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WHEN CAN THE STABLE ALGEBRA DETERMINE THE STRUCTURE OF A C^*-ALGEBRA ?**

WU LIANGSEN*

Abstract

Let A and B be C^*-algebras. Suppose that $\mathcal{K}$ is the algebra of all compact operators on a seperable Hilbert space, and α is an action on the stable algebra $\mathcal{K}\otimes A$ induced by $SU(\infty)$. It is proved that if A is α-invariant stable isomorphic to B, then there is a *-isomorphism between A and B. An analogous result is obtained by considering $O_n\otimes\mathcal{K}\otimes A$ in the place of $\mathcal{K}\otimes A$, where O_n is the Cuntz algebra $(3\le n<\infty)$.

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§1. Introduction

For any unital C^*-algebra A, we know that the elements of $K_0(A)$ are the formal difference of equivalent classes of projections in the matrix algebra over A. It is an interesting problem whether $K_0(A)$ can determine the structure of A. In particular case Elliot proved that if A and B are AF-algebras and $\sigma: K_0(A)\to K_0(B)$ is an isomorphism of scaled order groups, then there is an isomorphism $\phi: A\to B$ (see [6]). In general case, we can not expect so nice result, but we can raise a problem: when can the stable algebra $\mathcal{K}\otimes A$ determine the structure of C^*-algebra A?

In [1], we proved that if ψ is an α-invariant affine isomorphism between $M_n(A)_+^1$ and $M_n(B)_+^1$ $(n\ge 3)$, $\psi(0)=0$ (α is a natural action induced by $SU(n)$), then A is *-isomorphic to B.

Since $\mathcal{K}\otimes A=\overline{\cup_n M_n(A)}$, we can expect that an α-invariant affine isomorphism between $(\mathcal{K}\otimes A)_+^1$ and $(\mathcal{K}\otimes B)_+^1$ will give rise to a *-isomorphism between A and B (α is the natural action induced by $SU(\infty)$).

In section 2, we show that under α-invariant condition the order of unit ball of the stable algebra $\mathcal{K}\otimes A$ can determine the isomorphic class of the C^*-algebra A. As a consequence of the theorem, we prove that α-invariant stable isomorphism implies *-isomorphism.

In section 3, replacing $B(H)\otimes A$, where A is a von Neumann algebra, instead of $\mathcal{K}\otimes A$, we obtain analogous result.

In section 4, we consider when $O_n\otimes\mathcal{K}\otimes A$, where O_n is Cuntz algebra, can determine the *-isomorphic class of A.

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*Institute of Mathematics, East China Normal University, Shanghai 200062, China.
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§2. Stable Isomorphism and *-isomorphism

In this section,the main result is that α-invariant stable isomorphism implies *-isomorphism. First of all we give some notations.

Let $\mathcal{K}$ be the set of all compact operators on a seperable Hilbert space. $\mathcal{K}$ can be viewed as the closure of the union of $M_n(C)$, that is, $\mathcal{K} = \overline{\cup_n(M_n(C))}$, with embeddings from M_n to M_{n+1} by $a \to \text{diag}(a, 0)$.

$$\mathcal{K}_\infty(A) = \bigcup_{n=1}^{\infty}(M_n \otimes A).$$

$\mathcal{K} \otimes A$ is the closure of $\mathcal{K}_\infty(A)$.

$SU(\infty)$ is infinite unimodular unitary group defined by

$$SU(\infty) = \bigcup_{n=1}^{\infty} SU(n)$$

with embeddings from $SU(n)$ to $SU(n+1)$ by $a \to \text{diag}(a, 1)$.

Using $SU(\infty)$, we can define an action α on $\mathcal{K}_\infty(A)$ by

$$\alpha_u(x) = uxu^*, \ x \in \mathcal{K}_\infty(A), \ u \in SU(\infty),$$

which can be extended to an action on $\mathcal{K} \otimes A$. If needed, we can join an identity to $\mathcal{K} \otimes A$, on which α_u acts as an automorphism.

Theorem 2.1. *Let A and B be C^*-algebras. If ψ is an α-invariant affine isomorphism between $(\mathcal{K} \otimes A)^1_+$ and $(\mathcal{K} \otimes B)^1_+$, $\psi(0) = 0$, then A is *-isomorphic to B (α-invariance means that $\alpha\psi = \psi\alpha$).*

Proof. Using the affiness of ψ and $\psi(0) = 0$, we can extend ψ to an isometry between $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$.

Fixing an integer $n \geq 3$, there is an isomorphism κ from $\mathcal{K}$ to $M_n \otimes \mathcal{K}$. We have a diagram as follows:

$$\begin{array}{ccc} \mathcal{K} \otimes A & \xrightarrow{\psi} & \mathcal{K} \otimes B \\ \downarrow{\scriptstyle \kappa\otimes I} & & \downarrow{\scriptstyle \kappa\otimes I} \\ M_n \otimes \mathcal{K} \otimes A & \xrightarrow{\psi'} & M_n \otimes \mathcal{K} \otimes B \end{array}$$

in which

$$\psi' = (\kappa \otimes I) \circ \psi \circ (\kappa^{-1} \otimes I).$$

Since $SU(n) \otimes SU(\infty) \subset SU(\infty)$, ψ' is an α'-invariant map, where α' is a natural action induced by $SU(n) \otimes SU(\infty)$, that is,

$$\alpha_{u_1 \otimes u_2} : x \mapsto (u_1 \otimes u_2)(x)(u_1^* \otimes u_2^*),$$

$u_1 \in SU(n), u_2 \in SU(\infty), x \in M_n \otimes \mathcal{K} \otimes A$.

Next thing we should prove is $\psi'(M_n \otimes A) \subseteq M_n \otimes B$. We set $\alpha' = \alpha'_1 \otimes \alpha'_2$ in which α'_1 is an action induced by $SU(n)$, α'_2 is the one induced by $SU(\infty)$. α'_2-invariance implies that

$$(I_n \otimes u)\psi'(x)(I_n \otimes u^*) = \psi'(x).$$

$u \in SU(\infty), x \in M_n \otimes A$ (I_n is the identity in $M_n(\mathbb{C})$) such that $\psi'(x) \in M_n \otimes B$ as the identity representation of $\mathcal{K}$ is irreducible.

It follows that $\psi'(M_n \otimes A) \subseteq M_n \otimes B$ and ψ' is α'_1-invariant. By using the argument in [1], A is *-isomorphic to B.

Definition 2.1. *Suppose that A and B are C^*-algebras. If $\mathcal{K} \otimes A \cong \mathcal{K} \otimes B$, then A is called stable isomorphic to B.*

Theorem 2.2. *Assume that A and B are C^*-algebras. If A is α-invariant stable isomorphic to B, then there is a *-isomorphism between A and B.*

Proof. If ψ is an α-invariant *-isomorphism between $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$, then $\psi(0) = 0$ and ψ is an isometry between $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$. By Theorem 2.1, A is *-isomorphic to B.

§3. von Neumann Algebra Case

In this section, we discuss what will happen if A is a von Neumann algebra. $B(H) \otimes A$ is the tensor product of $B(H)$ and A.

In the same way we can define a natural action α on $B(H) \otimes A$ induced by $SU(\infty)$.

Theorem 3.1. *Let A and B be von Neumann algebras. ψ is an α-invariant affine isomorphism between $(B(H) \otimes A)^1_+$ and $B(H) \otimes B)^1_+$ with $\psi(0) = 0$. Then A is *-isomorphic to B.*

Proof. ψ can be viewed as an isometry from $B(H) \otimes A$ to $B(H) \otimes B$. Since $B(H)$ is a factor and for some integer $n \geq 3$, $M_n \otimes B(H) \cong B(H)$, like Theorem 2.1, we can prove that there is a *-isomorphism between A and B.

§4. Cuntz Algebra O_n Case

We have already studied when the tensor products $\mathcal{K} \otimes A$ and $M_{(n^\infty)} \otimes A$ [4] can determine the structure of C^*-algebra A, where $\mathcal{K}$ is a compact operator algebra on a separable Hilbert space and $M_{(n^\infty)}$ is a UHF algebra of type n^∞. It is natural to ask a question whether or not we can discuss $O_n \otimes \mathcal{K}$ instead of $\mathcal{K}$ and $M_{(n^\infty)}$, where O_n is a Cuntz algebra ($n < +\infty$), and obtain similar results.

We will review some notations and properties of Cuntz algebra O_n.

Let O_n be the C^*-algebra generated by n isometries $s_1, s_2, \ldots, s_n$ with $s_i^* s_i = 1$, $s_i s_i^* = p_i$, $p_1 + p_2 + \ldots + p_n = 1$. O_n were first studied by Cuntz[7].

Fix n with $1 < n < \infty$. Let $M_{(n^\infty)}$ be the UHF algebra with dimension group $\mathbb{Z}_{(n^\infty)}$. There is an automorphism Φ of $M_{(n^\infty)} \otimes \mathcal{K}$ with $(M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z} \cong O_n \otimes \mathcal{K}$, where Φ is an automorphism of $M_{(n^\infty)} \otimes \mathcal{K}$ induced by the shift to the left, such that $\Phi(X) = UXU^*$ ($X \in M_{(n^\infty)} \otimes \mathcal{K}$).

Set $SU(M_{(n^\infty)}) = \bigcup_k \phi_k(SU(n^k))$, where ϕ_k is the embedding from M_{n^k} to $M_{(n^\infty)}$ (see [4]).

By use of $SU(M_{(n^\infty)}) \otimes SU(\infty)$ we can define a natural action on $M_{(n^\infty)} \otimes \mathcal{K}$. Then the group G generated by $SU(M_{(n^\infty)}) \otimes SU(\infty)$ and U can induce an action on $O_n \otimes \mathcal{K}$, which is denoted by α.

Theorem 4.1. *Let A and B be C^*-algebras. If ψ is an α-invariant affine isomorphism between $(O_n \otimes \mathcal{K} \otimes A)^1_+$ and $(O_n \otimes \mathcal{K} \otimes B)^1_+$ with $\psi(0) = 0$, then $A \cong B$ ($3 \leq n < \infty$).*

Proof. Suppose that π is an irreducible representation of O_n so that π is faithful since O_n is simple. Since the identity representation of $\mathcal{K}$ is irreducible, $\pi \otimes I$ is an irreducible representation of $O_n \otimes \mathcal{K}$.

By [6, p.104], $M_n(O_n) \cong O_n$, so that ψ can be viewed as an α-invariant affine isomorphism ψ' from $(O_n \otimes \mathcal{K} \otimes M_n \otimes A)_+^1$ to $(O_n \otimes \mathcal{K} \otimes M_n \otimes B)_+^1$. Extend ψ' as an α-invariant linear isometry from $O_n \otimes \mathcal{K} \otimes M_n \otimes A$ to $O_n \otimes \mathcal{K} \otimes M_n \otimes B$ with $\psi(0) = 0$. We have

$$\begin{array}{ccc} O_n \otimes \mathcal{K} \otimes A & \xrightarrow{\psi} & O_n \otimes \mathcal{K} \otimes B \\ \downarrow{\scriptstyle \sigma \otimes I_1} & & \downarrow{\scriptstyle \sigma \otimes I_1} \\ O_n \otimes \mathcal{K} \otimes M_n \otimes A & \xrightarrow{\psi'} & O_n \otimes \mathcal{K} \otimes M_n \otimes A \\ \downarrow{\scriptstyle \tau \otimes I_2} & & \downarrow{\scriptstyle \tau \otimes I_2} \\ (M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z} \otimes M_n \otimes A & \xrightarrow{\psi'} & (M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z} \otimes M_n \otimes B \end{array}$$

where σ is an isomorphism from O_n to $M_n(O_n)$, τ is an isomorphism from $O_n \otimes \mathcal{K}$ to $(M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z}$, I_1 and I_2 are identity maps.

Since $G \otimes SU(n) \subset G$ (for $M_n(O_n) \cong O_n$), suppose that α' is a natural action on $M_{(n^\infty)} \otimes \mathcal{K} \times_\Phi \mathbb{Z} \otimes M_n \otimes A$ induced by $G \otimes SU(n)$, α_1' is an action on $M_{(n^\infty)} \otimes \mathcal{K} \times_\Phi \mathbb{Z}$ induced by G, and α_2' is another action on $M_n \otimes A$ induced by $SU(n)$, such that $\alpha' = \alpha_1' \otimes \alpha_2'$ and ψ' is an α'-invariant affine isometry from $(M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z} \otimes M_n \otimes A$ to

$$(M_{(n^\infty)} \otimes \mathcal{K}) \times_\Phi \mathbb{Z} \otimes M_n \otimes B.$$

α_1'-invariance means that

$$\psi'(UxU^*) = U\psi'(x)U^*, \quad x \in M_n \otimes A, \quad U \in G,$$

that is , $\psi'(x)U = U\psi'(x)$, $\forall x \in M_n \otimes A$, $U \in G$.

Without loss of generality, we can view $O_n \otimes \mathcal{K}$ and $\pi(O_n) \otimes \mathcal{K}$ as the same. Since

$$(\pi(O_n) \otimes \mathcal{K}) \cap (\pi(O_n) \otimes \mathcal{K})' = \mathbb{C}I,$$

$\psi'(x)U = U\psi'(x)$ implies that $\psi'(x) \in M_n \otimes B$. Now we have $\psi'(M_n \otimes A) \subset M_n \otimes B$, and ψ' is an α_2'-invariant affine isomorphism from $M_n \otimes A$ to $M_n \otimes B$, $\psi'(0) = 0$. From [1], $A \cong B$.

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