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STABILITY OF A PARABOLIC FIXED POINT OF REVERSIBLE MAPPINGS***

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Abstract

KAM theorem of reversible system is used to provide a sufficient condition which guarantees the stability of a parabolic fixed point of reversible mappings. The main idea is to discuss when the parabolic fixed point is surrounded by closed invariant curves and thus exhibits stable behaviour.

Keywords Stability, Reversible mapping, Parabolic fixed point, KAM theorem.
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§1. Introduction

In this paper, we consider a reversible mapping of the plane which has a fixed point and let λ_1,λ_2 be the eigenvalues of its linearization at this point. Since the mapping is real, λ_1 and λ_2 are either both real or complex conjugate to each other. If λ_1 and λ_2 are real and not ±1, the fixed point is called hyperbolic and it cannot be stable. If $\lambda_1 = \bar{\lambda}_2 \in S^1/\{\pm1\}$, the fixed point is called elliptic. In this situation, as well as the symplectic mappings, it is well known that the fixed point is surrounded by closed invariant curves and is stable under some nonresonance conditions [6,7]. Finally, if $\lambda_1 = \lambda_2 = 1$, the fixed point is called parabolic. In this case, Simo[8,9], Aharonov and Elias[1,2] have studied the stability of this point when the mapping is symplectic. In particular, Simo obtained a necessary and sufficient condition for symplectic mappings.

In this note, we will study the stability of a parabolic fixed point for reversible mappings. The main idea in this paper is the same as in the above-mentioned papers. We will discuss when the parabolic fixed point is surrounded by closed invariant curves and exhibits stable behaviour.

The basic tool to establish the existence of closed invariant curves is KAM theorem of reversible mappings[3-7].

Theorem. *Let*

$$A: \theta_1 = \theta + \gamma\rho + f(\theta,\rho), \quad \rho_1 = \rho + g(\theta,\rho),$$

$$G: \theta_1 = -\theta, \quad \rho_1 = \rho$$

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be mappings in the annulus $a \le \rho \le b, \theta \in S^1$, *where* f, g *are normal (that is,* f *and* g *are holomorphic and real analytic for real arguments) and periodic in* θ.

Assume that $AGA = G$ *throughout* D, *where* D *is a complex neighborhood of* $[a,b] \times S^1$ *in* C^2.

For $\epsilon > 0$, *there exists* $\delta > 0$, *depending only on* ϵ, D, *but not on* γ, *such that if*

$$|f|_D + |g|_D < \gamma\delta,$$

then the mappings A *and* G *have a common invariant curve*

$$\theta = \phi + \Phi_\omega^1(\phi), \quad \rho = \gamma^{-1}\omega + \Phi_\omega^2(\phi),$$

where ω *is a Diophantine number and* $|\Phi_\theta^1|, |\Phi_\theta^2| < \epsilon, \Phi_\omega^1, \Phi_\omega^2$ *are normal and periodic functions.*

Remark. The assumption that f and g are normal is not essential. In [6], it is only assumed that $f, g \in C^r$.

Similar to Aharonov and Elias[1], our study is devided into two parts. In section 2, we will study the existence of invariant curves around a finite fixed point. In section 3, we will deal with a parabolic fixed point at infinity.

§2. Invariant Curves Around a Finite Fixed Point

In this section, we consider a normal mapping

$$A : (x, y) \to (x + P(x, y), y + Q(x, y)) \tag{2.1}$$

in a neighborhood of $(0,0)$, where $P(x,y), Q(x,y) = o(r)$ as $r = (x^2 + y^2)^{\frac{1}{2}}$ tends to zero. Assume that A is reversible with respect to $G : (x, y) \to (-x, y)$.

From the equality $AGA = G$, it follows that

$$P \circ GA = P, \quad Q \circ GA = -Q. \tag{2.2}$$

We also assume that P and Q can be written in the form

$$\begin{aligned} P(x,y) &= p(x,y) + \hat{p}(x,y), \\ Q(x,y) &= q(x,y) + \hat{q}(x,y), \end{aligned} \tag{2.3}$$

and there are three positive constants α, β, d with $d > \alpha + \beta$ such that

$$\begin{aligned} p(e^{\alpha s}x, e^{\beta s}y) &= e^{(d-\beta)s}p(x,y), \\ q(e^{\alpha s}x, e^{\beta s}y) &= e^{(d-\alpha)s}q(x,y), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \hat{p}(e^{\alpha s}x, e^{\beta s}y) &= o(e^{(d-\beta)s}), \\ \hat{q}(e^{\alpha s}x, e^{\beta s}y) &= o(e^{(d-\alpha)s}), \end{aligned} \tag{2.5}$$

as $s \to -\infty$.

Remark. When p and q are homogeneous of degree $n \ge 2$, one can take $\alpha = \beta = 1, d = n + 1$ in (2.4).

Now we state our result in this section.

Theorem 2.1. *Consider the reversible mapping* A *defined by* (2.1) *with the conditions* (2.2)-(2.5). *The fixed point* $(0,0)$ *is stable if the following conditions hold:*

$$\alpha x q(x,y) - \beta y p(x,y) \ne 0 \tag{2.6}$$

for $(x, y) \neq (0,0)$,

$$p_x + q_y = 0. \tag{2.7}$$

Proof. From (2.3)-(2.5), we have

$$p(-x, y) = p(x, y), \quad q(-x, y) = -q(x, y). \tag{2.8}$$

Hence the differential equation

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y) \tag{2.9}$$

is reversible with respect to $G : (x, y) \to (-x, y)$.

Let $H(x, y) = \frac{\alpha}{d} xq(x, y) - \frac{\beta}{d} yp(x, y)$. Then the system (2.9) is a Hamiltonian system with H. Indeed, from (2.4), we have

$$\begin{aligned} \alpha x p_x(x, y) + \beta y p_y(x, y) &= (d - \beta) p(x, y), \\ \alpha x q_x(x, y) + \beta y q_y(x, y) &= (d - \alpha) q(x, y), \end{aligned} \tag{2.10}$$

and by (2.7),

$$\begin{aligned} \frac{\partial H}{\partial x} &= \frac{\alpha}{d} q(x, y) + \frac{\alpha}{d} x q_x(x, y) - \frac{\beta}{d} y p_x(x, y) \\ &= \frac{\alpha}{d} q(x, y) + \frac{1}{d} (\alpha x q_x(x, y) + \beta y q_y(x, y)) \\ &= (\frac{\alpha}{d} + \frac{d - \alpha}{d}) q(x, y) = q(x, y), \\ \frac{\partial H}{\partial y} &= -p. \end{aligned}$$

From (2.6), we may assume that

$$H(x, y) > 0 \tag{2.11}$$

for $(x, y) \neq (0,0)$.

Without loss of generality, we assume that the solution of (2.9) with initial condition $(0,1)$ exists and is denoted by $(S(t), C(t))$.

Since $H(e^{\alpha s} x, e^{\beta s} y) = e^{ds} H(x, y)$, and $H(x, y) > 0$, $H(x, y) = c > 0$ are closed curves. Hence $(S(t), C(t))$ is a periodic solution of (2.9) and let $T_0 > 0$ be its minimal period.

Now we give some properties of $(S(t), C(t))$:

(i) $S, \ C \in C^\infty$, $S(t + T_0) = S(t)$, $C(t + T_0) = C(t)$;

(ii) $C(-t) = C(t)$, $S(-t) = -S(t)$;

(iii) $H(S(t), C(t)) = -\frac{\beta}{d} p(0,1) > 0$.

We define a diffeomorphism $\Psi : R^+ \times S^1 \to R^2$ by

$$\Psi : x = \rho^\alpha S(\theta T_0), \quad y = \rho^\beta C(\theta T_0).$$

Then we have $G \circ \Psi(\rho, -\theta) = \Psi(\rho, \theta)$. Hence $\Psi^{-1} \circ A \circ \Psi$ is also reversible with respect to the involution $J : (\rho, \theta) \to (\rho, -\theta)$.

Let $(\rho_1, \theta_1) = \Psi^{-1} \circ A \circ \Psi(\rho, \theta)$ and $\Delta\rho = \rho_1 - \rho$, $\ \Delta\theta = \theta_1 - \theta$. From the definition of

Ψ, we have

$$\begin{aligned}\Delta\rho =& -\frac{1}{T_0\beta p(0,1)}\rho^{1-\alpha-\beta}[T_0\rho^\beta q(S(\theta T_0),C(\theta T_0))\hat{p}(x,y)\\ &- T_0\rho^\alpha p(S(\theta T_0),C(\theta T_0))\hat{q}(x,y)] + o(\rho^{d+1-\alpha-\beta})\\ =&o(\rho^{d+1-\alpha-\beta}),\\ \Delta\theta =& -\frac{1}{T_0\beta p(0,1)}\rho^{d-\alpha-\beta}[\alpha\rho^{\alpha-1}S(\theta T_0)(q(x,y)+\hat{q}(x,y))\\ &- \beta\rho^{\beta-1}C(\theta T_0)(p(x,y)+\hat{p}(x,y))] + o(\rho^{d-\alpha-\beta})\\ =&\frac{1}{T_0}\rho^{d-\alpha-\beta} + o(\rho^{d-\alpha-\beta}).\end{aligned}$$

Let us define $\Phi : R^+ \times S^1 \to R^+ \times S^1$ by $\theta = \theta$, $\rho = (T_0\mu)^{\frac{1}{d-\alpha-\beta}}$, and $(\mu_1,\theta_1) = \Phi^{-1}\circ\Psi^{-1}\circ A\circ\Psi\circ\Phi(\mu,\theta)$, $\Delta\mu = \mu_1 - \mu$.

One can easily see that $\Delta\mu = o(\mu^2)$ for $\mu > 0$ is sufficiently small. Hence we have

$$\mu_1 = \mu + f(\mu,\theta), \quad \theta_1 = \theta + \mu + g(\mu,\theta), \tag{2.12}$$

where $f(\mu,\theta) = o(\mu^2)$, $g(\mu,\theta) = o(\mu)$ are 1-periodic in θ. Moreover, the mapping (2.12) is reversible with respect to the involution $J : (\mu,\theta) \to (\mu,-\theta)$.

From the standard arguments, we can prove that there is a closed invariant curve of A in every small neighborhood of the origin and the fixed point is stable. The proof of Theorem 2.1 is completed.

§3. Invariant Curves Around a Fixed Point at Infinity

This section deals with the stability of a parabolic fixed point at infinity for reversible diffeomorphism.

Consider the mapping

$$A : (x,y) \to (x + P(x,y), y + Q(x,y)), \tag{3.1}$$

where $P(x,y), Q(x,y) \to 0$ as $x^2+y^2 \to \infty$, that is, when infinity is a parabolic fixed point. Like the previous section, we assume that A is reversible with respect to $G : (x,y) \to (-x,y)$ and $P(x,y), Q(x,y)$ can be written in the following form:

$$\begin{aligned}P(x,y) &= p(x,y) + \hat{p}(x,y),\\ Q(x,y) &= q(x,y) + \hat{q}(x,y),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}p(e^{\alpha s}x, e^{\beta s}y) &= e^{(d-\beta)s}p(x,y),\\ q(e^{\alpha s}x, e^{\beta s}y) &= e^{(d-\alpha)S}q(x,y),\end{aligned} \tag{3.3}$$

for some $\alpha,\beta > 0, d \le 0$. Moreover,

$$\begin{aligned}\hat{p}(e^{\alpha s}x, e^{\beta s}y) &= o(e^{(d-\beta)s}),\\ \hat{q}(e^{\alpha s}x, e^{\beta s}y) &= o(e^{(d-\alpha)s}),\end{aligned} \tag{3.4}$$

as $s \to \infty$.

Theorem 3.1. *Let the normal mapping* (3.1) *be given in a neighborhood of infinity and* $d < 0$. *Suppose that the conditions* (3.2)-(3.4) *are satisfied and* $AGA = G$ *for* $G : (x,y) \to$

$(-x, y)$. *The fixed point at infinity is surrounded by closed invariant curves provided the following assumptions hold:*

(i) $$p_x(x,y) + q_y(x,y) = 0, \tag{3.5}$$

(ii) $$\alpha x q(x,y) - \beta y p(x,y) \neq 0 \tag{3.6}$$

near infinity.

Proof. Like Theorem 2.1 of the previous section, one can prove that the system

$$\dot{x} = p(x,y), \quad \dot{y} = q(x,y), \tag{3.7}$$

is reversible with respect to $G : (x,y) \to (-x,y)$.

Let $H(x,y) = \frac{\alpha}{d} x q(x,y) - \frac{\beta}{d} y p(x,y)$. From (3.6), without loss of generality, we can assume $H(x,y) > 0$.

Suppose that $(S(t), C(t))$ is the solution of (3.7) with the initial value $(0,1)$. Since $H(e^{\alpha s}x, e^{\beta s}y) = e^{ds}H(x,y), (S(t), C(t))$ is a periodic solution of (3.7). Let $T_0 > 0$ be its minimal period.

Now we define a diffeomorphism $\Phi : R^+ \times S^1 \to R^2$ by

$$\Phi : x = \rho^{\frac{\alpha}{\alpha+\beta}} S(\theta T_0), \quad y = \rho^{\frac{\beta}{\alpha+\beta}} C(\theta T_0).$$

Like the proof of Theorem 2.1, we have

$$\Delta\rho = o(\rho^{\frac{d}{\alpha+\beta}}), \quad \Delta\theta = \rho^{\frac{d}{\alpha+\beta}-1} + o(\rho^{\frac{d}{\alpha+\beta}-1}),$$

as $\rho \to +\infty$.

Let $\mu = \frac{1}{T_0}\rho^{\frac{d}{\alpha+\beta}-1}$, $\theta = \theta$. Note that $\mu \to 0$ as $\rho \to +\infty$. Hence we have

$$\Delta\mu = o(\mu^2), \quad \Delta\theta = \mu + o(\mu).$$

The following proof is the same as that of Theorem 2.1. We omit it here.

Now, we study the case $d = 0$. In this case, under the conditions (3.3)-(3.6), we cannot confirm that the solutions of (3.7) are periodic.

Theorem 3.2. *Given the normal mapping* (3.1) *in a neighborhood of infinity, under the conditions* (3.3)-(3.6) *and* $d = 0$, *the fixed point at infinity is surrounded by closed invariant curves provided the system* (3.7) *has a periodic solution* $(S(t), C(t))$ *with initial value* $(S(0), C(0)) = (0,1)$.

Proof. We introduce a diffeomorphism $\Phi : R^+ \times S^1 \to R^2$ by

$$\Phi : x = \rho^{-\frac{\alpha}{\alpha+\beta}} S(\theta T_0), \quad y = \rho^{-\frac{\beta}{\alpha+\beta}} C(\theta T_0). \tag{3.8}$$

Clearly, $\rho \to 0^+$ as $x^2 + y^2 \to \infty$.

Under this diffeomorphism, the mapping (3.1) is transformed into the form:

$$\rho_1 = \rho + f(\rho,\theta), \quad \theta_1 = \theta + \frac{1}{T_0}\rho + g(\rho,\theta),$$

where $f(\rho,\theta) = o(\rho^2), g(\rho,\theta) = o(\rho)$ as $\rho \to 0^+$.

The rest of the proof is the same as that of Theorem 2.1. We omit it here.

We give now a sufficient condition for the existence of periodic solutions of the system (3.7). Since the mapping (3.1) is normal, from (3.3), we can take two positive integers m, n such that $\frac{\beta}{\alpha} = \frac{m}{n}$. Consider the following system

$$\dot{x} = n y^{2n-1}, \quad \dot{y} = -m x^{2m-1}. \tag{3.9}$$

It is a Hamiltonian system and a reversible system with respect to $G : (x, y) \to (-x, y)$. Suppose that $(u(t), v(t))$ is the solution of (3.9) with the initial value $(0, 1)$ and $T > 0$ is its minimal period.

From (3.9), we have $u^{2m}(t) + v^{2n}(t) = 1$.

Proposition. *Under the conditions* (3.3) *and* (3.6) *with* $d = 0, \frac{\beta}{\alpha} = \frac{m}{n}$, *the solutions of the* (3.7) *are periodic if the following assumption holds:*

$$\int_0^1 [mu^{2m-1}(\theta T)p(u(\theta T), v(\theta T)) + nv^{2n-1}(\theta T)q(u(\theta T), v(\theta T))]d\theta = 0.$$

Proof. Define diffeomorphism $\Phi : R^+ \times S^1 \to R^2$ by $\Phi : x = \rho^n u(\theta T), \quad y = \rho^m v(\theta T)$. Under this diffeomorphism Φ, the system (3.7) is transformed into the form:

$$\dot{\rho} = \frac{mu^{2m-1}(\theta T)p(u(\theta T), v(\theta T)) + nv^{2n-1}(\theta T)q(u(\theta T), v(\theta T))}{mn\rho^{m+n-1}},$$

$$\dot{\theta} = \frac{\lambda}{Tmn\rho^{m+n}},$$

where

$$\lambda = nu(\theta T)g(u(\theta T), v(\theta T)) - mv(\theta T)p(u(\theta T), v(\theta T)) \equiv \text{constant} \neq 0.$$

Hence $\rho(t) = \rho(0)e^{\frac{T}{\lambda}\int_0^\theta W(\theta)d\theta}$, where

$$W(\theta) = mu^{2m-1}(\theta T)p(u(\theta T), v(\theta T)) + nv^{2n-1}(\theta T)q(u(\theta T), v(\theta T)).$$

Clearly, if $\int_0^1 W(\theta)d\theta = 0$, the solutions of (3.7) are periodic. This proves the Proposition.

Remark. Using the method in this paper, we can also prove the Lyapunov stability of the fixed point (0,0) of the following area-preserving mapping $\Phi : R^2 \to R^2$

$$\Phi : x_1 = x + y^{2n}, \quad y_1 = y - (x + y^{2n})^{2m},$$

where $m, n \in N$ and $m, n \geq 2$.

The stability of the equilibrium point (0,0) of the following reversible equation

$$\dot{x} = 4y^3 + 2x^6 y + x^4 y^2 p(t), \quad \dot{y} = -6x^5 y^2 + 12x^{11} + x^3 y^3 q(t)$$

can be obtained, where $p(t), q(t)$ are periodic in t with period 1, and $p(-t) = p(t), \quad q(-t) = -q(t)$.

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