : \theta_1 = \theta_{10} \longleftrightarrow H_1 : \theta_1 = \theta_{1n}$. Under H_0, the density of $\mathbf{Y}$ is $\prod_{i=1}^{n} \varphi(y_i - \theta_{10} - g(t_i))$; under H_1, the density of $\mathbf{Y}$ is $\prod_{i=1}^{n} \varphi(y_i - \theta_{1n} - g(t_i))$.

Denote

$$\Gamma_n(\mathbf{Y}) = \prod_{i=1}^{n} \frac{\varphi(y_i - \theta_{10} - g(t_i))}{\varphi(y_i - \theta_{1n} - g(t_i))},$$

$$d_n = \exp\left\{\frac{n(1+\mu)\zeta^2}{2I^{-1}(\varphi)}\right\} \quad (\mu > 0).$$

According to Neyman-Pearson basic lemma, there exist $K_n > 0$, $\delta_n \in [0,1]$ such that

$$E_{\theta_{1n}}\{\Phi_n^*(\mathbf{Y})\} = \frac{1}{2},$$

where

$$\Phi_n^*(\mathbf{Y}) = \begin{cases} 1, & \Gamma_n(\mathbf{Y}) > K_n, \\ \delta_n, & \Gamma_n(\mathbf{Y}) = K_n, \\ 0, & \Gamma_n(\mathbf{Y}) < K_n. \end{cases}$$

Then

$$\begin{aligned} E_{\theta_{10}}\{\Phi_n^*(\mathbf{Y})\} &= \int \Phi_n^*(\mathbf{Y})\, dP_{n\theta_{10}} \geq \int_{\Gamma_n \leq d_n} \Phi_n^*(\mathbf{Y})\, dP_{n\theta_{10}} \\ &\geq \frac{1}{d_n} \int_{\Gamma_n(\mathbf{Y}) \leq d_n} \Phi_n^*(\mathbf{Y})\, \Gamma_n(\mathbf{Y})\, dP_{n\theta_{10}} = \frac{1}{d_n} \int_{\Gamma_n \leq d_n} \Phi_n^*(\mathbf{Y})\, dP_{n\theta_{1n}} \\ &= \frac{1}{d_n}\left[\frac{1}{2} - \int_{\Gamma_n(\mathbf{Y}) > d_n} \Phi_n^*(\mathbf{Y})\, dP_{n\theta_{1n}}\right]. \end{aligned}$$

If

$$\limsup_{n\to\infty} P_{\theta_{1n}}\{\Gamma_n(\mathbf{Y}) > d_n\} < \frac{1}{4}, \qquad (*)$$

then for n large enough $E_{\theta_{10}}\{\Phi_n^*(\mathbf{Y})\} \geq \frac{1}{4d_n}$. Set

$$\Phi_n(Y) = \begin{cases} 1, & |\tilde{h}_n - \theta_{10}| \geq \lambda'\varepsilon, \\ 0, & \text{otherwise}, \end{cases}$$

where constant $\lambda' \in (0,1)$. Because $|\theta_{1n} - \theta_1| = \zeta$ and $\{\tilde{h}_n\}$ are locally uniform consistent estimators, we have

$$\begin{aligned}
&\liminf_{n\to\infty} E_{\theta_{1n}}\{\Phi_n(\mathbf{Y})\}\\
=&\liminf_{n\to\infty} P_{\theta_{1n}}\{|\tilde{h}_n-\theta_{10}|>\lambda\zeta\}\\
=&\liminf_{n\to\infty} P_{\theta_{1n}}\{|\tilde{h}_n-\theta_{1n}+\theta_{1n}-\theta_{10}|\lambda'\zeta\}\\
\geq&\liminf_{n\to\infty} P_{\theta_{1n}}\{|\tilde{h}_n-\theta_{1n}|+|\theta_{1n}-\theta_{10}|>\lambda'\zeta\}\\
\geq&\liminf_{n\to\infty} P_{\theta_{1n}}\{|\tilde{h}_n-\theta_{1n}|<(\lambda-\lambda')\zeta\}\\
\geq&\liminf_{n\to\infty} P_{\theta_{1n}}\{|\tilde{h}_n-\theta_{1n}|<(\lambda-\lambda')\zeta\}=1.
\end{aligned}$$

So for n large enough

$$E_{\theta_{1n}}\{\Phi_n(\mathbf{Y})\}\geq\frac{1}{2}=E_{\theta_{1n}}\{\Phi_n^*(\mathbf{Y})\}.$$

According to Neyman-Pearson basic lemma, for n large enough,

$$E_{\theta_{10}}\{\Phi_n(\mathbf{Y})\}\geq E_{\theta_{10}}\{\Phi_n^*(\mathbf{Y})\}.$$

Therefore

$$P_{\theta_{10}}\{|\tilde{h}_n-\theta_{10}|>\lambda'\zeta\}\geq P_{\theta_{10}}\{|(\tilde{h}_n-\theta_{10})|>\lambda'\zeta\}=E_{\theta_{10}}\{\Phi_n(\mathbf{Y})\}\geq\frac{1}{4d_n}.$$

Hence

$$\begin{aligned}
&\frac{1}{n(\lambda'\zeta)^2}I^{-1}(\varphi)\log P_{\theta_{10}}\{|\tilde{h}_n-\theta_{10}|\geq\lambda'\zeta\}\\
\geq&\frac{1}{n(\lambda'\zeta)^2}I^{-1}(\varphi)\log\frac{1}{4d_n}\\
=&-\frac{1}{n(\lambda'\zeta)^2}I^{-1}(\varphi)\log d_n-\frac{1}{n(\lambda'\zeta^2)}I^{-1}(\varphi)\log 4\\
=&-\frac{(1+\mu)\zeta^2}{2(\lambda'\zeta)^2}-\frac{I^{-1}(\varphi)\log 4}{n(\lambda'\zeta)^2}.
\end{aligned}$$

Thus

$$\liminf_{\zeta\to 0}\liminf_{n\to\infty}\frac{I^{-1}(\varphi)}{n(\lambda'\zeta)^2}\log P_{\theta_{10}}\{|\tilde{h_n}-\theta_{10}|\geq\lambda'\zeta\}\geq-\frac{1+\mu}{2\lambda'^2}.$$

Letting $\mu\to 0$, $\lambda'\to 1$, we have

$$\liminf_{\zeta\to 0}\liminf_{n\to\infty}\frac{1}{n\zeta^2}I^{-1}(\varphi)\log P_{\theta_{10}}\{|\tilde{h}_n-\theta_{10}|>\zeta\}\geq-\frac{1}{2}.$$

Now we return to the proof of formula $(*)$

$$\limsup_{n\to\infty} P_{\theta_{1n}}\{\Gamma_n(\mathbf{Y})>d_n\}<\frac{1}{4}.$$

In fact

$$\begin{aligned}
P_{\theta_{1n}}\{\Gamma_n(\mathbf{Y})>d_n\}&=P_{\theta_{1n}}\left\{\prod_{i=1}^n\frac{\varphi(Y_i-\theta_{10}-g(t_i))}{\varphi(Y_i-\theta_{1n}-g(t_i))}>d_n\right\}\\
&=P_0\left\{\prod_{i=1}^n\frac{\varphi(Y_i)}{\varphi(Y_i+\Delta)}>d_n\right\}.
\end{aligned}\qquad(*_1)$$

According to Taylor formula, for sufficiently small $\zeta > 0$,

$$\sum_{i=1}^{n} \log \frac{\varphi(Y_i)}{\varphi(Y_i+\zeta)} = -\sum_{i=1}^{n} \left\{ \psi(Y_i)\zeta + \frac{1}{2}(\psi'(Y_i) + R_i(Y_i))\zeta^2 \right\}$$

where $R_i(Y_i) = \psi'(Y_i + \mu_i\zeta) - \psi'(Y_i), \quad 0 < \mu_i < 1$. Then

$$\begin{aligned} &P_{\theta_{1n}}\{\Gamma_n(Y) > d_n\} = P_0\left\{\prod_{i=1}^{n} \frac{\varphi(Y_i)}{\varphi(Y_i+\zeta)} > d_n\right\} \\ =&P_0\left\{\sum_{i=1}^{n} \log \frac{\varphi(Y_i)}{\varphi(Y_i+\zeta)} > \frac{n(1+\mu)\zeta^2}{2I^{-1}(\varphi)}\right\} \\ =&P_0\left\{-\sum_{i=1}^{n}[(\psi(Y_i)\zeta + \frac{1}{2}(\psi'(Y_i) + R_i(Y_i))\zeta^2] > \frac{n(1+\mu)\zeta^2}{2I^{-1}(\varphi)}\right\} \end{aligned}$$

$$\begin{aligned} =&P_0\left\{-\sum_{i=1}^{n}(\psi(Y_i)\zeta + \frac{1}{2}[\psi'(Y_i) + I(\varphi)]\zeta^2) \right. \\ &\left. -\frac{1}{2}\sum_{i=1}^{n} R_i(Y_i)\zeta^2 + \frac{1}{2}\sum_{i=1}^{n} I(\varphi)\zeta^2 > \frac{n(1+\mu)\zeta^2}{2I^{-1}(\varphi)}\right\} \\ \leq&P_0\left\{\frac{1}{2}\sum_{i=1}^{n} I(\varphi)\zeta^2 > \frac{n(1+\frac{\mu}{2})\zeta^2}{2I^{-1}(\varphi)}\right\} + P_0\left\{-\sum_{i=1}^{n}\psi(Y_i)\zeta > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\ &+ P_0\left\{-\sum_{i=1}^{n}\frac{1}{2}[\psi'(Y_i) + I(\varphi)]\zeta^2 > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\ &+ P_0\left\{-\frac{1}{2}\sum_{i=1}^{n} R_i(Y_i)\zeta^2 > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\ =&P_1 + P_2 + P_3 + P_4. \end{aligned}$$

$$
\begin{aligned}
P_1 &= P_0\left\{\frac{1}{2}\sum_{i=1}^{n} I(\varphi)\zeta^2 > \frac{n(1+\frac{\mu}{2})\zeta^2}{2I^{-1}(\varphi)}\right\} \\
&= P_0\Big\{\frac{1}{2}I(\varphi)\zeta^2 > \frac{(1+\frac{\mu}{2})\zeta^2 I(\varphi)}{2}\Big\} = 0, \\
P_2 &= P_0\left\{-\sum_{i=1}^{n}\psi(Y_i)\zeta > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\
&\le \frac{\zeta^2 nE\psi^2}{\frac{n^2\mu^2\zeta^4}{144}I^2(\varphi)} = \frac{144}{n\mu^2\zeta^2 I(\varphi)} \to 0, \\
P_3 &= P_0\left\{-\sum_{i=1}^{n}\frac{1}{2}[\psi'(Y_i)+I(\varphi)]\zeta^2 > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\
&\le E_0\{\sum_{i=1}^{n}[\psi'(Y_i)+I(\varphi)]^2\}\cdot\frac{36}{n^2\mu^2 I^2(\varphi)} \\
&= \frac{36E_0\big[\psi'(Y_1)+I(\varphi)\big]^2}{n\mu^2 I^2(\varphi)} \to 0, \\
P_4 &= P_0\left\{-\frac{1}{2}\sum_{i=1}^{n}R_i(Y_i)\zeta^2 > \frac{n\mu\zeta^2}{12I^{-1}(\varphi)}\right\} \\
&\le P_0\Big\{\max_{1\le i\le n}|R_i(Y_i)|\sum_{i=1}^{n}\zeta^2 > \frac{n\mu\zeta^2}{6|I^{-1}(\varphi)|}\Big\} \\
&\le \frac{6I^{-1}(\varphi)}{n\mu\zeta^2}\cdot n\zeta^2 E_0\{\max_{1\le i\le n}|R_i(Y_i)|\} = \frac{6I^{-1}(\varphi)}{\mu}E_0\{\max_{1\le i\le n}|R_i(Y_i)|\}.
\end{aligned}
$$

Due to Condition 3, letting ζ be sufficiently small, we have

$$
E_0\{\max_{1\le i\le n}|R_i(Y_i)|\} \le \int \sup_{|h|<\zeta}|\psi'(y+h)-\psi'(y)|\varphi(y)\,dy \le \frac{\mu I(\varphi)}{24},
$$

so $\lim\limits_{n\to\infty} P_4 \le \frac{1}{4}$.
Hence

$$
\limsup_{n\to\infty} P_{\theta_{1n}}\{\Gamma_n(\mathbf{Y}) > d_n\} < \frac{1}{4}.
$$

Condition 5. There exists a $t_0>0$ such that

$$
\int e^{t_0|\psi(x)|}\varphi(x)\,dx < \infty, \qquad \int e^{t_0|\psi'(x)|}\varphi(x)\,dx < \infty.
$$

Condition 6. There exist measurable function $h(x)>0$, nondecreasing function $\gamma(t)>0$, $t>0$, $\lim\limits_{t\to 0^+}\gamma(t)=0$, and $|\psi(x+t)-\psi(x)| \le h(x)\gamma(t)$ whenever $|t|\le|t_0|$. Meanwhile $\int e^{h(x)}\varphi(x)\,dx<\infty$.

Condition 7. MLE $\hat\theta_{1ML}$ exists, and for each $\zeta>0$, $\theta_{10}\in R^1$, there exist constsnts $K=K(\zeta,\theta_{10})$, $\rho=\rho(\zeta,\theta_{10})>0$, such that

$$
P_{\theta_{10}}\Big\{\big|\hat\theta_{1ML}-\theta_{10}\big|>\zeta\Big\} \le Ke^{-n\rho\zeta^2}.
$$

Theorem 2.2. *If conditions* 1-7 *hold, then for each* $\theta_{10} \in R^1$

$$\limsup_{\zeta\to 0}\limsup_{n\to\infty}\frac{1}{n\zeta^2}\log P_{\theta_{10}}\Big\{|\hat\theta_{1ML}-\theta_{10}|>\zeta\Big\}\le -\frac{I}{2}.$$

Proof. From the definition of MLE and differentiability of φ, we know

$$\sum_{i=1}^n \psi(Y_i-\hat\theta_{1ML}-\hat g(T_i))=0.$$

On the other hand

$$\begin{aligned}&\psi(Y_i-\hat\theta_{1ML}-\hat g(T_i))-\psi(Y_i-\theta_1-g(T_i))\\ =&\psi'(Y_i-\theta_1^*-g^*(T_i))\Big\{(\hat\theta_{1ML}-\theta_1)+(\hat g(T_i)-g(T_i))\Big\}.\end{aligned}$$

So

$$\begin{aligned}&\sum_{i=1}^n \psi(Y_i-\hat\theta_{1ML}-\hat g(T_i))\\ =&\sum_{i=1}^n \psi(Y_i-\theta_1-g(T_i))-\sum_{i=1}^n\psi'(Y_i-\theta_1^*-g^*(T_i))\Big[(\hat\theta_{1ML}-\theta_1)+(\hat g(T_i)-g(T_i))\Big]=0,\end{aligned}$$

where θ_1^* lies between $\hat\theta_{1ML}$ and θ_1, $g^*(T_i)$ lies between $g(T_i)$ and $\hat g(T_i)$. Denote

$$R_i(Y_i,T_i)=\psi'(Y_i-\theta_1^*-g^*(T_i))-\psi'(Y_i-\theta_1-g(T_i)).$$

So

$$\psi'(Y_i-\theta_1^*-g^*(T_i))=\psi'(Y_i-\theta_1-g(T_i))+R_i(Y_i,T_i),$$

$$\begin{aligned}&\sum_{i=1}^n \psi(Y_i-\theta_1-g(T_i))\\ =&\sum_{i=1}^n\Big[\psi'(Y_i-\theta_1-g(T_i))+R_i(Y_i,T_i)\Big]\big[(\hat\theta_{1ML}-\theta_1)+(\hat g(T_i)-g(T_i))\big].\end{aligned}$$

Denote

$$\begin{aligned}
R_1^{**} =& \frac{1}{n}\sum_{i=1}^n \big[I + \psi'(Y_i - \theta_1 - g(T_i))\big],\\
R_2^{**} =& \frac{1}{n}\sum_{i=1}^n R_i(Y_i, T_i) \Longrightarrow \frac{1}{n}\sum_{i=1}^n \psi(Y_i - \theta_1 - g(T_i))\\
=& \Big\{-I + \frac{1}{n}\sum_{i=1}^n \Big[I + \psi'(Y_i - \theta_1 - g(T_i))\\
&+ R_i(Y_i, T_i)\Big]\Big\}\Big[(\hat{\theta}_{1ML} - \theta_1) + (\hat{g}(T_i) - g(T_i))\Big]\\
=& \{-I + R_1^{**} + R_2^{**}\}(\hat{\theta}_{1ML} - \theta_1) + \frac{1}{n}\sum_{i=1}^n \psi(Y_i - \theta_1^* - g^*(T_i))(\hat{g}(T_i) - g(T_i))\\
\Longrightarrow& \hat{\theta}_{1ML} - \theta_1 = (-I + R_1^{**} + R_2^{**})^{-1}\Big[\frac{1}{n}\sum_{i=1}^n \psi'(Y_i - \theta_1 - g(T_i))\Big]\\
&- \frac{1}{n}\sum_{i=1}^n \psi'(Y_i - \theta_1^* - g^*(T_i))(\hat{g}(T_i) - g(T_i)).
\end{aligned}$$

Let α be sufficiently small. When $|R_1^{**} + R_2^{**}| < \alpha$, $|-I + R_1^{**} + R_2^{**}| \neq 0$, and so $(-I+R_1^{**}+R_2^{**})^{-1}$ exists, denote $-(I^{-1}+\tilde{W})$. According to continuity, there is a nondecrease function $\eta(\alpha) > 0$ such that when $|R_1^{**} + R_2^{**}| < \alpha$, $|\tilde{W}| < \eta(\alpha)$ and $\lim_{\alpha\to 0}\eta(\alpha) = 0$. Suppose $|R_1^{**} + R_2^{**}| < \alpha$. Then

$$\begin{aligned}
\hat{\theta}_{1ML} - \theta_1 =& -(I^{-1} + \tilde{W})\big[\frac{1}{n}\sum_{i=1}^n \psi(Y_i - \theta_1 - g(T_i))\big]\\
&- \frac{1}{n}\sum_{i=1}^n \psi'(Y_i - \theta_1^* - g^*(T_i))(\hat{g}(T_i) - g(T_i)),
\end{aligned}$$

so for every $0 < \lambda < \frac{1}{4}$

$$
\begin{aligned}
&P_{\theta_{10}}\Big\{\Big|(\hat{\theta}_{1ML} - \theta_{10})\Big| > \zeta\Big\} \\
\le & P_{\theta_{10}}\Big\{\Big|\frac{1}{n}\sum_{i=1}^{n}\psi(Y_i - \theta_{10} - g(T_i))\Big| > (1-2\lambda)I\zeta\Big\} \\
&+ P_{\theta_{10}}\Big\{\Big|\frac{1}{n}\sum_{i=1}^{n}\tilde{W}\psi(Y_i - \theta_{10} - g(T_i))\Big| > \lambda\zeta\Big\} \\
&+ P_{\theta_{10}}\Big\{\frac{1}{n}\sum_{i=1}^{n}|\hat{g}(T_i) - g(T_i))|\Big|\psi'(Y_i - \theta_1^* - g^*(T_i))\Big| > \lambda\zeta\Big\} \\
\le & P_{\theta_{10}}\Big\{\frac{1}{n}\sum_{i=1}^{n}\psi(Y_i - \theta_{10} - g(T_i))| > (1-2\lambda)I\zeta\Big\} \\
&+ P_{\theta_{10}}\Big\{|\sum_{i=1}^{n}\frac{1}{n}\psi(Y_i - \theta_{10} - g(T_i))| > \frac{\lambda\zeta}{\eta(2\alpha)}\Big\} \\
&+ P_{\theta_{10}}\Big\{|R_1^{**}| > \alpha\} + P_{\theta_{10}}\{|R_2^{**}| > \alpha\} \\
&+ P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i) - g(T_i))|\Big|\psi'(Y_i - \theta_1^* - g^*(T_i))\Big| > \lambda\zeta\Big\} \\
= & P_1 + P_2 + P_3 + P_4 + P_5.
\end{aligned}
$$

First we estimate P_1. Denote

$$
\begin{aligned}
W_i &= \psi(Y_i - \theta_{10} - g(T_i)), \\
a_{in} &= \frac{1}{n}, \\
A_n &= \sum_{i=1}^{n} a_{in}^2 = \frac{1}{n}.
\end{aligned}
$$

Then

$$
E_{\theta_{10}}W_i = 0, \qquad E_{\theta_{10}}W_i^2 = I, \quad \frac{|a_{in}|}{A_n} = 1 = \hat{A}.
$$

So

$$
P_1 \le 2\exp\big\{-\frac{n(1-2\lambda)^2\zeta^2 I^2}{2I}(1 + O_1(\zeta))\big\},
$$

$|O_1(\zeta)| < CWB_1\zeta$, B_1 depends on ζ, not n.

Denote

$$
\begin{aligned}
W_i &= \psi(Y_i - \theta_{10} - g(T_i)), \\
a_{in} &= \frac{1}{n}, \\
A_n &= \frac{1}{n}, \qquad \hat{A} = 1.
\end{aligned}
$$

$$
P_2 \le P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}\psi(Y_i - \theta_{10} - g(T_i))\Big| > \frac{\lambda\zeta}{\eta(2\alpha)}\Big\},
$$

so

$$P_2 \le 2\exp\Big\{-\frac{2\lambda^2\zeta^2}{2I\eta^2(\alpha)}(1+O_2(\zeta))\Big\}$$

$|O_2(\zeta)| < B_2\eta^{-1}(2\alpha)\zeta$. Due to $\eta(\alpha)\to 0$, let α be small enough so that

$$\eta^2(2\alpha) \le \frac{\lambda^2}{(1-2\lambda)^2 I^2} \Longrightarrow \frac{\lambda^2\zeta^2}{2I\eta^2(2\alpha)} \ge \frac{(1-2\lambda)^2\zeta^2 I}{2}.$$

So

$$P_2 \le 2\exp\Big\{-\frac{n(1-2\lambda)^2\zeta^2 I}{2}(1+O_2(\zeta))\Big\}.$$

$$P_3 = P_{\theta_{1_0}}\{|R_1^{**}| > \alpha\} = P_{\theta_{1_0}}\Big\{\Big|\sum_{i=1}^n \frac{1}{n}\big(I+\psi'(Y_i-\theta_{10}-g(T_i))\big)\Big| > \alpha\Big\}.$$

Let

$$W_i = I+\psi'(Y_i-\theta_{10}-g(T_i)) \quad (\text{so} \quad EW_i=0, \ \ \tilde\sigma^2 = EW_i^2>0),$$
$$a_{in} = \frac{1}{n}, \qquad A_n = \frac{1}{n}, \qquad \hat A = 1.$$

$$P_3 \le 2\exp\Big\{-\frac{n\alpha^2}{2}(1+O_3(\alpha))\Big\},$$

$|O_3(\alpha)| \le B_3\alpha$, B_1 only depends on $\psi'(\epsilon)$, not on n. Let ζ be small enough so that

$$P_3 \le 2\exp\Big\{-\frac{(1-2\lambda)^2\zeta^2 I}{2}\}.$$

$$\begin{aligned}|R_i(Y_i,T_i)| =& |\psi'(Y_i-\theta_1^*-g^*(T_i)-\psi'(Y_i-\theta_{10}-g(T_i))| \\ \le& h(Y_i-\theta_{10}-g(T_i))\gamma(({\theta_1}^*-\theta_{10})-(g^*(T_i)-g(T_i))).\end{aligned}$$

Denote $h_0 = E_{\theta_{1_0}}h$.

$$\begin{aligned}P_4 =& P_{\theta_{1_0}}\Big\{\Big|\frac{1}{n}\sum_{i=1}^n R_i(Y_i,T_i)\Big| > \alpha\Big\} \\ \le& P_{\theta_{1_0}}\Big\{\Big|\sum_{i=1}^n \frac{1}{n}h(Y_i-\theta_{10}-g(T_i))\gamma(({\theta_1}^*-\theta_{10})-(g^*(T_i)-g(T_i)))\Big| \ge \alpha\Big\} \\ \le& P_{\theta_{1_0}}\Big[\{\bigcup_{i=1}^n |(\hat g(T_i)-g(T_i))| > \zeta\Big\} + P_{\theta_{1_0}}\{|\theta_{1ML}-\theta_{10}| \ge \zeta\} \\ &+ P_{\theta_{1_0}}\Big\{\Big|\sum_{i=1}^n \frac{1}{n}h(Y_i-\theta_{10}-g(T_i))\gamma(2\zeta)\Big| \ge \alpha\Big\}.\end{aligned}$$

According to condition 7

$$P_{\theta_{1_0}}\{|\theta_{1ML}-\theta_{10}| \ge \zeta\} \le K(\zeta,\theta_{10})\exp\{-\rho(\zeta,\theta_{10})\zeta^2\}.$$

Let ζ be small enough.

$$\zeta \le \Big(\frac{2\rho}{(1-2\lambda)^2 I}\Big)^{\frac{1}{2}} \Longrightarrow \rho(\zeta,\theta_{10}) \ge \frac{(1-2\lambda)^2 I}{2}\zeta^2.$$

$$P_{\theta_{1_0}}\{|\theta_{1ML}-\theta_{10}| \ge \zeta\} \le K(\zeta,\theta_{10})\exp\Big\{-\frac{n(1-2\lambda)^2 I}{2}\zeta^2\Big\}.$$

$$\begin{aligned}&P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}h(Y_i-\theta_{10}-g(T_i))\gamma(2\zeta)\Big|\geq\alpha\Big\}\\=&P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}[h(Y_i-\theta_{10}-g(T_i))-h_0+h_0]\Big|\geq\frac{\alpha}{\gamma(2\zeta)}\Big\}\\\leq&P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}[h(Y_i-\theta_{10}-g(T_i))-h_0]\Big|\geq\frac{\alpha}{2\gamma(2\zeta)}\Big\}.\end{aligned}$$

Let

$$\begin{aligned}&W_i=h(Y_i-\theta_{10}-g(T_i))-h_0,\\&a_{in}=\frac{1}{n},\qquad A_n=\frac{1}{n},\qquad \hat{A}=1,\\&\sigma_h^2=E_{\theta_{10}}[h(Y_i-\theta_{10}-g(T_i))-h_0]^2.\end{aligned}$$

So

$$\begin{aligned}&P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}[h(Y_i-\theta_{10}-g(T_i))-h_0]\Big|\geq\frac{\alpha}{2\gamma(2\zeta)}\Big\}\\\leq&2\exp\Big\{-\frac{\Big(\frac{\alpha}{2\gamma(2\zeta)}\Big)^2}{2\sigma_h^2}\Big(1+O_4(\frac{\alpha}{2\gamma(2\zeta)})\Big)\Big\},\end{aligned}$$

$$\Big|O_4\Big(\frac{\alpha}{2\gamma(2\zeta)})\Big|\leq B_4\frac{\alpha}{2\gamma(2\zeta)},$$

B_4 depends on $h(\epsilon_1)$ only, but not on n. Let ζ be small enough. Then

$$P_{\theta_{10}}\Big\{\Big|\sum_{i=1}^{n}\frac{1}{n}[h(Y_i-\theta_{10}-g(T_i))-h_0]\Big|\geq\frac{\alpha}{2\gamma(2\zeta)}\Big\}\leq 2\exp\Big\{-\frac{n(1-2\lambda)^2I\zeta^2}{2}\Big\}.$$

From Definition 2.1 we know

$$P_{\theta_{10}}\{\bigcup_{i=1}^{n}|(\hat{g}(T_i)-g(T_i))|>\zeta\}=0.$$

So

$$P_4\leq(2+K)\exp\Big\{-\frac{n(1-2\lambda)^2I\zeta^2}{2}\Big\}.$$

For P_5,

$$\begin{aligned}&P_{\theta_{10}}\Big\{\frac{1}{n}\sum_{i=1}^{n}|\hat{g}(T_i)-g(T_i)|\big|\psi'(Y_i-\theta_1^*-g^*(T_i))\big|>\lambda\zeta\Big\}\\=&P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i)-g(T_i)|\big|R_i(Y_i,T_i)+\psi'(Y_i-\theta_{10}-g(T_i))\big|>\lambda\zeta\Big\}\\\leq&P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i)-g(T_i)|\big|R_i(Y_i,T_i)\big|>\frac{\lambda}{2}\zeta\Big\}\\&+P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i)-g(T_i)|\big|\psi'(Y_i-\theta_{10}-g(T_i))\big|>\frac{\lambda}{2}\zeta\Big\}.\end{aligned}$$

According to (1.1) and the processes of the proof of P_4, we know

$$P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i)-g(T_i)|\big|R_i(Y_i,T_i)\big|>\frac{\lambda}{2}\zeta\Big\}\le 2\exp\Big\{-\frac{n(1-2\lambda)^2I\zeta^2}{2}\Big\}.$$

Denote

$$\begin{aligned}
h_g &= E\Big\{\Big|\psi'(Y_i-\theta_{10}-g(T_i))\Big|\Big|\mathbf{T}\Big\},\\
\sigma_g^2 &= Var\Big\{\Big|\psi'(Y_i-\theta_{10}-g(T_i))\Big|\Big|\mathbf{T}\Big\}\le\tilde{\sigma}^2,\\
a_{in} &= \frac{1}{n}|\hat{g}(T_i)-g(T_i)|,\\
W_i &= \big|\psi'(Y_i-\theta_{10}-g(T_i))\big|-h_g.
\end{aligned}$$

So

$$\sum_{i=1}^{n}a_{in}^2=\sum_{i=1}^{n}|\hat{g}(T_i)-g(T_i))|^2n^{-2}\le n^{-2}M_n^{-2p}=A_n,\quad \hat{A}=1.$$

Due to (1.1), let n be sufficiently large, then $\big|\hat{g}(T_i)-g(T_i)\big|h_g<\frac{\lambda}{4}\zeta$,

$$\begin{aligned}
&P_{\theta_{10}}\Big\{\sum_{i=1}^{n}\frac{1}{n}|\hat{g}(T_i)-g(T_i)|\big|\psi'(Y_i-\theta_{10}-g(T_i))\big|>\frac{\lambda}{2}\zeta\Big|\mathbf{T}\Big\}\\
\le&P_{\theta_{10}}\Big\{\sum_{i=1}^{n}a_{in}\big|\psi'(Y_i-\theta_{10}-g(T_i))-h_g\big|>\frac{\lambda}{4}\zeta\Big\}\\
\le&2\exp\Big\{-\frac{(\frac{\lambda}{4}\zeta)^2}{2\sigma_g^2A_n^2}\big(1+O_5(\zeta)\big)\Big\}.
\end{aligned}$$

$|O_5(\zeta)|<B_5\lambda\zeta$, B_5 depends $\psi(\epsilon_1)$, not n. Obviously

$$\frac{(\frac{\lambda}{4}\zeta)^2}{\tilde{\sigma}_g^2M_n^{-2p}n^{-2}}>\frac{n(1-2\lambda)^2I\zeta^2}{2}.$$

So

$$P_5<(K+4)\exp\Big\{-\frac{(1-2\lambda)^2I\zeta^2}{2}\Big\}.$$

In all, first let δ be small, then let α small, at last let ζ be sufficiently small, thus

$$\begin{aligned}
P_{\theta_{10}}\Big\{|\theta_{1ML}-\theta_{10}|>\zeta\Big\}&\le P_1+P_2+P_3+P_4+P_5\\
&\le 2(K+5)\exp\Big\{-\frac{n(1-2\lambda)^2I\zeta^2}{2}(1+O(\zeta))\Big\},
\end{aligned}$$

$$O(\zeta)=\min\{O_1(\zeta),O_2(\zeta),O_3(\zeta),O_4(\zeta),O_5(\zeta)\},\quad |O(\zeta)|\le B^*(\eta^{-1}(2\alpha)+1)\}.$$

So

$$\limsup_{\zeta\to0}\limsup_{n\to\infty}\frac{1}{n\zeta^2}\log P_{\theta_{10}}\Big\{|\theta_{1ML}-\theta_{10}|>\zeta\Big\}\le-\frac{(1-2\lambda)^2I}{2}.$$

Letting $\lambda\to0$, we come to the conclusion

$$\limsup_{\zeta\to0}\limsup_{n\to\infty}\frac{1}{n\zeta^2}\log P_{\theta_{10}}\Big\{|\theta_{1ML}-\theta_{10}|>\zeta\Big\}\le-\frac{I}{2}.$$

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