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HOPF ALGEBRAIC APPROACH TO THE n LINEARLY RECURSIVE SEQUENCES**

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Abstract

It is proved that a linearly recursive sequence of n indices over field F ($n \geq 1$) is automatically a product of n linearly recursive sequences of 1-index over F by the theory of Hopf algebras. By the way, the correspondence between the set of linearly recursive sequences of 1-index and $F[X]^\circ$ is generalized to the case of n-index.

Keywords Hopf algebra, Linearly recursive sequence, Recursive relation.
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§1. Introduction

The purpose of the paper is to present a study of linearly recursive sequences of n-index over a field F from the aspects of Hopf algebras. It is based on the realization, which E. J. Taft suggested in [1], that there exists a one to one correspondence between the set of linearly recursive sequences of 1-index over a field F and the continuous dual of polynomial algebra $F[X]^\circ$. Since we do not assume one is familiar to the theory of Hopf algebras, in this section we mainly give a short summary of the terminology and the results which will be used in the sequal. First of all, we offer the concept of a linearly recursive sequence of n-index over field F. For the sake of conciseness and convienence, we only mentione the case when $n = 2$.

Throughout this paper, we have to mention, algebras and coalgebras are all taken over a given field F, N denotes the set of natural numbers and Z^+ the set of non-negative integers. Linearly recursive sequence always means homogeneous linearly recursive sequence since each linearly recursive sequence can determine uniquely a homogeneous linearly recursive sequence.

Definition 1.1. *Suppose $S_{i,j} \in F$. $\{S_{i,j}\}$ $(i, j \in Z^+)$ is called a linearly recursive sequence of 2-index over F if it can be obtained by the following relations:*

$$\begin{aligned} S_{n+i_1,m+j_1} &= a^{(1)}_{i_1,j_1} S_{n+i_1,m+j_1-1} + a^{(1)}_{i_1-1,j_1} S_{n+i_1-1,m+j_1} + \cdots + a^{(1)}_{0,0} S_{n,m}, \\ S_{n+i_2,m+j_2} &= a^{(2)}_{i_2,j_2} S_{n+i_2,m+j_2-1} + a^{(2)}_{i_2-1,j_2} S_{n+i_2-1,m+j_2} + \cdots + a^{(2)}_{0,0} S_{n,m}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_{n+i_r,m+j_r} &= a^{(r)}_{i_r,j_r} S_{n+i_r,m+j_r-1} + a^{(r)}_{i_r-1,j_r} S_{n+i_r-1,m+j_r} + \cdots + a^{(r)}_{0,0} S_{n,m}. \end{aligned} \tag{1.1}$$

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with the following initial conditions:

$$S_{i,j} = k_{i,j}, \ (i,j) \in L \times L, \ k_{i,j} \in F. \tag{1.2}$$

Here $L \times L$ is a subset of finite set:

$$\{(i_1,j_1),(i_1-1,j_1),\cdots,(i_r,j_r-1),(i_r-1,j_r),\cdots,(0,0)\}.$$

The sequences above are the natural generalization of that of one index. They have the finite number of linear relations and initial conditions which decide the whole sequence. The example below is a sequence of two-index.

$$\begin{aligned} S_{n+2,m} &= 2S_{n+1,m} - S_{n,m}, \\ S_{n+1,m+1} &= S_{n,m+1} + 2S_{n+1,m} - S_{n,m}, \\ S_{n,m+2} &= 4S_{n,m+1} - 3S_{n,m}, \end{aligned}$$

whose initial conditions are $S_{0,0} = 1$, $S_{1,0} = 2$, $S_{0,1} = 3$.

Definition 1.2. *Let $\{S_{i,j}\}$ $(i,j \in Z^+)$ be a linearly recursive sequence of 2-index over F described above. We call the following the set of characteristic polynomials associated with the sequence $\{S_{i,j}\}$ $(i,j \in Z^+)$:*

$$\begin{aligned} f_1 &\equiv X_1^{i_1}X_2^{j_1} - a^{(1)}_{i_1,j_1-1}X_1^{i_1}X_2^{j_1-1} - \cdots - a^{(1)}_{0,0}, \\ f_2 &\equiv X_1^{i_2}X_2^{j_2} - a^{(2)}_{i_2,j_2-1}X_1^{i_2}X_2^{j_2-1} - \cdots - a^{(2)}_{0,0}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_r &\equiv X_1^{i_r}X_2^{j_r} - a^{(r)}_{i_r,j_r-1}X_1^{i_r}X_2^{j_r-1} - \cdots - a^{(r)}_{0,0}. \end{aligned} \tag{1.3}$$

Suppose that $F[X_1,X_2,\cdots,X_n]$ is the polynomial algebra of n variables. We know that $F[X_1,X_2,\cdots,X_n]$ is in fact a Hopf algebra, whose comultiplication, counit and antipode are described as:

$$\Delta(X^n) = \sum_{i=0}^{n} \binom{n}{i} X^i \otimes X^{n-i}, \tag{1.4}$$

$$\varepsilon(X^n) = \delta_{n,0}; \quad S(X^n) = (-1)^n X^n,$$

$F[X_1,X_2,\cdots,X_n]^\circ$. The continuous dual of $F[X_1,X_2,\cdots,X_n]$ is defined as (see [2])

$$F[X_1,X_2,\cdots,X_n]^\circ = \left\{ f \in F[X_1,X_2,\cdots,X_n]^* \,\middle|\, \begin{array}{l} \text{Ker } f \text{ contains a cofinite} \\ \text{ideal of } F[X_1,X_2,\cdots,X_n]. \end{array} \right\}.$$

Observing that

$$F[X_1,X_2,\cdots,X_n]^* = F[[X_1],[X_2],\cdots,[X_n]]$$

and $F[X_1,X_2,\cdots,X_n]^\circ$ is a subalgebra of $F[X_1,X_2,\cdots,X_n]^*$, we learn that $F[X_1,X_2,\cdots,X_n]$. The multiplication and antipode of $F[X_1,X_2,\cdots,X_n]^\circ$ are given by

$$X_i^l * X_i^k = \binom{l+k}{k} X_i^{l+k}; \quad X_i * X_j = X_j * X_i \ (i \neq j), \ (i = 1,2,\cdots,n), \ i,k \in Z^+,$$

$$\langle S(f), a\rangle = \langle f, S(a)\rangle.$$

At the end of this section, we present a proposition, which Taft suggested in [1], as the initial point of our paper.

Proposition 1.1. *There exists a one to one correspondence between the set of linearly recursive sequences of 1-index over F and $F[X]^\circ$.*

Proof. Suppose that $\{S_i\}$ $(i \in Z^+)$ is a linearly recursive sequence of 1-index over F and $f(x)$ is its associated characteristic polynomial. Clearly $f(x) \neq 0$ and the ideal $I = (f(x))$ is cofinite. Let

$$g(X) = \sum_{i=0}^{\infty} S_i X^i.$$

Now we let $g(x)$ correspond to $\{S_i\}$ $(i \in Z^+)$. It is apparent that $g(X) \in F[X]^*$. Furthermore $g(X) \in F[X]^\circ$ since $\langle g(X), f(x)\rangle = 0$, and then $I \subseteq \text{Ker } g(X)$. Conversely let $g(X) \in F[X]^\circ$. Then there exists a cofinite ideal of $F[X]$, namely I, such that $I \subseteq \text{Ker } g(X)$. We could write $I = (f(x))$, where $f(x)$ is a nonzero monic polynomial over F since $F[X]$ is a principal ideal domain. Hence the only thing left is to show $\{S_i\}$ $(i \in Z^+)$, which consists of all coefficients of $g(X)$, is exactly the linearly recursive sequence associated with characteristic polynomial $f(x)$.

Let $f(x) = x^r - a_{r-1}x^{r-1} - a_{r-2}x^{r-2} - \cdots - a_0$. It is easy to see that

$$S_{r+n} = a_{r-1}S_{r+n-1} + a_{r-2}S_{r+n-2} + \cdots + a_0 S_n (n \geq 0) \tag{1.5}$$

because $x \cdot f(x) \in I \subseteq \text{Ker } g(X)$ for all $n \geq 0$. This derives our conclusion.

§2. Main Results

In this section, we mainly characterize the linearly recursive sequences involving multivariable relations over F. The first thing is to study the family of linearly recursive sequences. Let $f(x)$ be a monic polynomial of positive degree. We denote the set of all homogeneous linear recurring sequences over F with characteristic polynomial $f(x)$ by $S(f(x))$. In other words, $S(f(x))$ consists of all sequences in F satisfying the homogeneous linear recurrence relation (1.4) determined by $f(x)$. We have

Proposition 2.1.

(1) $S(f(x)) = (f(x))^\perp$, *where* $(f(x))^\perp$ *means the annihilator of ideal* $(f(x))$ *in* $F[X]^\circ$.

(2) *If* $\deg(f(x)) = k$ *and* F *is a finite field of* q *elements, then* $S(f(x))$ *has exactly* q^k *elements.*

Proof. (1) Let $\{S_i\}$ $(i \in Z^+)$ be a linearly recursive sequence whose monic characteristic polynomial is $f(x)$. By Proposition 1.1, there exists a $g \in F[X]^\circ$ corresponding to $\{S_i\}$ $(i \in Z^+)$ such that $g(X^n) = S_n$ and

$$g((f(x)) = 0 \Rightarrow g \in (f(x))^\perp.$$

The other direction is apparent.

(2) By (1), we have $S(f(x)) = (f(x))^\perp$. This means that $(f(x))^\perp$ is a subcoalgebra of $F_q[X]^\circ$ and of course a vector space over F_q. Since $\deg(f(x)) = k$, the codimension of $(f(x))$ must be

$$k \Rightarrow \dim_{F_q} S(f(x)) = k.$$

But F_q is a finite field of q elements, this implies that the vector space $S(f(x))$ has exactly q^k elements.

Since $(f(x))$ is an ideal of $F[X]$, $(f(x))^{\perp}$ must be a subcoalgebra of $F_q[X]^{\circ}$ (see [4] Chapter 2). The proposition above tells us that $S(f(x))$ is indeed a subcoalgebra, by this realization, we can establish the following theorems concerning the product of families of linearly recursive sequences of 1-index, which generalizes the Theorem 8.65 in [3] to an arbitrary field F. If σ is the sequence of elements $s_0, s_1, \cdots$ of F and τ is the sequence of elements $t_0, t_1, \cdots$ of F, then the product $\sigma\tau$ has terms $s_0t_0, s_1t_1, \cdots$. Analougously, one defines the product of any finite number of sequences. Let S be the vector space over F consisting of all sequences of elements of F, under the usual addition and scalar multiplication of the sequences. For nonconstant monic polynomials $f_1, f_2, \cdots, f_h$, let $S(f_1(x))\cdots S(f_h(x))$ be the subspace spanned by all products $\sigma_1\cdots\sigma_h$ with $\sigma_i \in S(f_i(x))$ $(1 \le i \le h)$.

Theorem 2.1. *If $f_1(x), \cdots, f_h(x)$ are nonconstant monic polynomials over F, then there exists a nonconstant monic polynomial $g(x) \in F[X]$ such that*

$$S(f_1(x))\cdots S(f_h(x)) = S(g(x)).$$

Before we prove the theorem, we first define another coalgebraic structure on $F[X]$. Set,

$$\Delta(X^i) = X^i \otimes X^i, \quad \varepsilon(X^i) = 1$$

for $i \in Z^+$, i.e., the elements of $F[X]$ are all group-like elements. One could prove that $F[X]$ is a coalgebra under this structure and furthermore a bialgebra with the original algebraic structure. Suppose that f, g are two elements of $F[X]^{\circ}$, which correspond to the linearly recursive sequences $\{s_i\}$ $(i \in Z^+)$ and $\{t_i\}$ $(i \in Z^+)$ respectively by Proposition 1.1. Then

$$\begin{aligned} s_it_i &= \langle f, X^i\rangle\langle g, X^i\rangle \\ &= \langle f \otimes g, X^i \otimes X^i\rangle \\ &= \langle f \otimes g, \Delta(X^i)\rangle \\ &= \langle fg, X^i\rangle. \end{aligned}$$

So the product of linearly recursive sequences coheres with that of algebra $F[X]^{\circ}$.

Proof of Theorem 2.1. By the remark above, $F[X]^{\circ}$ is a bialgebra and is closed under the product of algebra. Thus $S(f_i(x))S(f_j(x)) \subseteq F[X]^{\circ}$. Hence the only remaining thing is to prove that $S(f_1(x))\cdots S(f_h(x))$ is also a subcoalgebra of $F[X]^{\circ}$ when $h = 2$. In this case, because $S(f_1(x))$, $S(f_2(x))$ are subcoalgebra of $F[X]^{\circ}$, we obtain

$$\Delta(S(f_1(x))) \subseteq S(f_1(x)) \otimes S(f_1(x)),$$

$$\Delta(S(f_2(x))) \subseteq S(f_2(x)) \otimes S(f_2(x)).$$

Theorefore,

$$\begin{aligned} &\Delta(S(f_1(x))S(f_2(x))) \\ =&\Delta(S(f_1(x)))\Delta(S(f_2(x))) \\ \subseteq&(S(f_1(x)) \otimes S(f_1(x)))\cdots(S(f_2(x)) \otimes S(f_2(x))) \\ =&S(f_1(x))S(f_2(x)) \otimes S(f_1(x))S(f_2(x)). \end{aligned}$$

Thus $S(f_1(x))S(f_2(x))$ is a subcoalgebra of $F[X]^{\circ}$ and by Proposition 2.1 there exists an ideal I of $F[X]$ such that $S(f_1(x))S(f_2(x)) = I^{\perp}$. But $F[X]$ is a p.i.d, there exists a monic polynomial $g(x)$ such that $g(x) \neq 0$ and $I = (g(x))$, i.e.,

$$S(f_1(x))S(f_2(x)) = I^{\perp} = (g(x))^{\perp} = S(g(x))$$

and this completes our proof.

The product of linearly recursive sequences defined above is so-called Hadarmard product. Larson and Taft studied this kind of products of linearly recursive sequences by the method of Hopf algebras in [4]. Now we consider another kind of products of linearly recursive sequences, called Hurwitz product. If σ is a sequence of elements $s_0, s_1, \cdots$ of F and τ is a sequence of elements $t_0, t_1, \cdots$ of F, then the Hurwitz product of σ and τ, $\sigma \circ \tau$ has elements $\{s_i t_j\}$. One can find it is a linearly recursive sequence of 2-index. Our next theorem shows that every linearly recursive sequence of n-index is a Hurwitz product of n linearly recursive sequences of 1-index.

We first give a proposition concerning the charactristic polynomial set of linearly recursive sequences of n-index.

Proposition 2.2. *If $\{S_{i_1 i_2}\}$ $(i_1, i_2 \in Z^+)$ is a linearly recursive sequence with charactristic polynomial set $f_1, f_2, \cdots, f_n$, then the ideal generated by $f_1, f_2, \cdots, f_n$ is a cofinite ideal of $F[X_1, X_2]$.*

Proof. Let $I = (f_1, f_2, \cdots, f_n)$. For every $S_{k,l} \in \{S_{i_1 i_2}\}$ $(i_1, i_2 \in Z^+)$, by observing (1.1), we learn that each $S_{k,l}$ can be represented as a linear combination of its initials $\{S_{i_1 i_2}\}$ $(i_1, i_2 \in Z^+)$. But

$$S_{k,l} = \sum_{(i,j)\in L} k_{i,j} S_{i,j} \iff X_1^k X_2^l - \sum_{(i,j)\in L} k_{i,j} X_1^i X_2^j \in I,$$

this shows that $F[X_1, X_2]/I$ is spanned by $\{X_1^i X_2^j | (i,j) \in L \times L\}$. Since we only have finite number of initial conditions, $\dim_k F[X_1, X_2]/I < \infty$, i.e., I is a cofinite ideal of $F[X_1, X_2]$ and this completes our proof.

By the proposition above we can establish the following

Theorem 2.2. *There is a one to one correspondence between the set of linearly recursive sequences of n-index over field F and $F[X_1, X_2, \cdots, X_n]^\circ$.*

Proof. We only consider the case $n = 2$.

First, for each $f \in F[X_1, X_2]^\circ$, one can write f as $f = \sum S_{i,j} X_1^i X_2^j$ since

$$F[X_1, X_2]^\circ \subseteq F[[X_1], [X_2]] = F[X_1, X_2]^*.$$

But

$$F[X_1, X_2] \cong F[X] \otimes F[X] \quad \text{and} \quad (F[X] \otimes F[X])^\circ \cong F[X]^\circ \otimes F[X]^\circ.$$

Thus

$$F[X_1, X_2]^\circ \cong F[X]^\circ \otimes F[X]^\circ$$

and thereby there exist $f_1, f_2 \in F[X]^\circ$ such that $f = f_1 \otimes f_2$. By Proposition 1.1, we can assume that f_1, f_2 correspond to linearly recursive sequences of 1-index $\{s_i\}$ $(i \in Z^+)$ and

$\{t_i\}$ ($i \in Z^+$), respectively. Then

$$\begin{aligned} S_{i,j} &= \langle f, X_1^i X_2^j \rangle \\ &= \langle f_1 \otimes f_2, X^i \otimes X^j \rangle \\ &= \langle f_1, X^i \rangle \langle f_2, X^j \rangle \\ &= s_i t_j. \end{aligned}$$

Thus if we set f to correspond to $\{S_{i_1 i_2}\}$, then

$$\{S_{i_1 i_2}\} = \{S_{i_1} t_{i_2}\} = f_1 \circ f_2.$$

Clearly $\{s_{i_1} t_{i_2}\}$ is a linearly recursive sequence. Suppose that $\{s_i\}$ ($i \in Z^+$) has the characteristic polynomial $h_1(x)$ and initial conditions $s_i = l_i$ ($i \in L_1$), and $\{t_i\}$ ($i \in Z^+$) has the characteristic polynomial $h_2(x)$ and initial conditions $t_i = m_i$ ($i \in L_2$), where L_1, L_2 are finite label sets. Then the characteristic polynomial set of f is $h_1(x)$, $h_2(x)$, and the initial conditions of f are

$$s_{i_1} t_{i_2} = l_{i_1} m_{i_2} \quad (i_1 \in L_1, i_2 \in L_2).$$

Conversely, if $\{S_{i_1 i_2}\}$ ($i_1, i_2 \in Z^+$) is a linearly recursive sequence of 2-index whose set of characteristic polynomials is $f_1, f_2, f_3, \cdots, f_r$, let $I = (f_1, f_2, f_3, \cdots, f_r)$, then by Proposition 2.2, I is a cofinite ideal of $F[X_1, X_2]$. Let

$$f = \sum_{(i,j) \in Z^+} S_{i,j} X_1^i X_2^j.$$

Thus it suffices to show that for each pair $(s,t) \in Z^+ \times Z^+$,

$$\langle f, X_1^s X_2^t \cdot f_i \rangle = 0 \quad (i = 1, 2, \cdots, r).$$

By computation, we have

$$\langle f, X_1^s X_2^t \cdot f_i \rangle = S_{s+i_1, t+j_1} - \sum_{(u,v) < (i_1, j_1)} a_{u,v} S_{s+i_1, t+j_1} = 0.$$

Since this is our recursive relations described in (1.1), and then $\Rightarrow f \in F[X_1, X_2]^\circ$.

From the proof of Theorem 2.2, we obtain

Theorem 2.3. *If $\{S_{i_1 i_2 \cdots i_n}\}$ is a linearly recursive sequence of n-index over F, then there must exsit n linearly recursive sequences of 1-index $f_1, f_2, f_3, \cdots, f_n$ over F such that* $\{S_{i_1 i_2 \cdots i_n}\} = f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n$.

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