Chin. Ann. of Math.
15B: 2(1994),157-164.

A NOTE ON THE RELATIVE CANONICAL IMAGE OF A NON-HYPERELLIPTIC FIBRATION OF GENUS 4**

CHEN ZHIJIE*

Abstract

This paper investigates the relative 1-canonical images of non-hyperelliptic fibrations of genus 4. It is proved that if a fibre of the relative 1-canonical image Σ is not a complete intersection in $\mathbb{P}^3$, then the variety Σ cannot be smooth on this fibre. Moreover, two examples are given to show the occurrence of such cases.

Keywords Genus, Non-hyperelliptic fibration, Canonical image, Algebraic variety.
1991 MR Subject Classification 14D, 14H.

The relative canonical maps and relative canonical images have been successfully used by Horikawa[2] and Xiao[5] in studying the fibrations of genus 2. As the canonical curve plays an important role in the investigation of non-hyperelliptic curves, it seems natural that the relative 1-canonical images might be useful in studying the non-hyperelliptic fibrations. The genus 3 fibrations have been studied by Lopes[3]. In this paper we will investigate relative 1-canonical images of non-hyperelliptic fibrations of genus 4. We will prove that if a fibre of the relative 1-canonical image Σ is not a complete intersection in $\mathbb{P}^3$, then the variety Σ cannot be smooth on this fibre (Theorem 1). Finally two examples are given to show the occurrence of such cases.

The base field is the complex number field $\mathbb{C}$. A fibration $f: S \longrightarrow C$ is a surjective morphism with connected fibres, where S is a smooth projective surface, C is a smooth projective curve. The genus g of a general fibre of f is called the genus of f. We always assume that f is a relative minimal fibration, i.e., none of its fibres contains (-1)-curves. The invertible sheaf $\omega_{S/C} = \omega_S \otimes f^*\omega_C^\vee$ on S is called the dualising sheaf of f. Let $\mathcal{L}$ be a sufficiently ample sheaf on C. The natural morphism $f^*\mathcal{E} = f^*f_*\omega_{S/C} \otimes f^*\mathcal{L} \longrightarrow \omega_{S/C} \otimes f^*\mathcal{L}$ induces a rational map Φ:

$$\begin{array}{ccc} S & \overset{\Phi}{\dashrightarrow} & P = \mathbf{P}(\mathcal{E}) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & C & \end{array}$$

The rational map Φ is called a relative canonical map, the closed subvariety $\Sigma = \overline{\Phi(S)}$ is called a relative 1-canonical image.

Manuscript received February 18, 1992. Revised May 2, 1992.
*Department of Mathematics, East China Normal University, Shanghai 200062, China.
**Project supported by the National Natural Science Foundation of China.

A fibration f is non-hyperelliptic if its general fibre is a non-hyperelliptic curve with $g \geq 3$.

From now on, we assume $g = 4$.

Proposition 1. *Let $f : S \longrightarrow C$ be a non-hyperelliptic fibration of genus 4. When the invertible sheaf $\mathcal{L}$ on C is sufficiently ample, then for any point $p \in C$ there exists a relative quadratic hypersurface Q and a relative cubic hypersurface V in the projective space bundle $P = \mathbf{P}(f_*\omega_{S/C} \otimes \mathcal{L})$ such that the relative 1-canonical image $\Sigma \subseteq Q \cap V$ and $Q_p \not\subset V_p$.*

Proof. We set $\mathcal{E} = f_*\omega_{S/C} \otimes \mathcal{L}$. Let $\mathcal{I}$ be the ideal sheaf of Σ. Consider the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{O}_P(2)$ and taking direct image, we obtain

$$0 \longrightarrow \pi_*\mathcal{I}(2) \longrightarrow \pi_*\mathcal{O}_P(2) \longrightarrow \pi_*\mathcal{O}_\Sigma(2).$$

Denote the invertible sheaf $\pi_*\mathcal{I}(2)$ by $\mathcal{M}$. Then we have an inclusion

$$0 \longrightarrow \mathcal{O}_P(-2) \otimes \pi^*\mathcal{M} \longrightarrow \mathcal{I}. \tag{1}$$

Since $\mathcal{I} \subset \mathcal{O}_P$, we have

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P(2) \otimes \pi^*\mathcal{M}^{-1}.$$

Denote the image of $1 \in H^0(\mathcal{O}_P)$ by $q \in H^0(\mathcal{O}_P(2) \otimes \pi^*\mathcal{M}^{-1})$ and let $Q = \operatorname{Div} q$. Then Q is a relative quadratic hypersurface in P and $\mathcal{O}_P(-Q) = \mathcal{O}_P(-2) \otimes \pi^*\mathcal{M}$. Hence $\Sigma \subset Q$.

On the other hand, tensoring the sequence (1) by $\mathcal{O}_P(3)$, we obtain another inclusion

$$0 \longrightarrow \mathcal{O}_P(1) \otimes \pi^*\mathcal{M} \longrightarrow \mathcal{I}(3).$$

So the sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{M} \longrightarrow \pi_*\mathcal{I}(3)$$

is exact as well. Denote the quotient sheaf by $\mathcal{Q}$, i.e.,

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{M} \xrightarrow{\alpha} \pi_*\mathcal{I}(3) \xrightarrow{\beta} \mathcal{Q} \longrightarrow 0.$$

Since the invertible sheaf $\mathcal{L}$ is sufficiently ample, we may assume that $H^1(\mathcal{E} \otimes \mathcal{M}) = 0$, $H^0(\mathcal{Q}) \neq 0$, and $\mathcal{Q}$ is generated by global sections. So we can find a global section $s \in H^0(\mathcal{Q})$ such that $s_p \neq 0$. Let $\beta^*(s) \in H^0(\pi_*\mathcal{I}(3))$ be the inverse image of s. Then $\beta^*(s) \notin \alpha(H^0(\mathcal{E} \otimes \mathcal{M}))$ and $\beta^*(s)$ defines an inclusion

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \pi_*\mathcal{I}(3).$$

This morphism induces an inclusion

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P(3).$$

Let $t \in H^0(\mathcal{O}_P(3))$ be the image of $1 \in H^0(\mathcal{O}_P)$ and let $V = \operatorname{Div} t$ which is a relative cubic hypersurface. Then we have $\Sigma \subset V$ and $Q_p \not\subset V_p$ because $\beta^*(s)_p \notin \alpha_p((\mathcal{E} \otimes \mathcal{M})_p)$.

In fact, for a fixed $\mathcal{L}$, the relative quadric Q is unique, but the relative cubic V need not be unique just like in the case of canonical curves of genus 4.

Corollary 1. *If Q_p is irreducible, then Σ_p is a complete intersection in $P_p \cong \mathbb{P}^3$.*

Theorem 1. *If the fibre Σ_p of the relative 1-canonical image is not a complete intersection in $P_p \cong \mathbb{P}^3$, then the variety Σ cannot be smooth on this fibre.*

Proof. Let $(\Delta, u) \subset C$ be an analytic neighborhood centered at p. Denote the homogeneous coordinates of the projective space $\mathbb{P}^3_\Delta \cong \pi^{-1}(\Delta)$ by $x = (x_1 : x_2 : x_3 : x_4)$. Then $q|_{\mathbb{P}^3_\Delta} = q(x,u)$, $t|_{\mathbb{P}^3_\Delta} = t(x,u)$, and the greatest common divisor $(q(x,0), t(x,0))$ is a homogeneous linear polynomial. There are two possibilities for the linear factors of $q(x,0)$.

Case I. The quadratic polynomial $q(x,0)$ has two different linear factors. By coordinate transformation, we can assume

$$q(x,u) = x_1x_2 + \sum_{i\geq 1} a_i(x)u^i,$$

$$t(x,u) = x_1b_0(x) + \sum_{i\geq 1} b_i(x)u^i,$$

where x_2 and $b_0(x)$ are coprime. All the homogeneous polynomials $a_i(x)$ (resp. $b_i(x)$) are of degree 2 (resp. 3). Let I_p denote the homogeneous ideal of Σ_p in $\mathbb{P}^3$. Then x_1x_2, $x_1b_0(x) \in I_p$. If there are homogeneous polynomials (with respect to x)

$$\alpha(x,u) = \sum_{k=0}^{m-1} \alpha_k(x)u^k,$$

$$\beta(x,u) = \sum_{k=0}^{m-1} \beta_k(x)u^k,$$

such that

$$u^m p(x,u) = q(x,u)\alpha(x,u) - t(x,u)\beta(x,u),$$

and $p(x,0) \neq 0$, then we have $p(x,0) \in I_p$. It is not difficult to see that in this case there are polynomials $\gamma_k(x)$ $(k = 0, \ldots, m-1)$ such that for $k = 0, \cdots, m-1$,

$$x_1^k\alpha_k(x) = \sum_{i=0}^{k-1} x_1^{k-i-1} b_{k-i}\gamma_i(x) + b_0\gamma_k(x),$$

$$x_1^k\beta_k(x) = \sum_{i=0}^{k-1} x_1^{k-i-1} a_{k-i}\gamma_i(x) + x_2\gamma_k(x),$$

$$p(x,0) = \sum_{i=0}^{m-1} \big(b_0(x)a_{m-i}(x) - x_2b_{m-i}(x)\big)\gamma_i(x)/x_1^i.$$

Hence any element in I_p can be generated by x_1, x_2 and $b_0(x)$. This implies that any point $(x_1 : x_2 : x_3 : x_4, u) \in \mathbb{P}^3_\Delta$ satisfying $x_1 = x_2 = u = b_0(x) = 0$ must be in Σ. Moreover, since Σ_p must be connected, we can always find a $p(x,u)$ such that Σ is defined by $q(x,u)$, $t(x,u)$ and $p(x,u)$.

Now let us suppose

$$q(x,u) = x_1\big(x_2 + u^{n_1}f_1(x,u)\big) + u^{m_1}f_2(x_2,x_3,x_4,u), \tag{2}$$

$$t(x,u) = x_1\big(g(x_1,x_3,x_4) + u^{n_2}h_1(x,u)\big) + u^{m_2}h_2(x_2,x_3,x_4,u), \tag{3}$$

where m_1, $m_2 > 0$ and either $f_1(x,u) = 0$ (resp. $h_1(x,u) = 0$) or $n_1 > 0$, $f_1(x,0) \neq 0$ (resp. $n_2 > 0$, $h_1(x,0) \neq 0$).

We distinguish four cases.

(i) $m_1, m_2 > 1$. Then the jacobian matrix

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=x_2=u=g(x)=0} = (0).$$

Thus Σ is singular on these points.

(ii) $m_2 = 1$ and $m_1 > 1$. If $\deg\big(g(0,x_3,x_4), h_2(0,x_3,x_4,0)\big) > 0$, then

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=x_2=u=g(x)=h_2(x,0)=0} = (0).$$

In this case, the third equation is

$$\begin{aligned}p(x,u) =& x_2h_2(x,u) + u^{n_1}h_2(x,u)f_1(x,u) - u^{m_1-1}f_2(x,u)g(x)\\ &- u^{m_1+n_2-1}f_2(x,u)h_1(x,u).\end{aligned}$$

If $m_1 > 2$, then

$$\left.\frac{\partial(q,p)}{\partial(x,u)}\right|_{x_1=x_2=u=h_2(x,0)=0} = (0).$$

If $m_1 = 2$ and $\deg\big(f_2(0,x_3,x_4,0), h_2(0,x_3,x_4,0)\big) > 0$, then

$$\left.\frac{\partial(q,p)}{\partial(x,u)}\right|_{x_1=x_2=u=f_2(x,0)=h_2(x,0)=0} = (0).$$

(iii) $m_1 = 1$ and $m_2 \geq 1$. If $\deg\big(g(0,x_3,x_4), f_2(0,x_3,x_4,0)\big) > 0$, then

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=x_2=u=g(x)=f_2(x,0)=0} = (0).$$

Now assume $\big(g(0,x_3,x_4), f_2(0,x_3,x_4,0)\big) = 1$, hence $g(0,x_3,x_4) \neq 0$. In this case, the third equation is

$$\begin{aligned}p(x,u) =& g(x)f_2(x,u) - u^{m_2-1}x_2h_2(x,u) - u^{n_2}f_2(x,u)h_1(x,u)\\ &- u^{m_2+n_1-1}h_2(x,u)f_1(x,u).\end{aligned} \tag{4}$$

If $f_2(x,0)$ has a singularity at any point satisfying $x_1 = x_2 = u = f_2(x,0) = 0$, then

$$\left.\frac{\partial(q,p)}{\partial(x,u)}\right|_{x_1=x_2=u=f_2(x,0)=0} = (0).$$

If $g(x)$ has a singularity at any point satisfying $x_1 = x_2 = u = g(x) = 0$, then

$$\left.\frac{\partial(t,p)}{\partial(x,u)}\right|_{x_1=x_2=u=g(x)=0} = (0).$$

(iv) $m_1 = m_2 = 1$. If

$$g(0,x_3,x_4)f_2(x_2,x_3,x_4,0) - x_2h_2(x_2,x_3,x_4,0) = 0,$$

then we have $x_2 \mid f_2(x_2,x_3,x_4,0)$ and $g(0,x_3,x_4) \mid h_2(x_2,x_3,x_4,0)$. Hence

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=x_2=u=g(x)=0} = (0).$$

Now assume

$$g(0,x_3,x_4)f_2(x_2,x_3,x_4,0) - x_2h_2(x_2,x_3,x_4,0) \neq 0.$$

Then the third equation is

$$p(x,u) = g(x)f_2(x,u) - x_2h_2(x,u) + u^{n_2}f_2(x,u)h_1(x,u) - u^{n_1}f_1(x,u)h_2(x,u). \tag{5}$$

If $g(x)f_2(x,0) - x_2h_2(x,0)$ has a singularity at any point satisfying $x_1 = x_2 = u = g(x) = 0$, then

$$\text{rank } \frac{\partial(q,t,p)}{\partial(x,u)}\bigg|_{x_1=x_2=u=g(x)=0} \leq 1.$$

Therefore three cases remain to be checked. Suppose that Σ is smooth at any point of Σ_p. Then the corresponding fibre S_p of f is a blowing down of Σ_p.

(a) $m_1 = 2$, $m_2 = 1$ and

$$\big(g(0,x_3,x_4), h_2(0,x_3,x_4,0)\big) = 1.$$

Let Γ_1, Γ_2 and Γ_3 denote the curves defined by the following equations respectively

$$\begin{cases} x_2 = u = 0, \\ g = 0, \end{cases} \qquad \begin{cases} x_1 = u = 0, \\ h_2 = 0, \end{cases} \qquad \begin{cases} u = 0, \\ x_1 = x_2 = 0. \end{cases}$$

By the assumption, we have

$$\Sigma_p = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

and

$$\Gamma_1\Gamma_3 = 2, \qquad \Gamma_2\Gamma_3 = 3, \qquad \Gamma_1\Gamma_2 = 0.$$

Since $\Gamma_3\Sigma_p = 0$ and $p_a(\Gamma_3) = 0$, we obtain

$$\Gamma_3^2 = -5, \qquad K_{\Sigma_p}\Gamma_3 = 3.$$

Moreover, as $\Gamma_1^2 = -2$, $\Gamma_2^2 = -3$, if Σ_p is reduced, then $\Sigma_p = S_p$. If Σ_p is 2-connected, then the canonical divisor K_S is base-point-free on $S_p = \Sigma_p$ (see e.g. [3]). Hence $K_S\Gamma_3 = 1$, a contradiction. This means Σ_p is not reduced. Since $\Gamma_1^2 = -2$ and $\Gamma_2^2 = -3$, these components are not multiples. Thus the only possibility is $\Gamma_2 = 2\Gamma_2' + \Gamma_2''$, $\Gamma_2' \neq \Gamma_2''$. Since ${\Gamma_2'}^2 = -1$, ${\Gamma_2''}^2 = -5$, $p_a(\Gamma_2') = p_a(\Gamma_2'') = 0$, we get $K_{\Sigma_p}\Gamma_2' = -1$, $K_{\Sigma_p}\Gamma_2'' = 0$. As $K_{\Sigma_p}\Gamma_1 = 0$, we get $K_{\Sigma_p}\Sigma_p = 2 \neq 6$, a contradiction.

(b) $m_1 = 1$, $m_2 > 1$, $g(x)$ is smooth at points $x_1 = x_2 = u = g(x) = 0$, $f_2(x,0)$ is smooth at points $x_1 = x_2 = u = f_2(x,0) = 0$, and

$$\big(f_2(0,x_3,x_4,0), g(0,x_3,x_4)\big) = 1.$$

In this case the third equation $p(x,u)$ is given in formula (4). Let Γ_1, Γ_2 and Γ_3 denote the curves defined by the following equations respectively

$$\begin{cases} x_2 = u = 0, \\ g = 0, \end{cases} \qquad \begin{cases} x_1 = u = 0, \\ g = 0, \end{cases} \qquad \begin{cases} x_1 = u = 0, \\ f_2 = 0. \end{cases}$$

By the assumption, we have

$$\Sigma_p = \Gamma_1 + \Gamma_2 + \Gamma_3$$

and

$$\Gamma_1\Gamma_2 = 2, \qquad \Gamma_2\Gamma_3 = 4, \qquad \Gamma_1\Gamma_3 = 0.$$

By the assumption, $g(x)$ is not a square of a linear polynomial. Therefore Γ_1 and Γ_2 are reduced, $p_a(\Gamma_1) = p_a(\Gamma_2) = 0$. An easy computation yields $\Gamma_1^2 = -2$, $\Gamma_2^2 = -6$, $\Gamma_3^2 = -4$. If Σ_3 is reduced, then $S_p = \Sigma_p$. Since $K_{\Sigma_p}\Gamma_2 = 4$, Σ_p cannot be 2-connected. This implies

$\Gamma_3 = 2\Gamma_3'$. By computation we get $K_{\Sigma_p}\Gamma_3' = -1$, $K_{\Sigma_p}\Gamma_1 = 0$. Thus $K_{\Sigma_p}\Sigma_p = 2 \neq 6$, a contradiction.

(c) $m_1 = m_2 = 1$, $g(x)f_2(x,0) - x_2h_2(x,0)$ is smooth at points $x_1 = x_2 = u = g(x) = 0$. In this case the third equation $p(x,u)$ is given in formula (5). Let Γ_1, Γ_2 denote the curves defined by the following equations respectively

$$\begin{cases} x_2 = u = 0, \\ g = 0, \end{cases} \qquad \begin{cases} x_1 = u = 0, \\ gf_2 - x_2h_2 = 0. \end{cases}$$

Then

$$\Sigma_p = \Gamma_1 + \Gamma_2, \qquad \Gamma_1\Gamma_2 = 2, \qquad \Gamma_1^2 = \Gamma_2^2 = -2.$$

If the divisor Σ_p is reduced, then $S_p = \Sigma_p$ and Σ_p is 2-connected. That means $K_S\Gamma_1 = 2$, $K_S\Gamma_2 = 4$. But $p_a(\Gamma_1) = 1 + (K_S\Gamma_1 + \Gamma_1^2)/2 = 1$, a contradiction.

Now suppose that Σ_p is not reduced. Since $\Gamma_1^2 = \Gamma_2^2 = -2$, the divisors Γ_1 or Γ_2 cannot be multiple divisors. Assume $\Gamma_2 = 3\Gamma_2' + \Gamma_2''$. Then

$$\Gamma_2^2 = 9\Gamma_2'^2 + \Gamma_2''^2 + 6\Gamma_2'\Gamma_2'' = 9\Gamma_2'^2 + \Gamma_2''^2 + 6 \leq -4,$$

which is impossible.

If $\Gamma_2 = 2\Gamma_2' + \Gamma_2'' + \Gamma_2'''$ with $\Gamma_2'' \neq \Gamma_2'''$, then

$$\Gamma_2^2 = 4\Gamma_2'^2 + \Gamma_2''^2 + \Gamma_2'''^2 + 10 = -2.$$

Since $p_a(\Gamma_2') = p_a(\Gamma_2'') = p_a(\Gamma_2''') = 0$, in any case we cannot obtain the equality $K_{\Sigma_p}\Sigma_p = 6$.

If $\Gamma_2 = 2\Gamma_2' + \Gamma_2''$, then the equality

$$\Gamma_2^2 = 4\Gamma_2'^2 + \Gamma_2''^2 + 8 = -2$$

also yields a contradiction.

Case II. The quadratic polynomial $q(x,0)$ is a square. By coordinate transformation, we can assume

$$q(x,u) = x_1\big(x_1 + u^{n_1}f_1(x,u)\big) + u^{m_1}f_2(x_2,x_3,x_4,u),$$
$$t(x,u) = x_1\big(g(x_2,x_3,x_4) + u^{n_2}h_1(x,u)\big) + u^{m_2}h_2(x_2,x_3,x_4,u).$$

Then we distinguish two cases.

(i) $m_1 \leq m_2$. In this case the third equation is

$$p(x,u) = f_2(x_2,x_3,x_4,u)g(x_2,x_3,x_4) - u^{m_2-m_1}x_1h_2(x,u) + u^{n_2}h_1(x,u)f_2(x,u) - u^{n_1+m_2-m_1}f_1(x,u)h_2(x,u).$$

If $m_2 > 1$, we have

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=u=g(x)=f_2(x,0)=0} = (0).$$

If $m_1 = m_2 = 1$ and $\deg d(x_2,x_3,x_4) > 0$, where $d(x) = \big(g(x_2,x_3,x_4), f_2(x_2,x_3,x_4,0)\big)$, then

$$\left.\frac{\partial(q,t)}{\partial(x,u)}\right|_{x_1=u=d(x)=h_2(x,0)=0} = (0).$$

(ii) $m_1 > m_2$. In this case, we can show that any homogeneous polynomial $r(x,0)$ satisfying

$$u^m r(x,u) = q(x,u)\alpha(x,u) - t(x,u)\beta(x,u)$$

with $m < m_2$ must be generated by x_1^2 and $x_1 g(x)$. When $m = m_2$, we obtain

$$r(x,u) = \big(x_1 h_2(x,u) + u^{n_1} h_2(x,u) f_1(x,u) - u^{m_1-m_2} f_2(x,u) g(x) - u^{n_2+m_1-m_2} f_2(x,u) h_1(x,u)\big)\gamma(x).$$

Thus we can see that the elements of the ideal I_p can be generated by x_1, $g(x)$ and $h_2(x,0)$. This implies that any point $(x_1 : x_2 : x_3 : x_4, u) \in \mathbb{P}^3_\Delta$ satisfying $x_1 = u = g(x) = h_2(x,0) = 0$ must be in Σ. Moreover, since Σ_p is connected, we can always find a $p(x,u)$ such that Σ is defined by $q(x,u)$, $t(x,u)$ and $p(x,u)$. The following equality

$$\frac{\partial(q,t)}{\partial(x,u)}\bigg|_{x_1=u=g(x)=h_2(x,0)=0} = (0)$$

shows that Σ cannot be smooth in this case.

Now there remains only one case to be checked. That is, $m_1 = m_2 = 1$ and

$$\big(g(x_2,x_3,x_4), f_2(x_2,x_3,x_4,0)\big) = 1.$$

In this case, the third equation is

$$p(x,u) = g(x_2,x_3,x_4) f_2(x_2,x_3,x_4,u) - x_1 h_2(x,u) + u^{n_2} h_1(x,u) f_2(x,u) - u^{n_1} f_1(x,u) h_2(x,u).$$

Let Γ_1 and Γ_2 denote the curves defined by the following equations respectively

$$\begin{cases} x_1 = u = 0, \\ g = 0, \end{cases} \qquad \begin{cases} x_1 = u = 0, \\ f_2 = 0. \end{cases}$$

By the assumption, we have

$$\Sigma_p = 2\Gamma_1 + \Gamma_2,$$

and

$$\Gamma_1\Gamma_2 = 4, \qquad \Gamma_1^2 = -2, \qquad \Gamma_2^2 = -8.$$

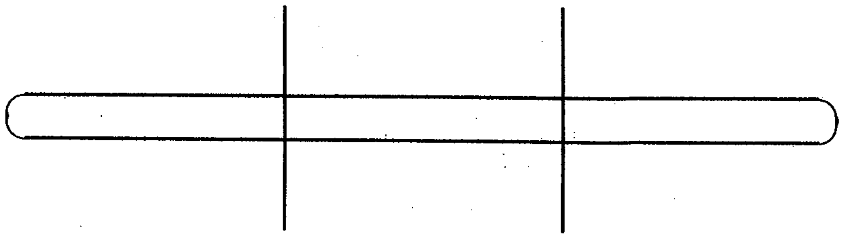
Hence Γ_1 and Γ_2 cannot be multiple divisors. Then we can check that $S_p = \Sigma_p$ is 2-connected. But $K_S\Gamma_1 = 0$ and $K_S\Gamma_2 = -6$, this contradicts $K_S S_p = 6$.

Now we will give two examples showing that Σ_p need not be a complete intersection in $P_p \cong \mathbb{P}^3$.

Example 1. Let $C = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $\mathcal{E} = \mathcal{O}_C^{\oplus 4}$. Then $P = \mathbb{P}(\mathcal{E}) = \mathbb{P}^3 \times \mathbb{P}^1$ and $\pi : P \longrightarrow C$ is the projection. Let $p = 0 \in C$. The following global sections:

$$q(x,u) = x_1x_2 + u(x_1^2 + x_3^2 + x_4^2) \in H^0(\mathcal{O}_P(2) \otimes \pi^*\mathcal{O}_C(p)),$$
$$t(x,u) = x_1x_3^2 + ux_2(2x_2^2 + x_4^2) \in H^0(\mathcal{O}_P(3) \otimes \pi^*\mathcal{O}_C(p)),$$
$$h(x) = x_2^2(2x_2^2 + x_4^2) - x_3^2(x_1^2 + x_3^2 + x_4^2) \in H^0(\mathcal{O}_P(4)),$$

define a subvariety $\Sigma \subseteq P$. Outside the singular line $u = x_2 = x_4 = 0$, the projective variety Σ is smooth. After desingularization, we obtain a non-hyperelliptic fibration of genus 4 $f : S \longrightarrow C$ whose relative 1-canonical image is Σ. The singular fibre S_p has the configuration



where

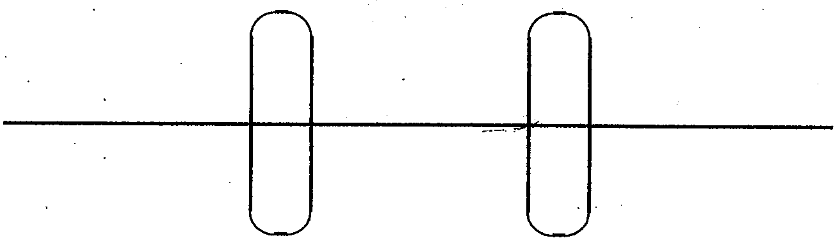
$$K_S\Gamma_1 = 4, \quad \Gamma_1^2 = -2, \quad p_a(\Gamma_1) = 2,$$
$$K_S\Gamma_2 = 2, \quad \Gamma_2^2 = -4, \quad p_a(\Gamma_2) = 0,$$
$$K_S\Gamma_3 = 0, \quad \Gamma_3^2 = -2, \quad p_a(\Gamma_3) = 0.$$

Note that the general fibre of f is a non-hyperelliptic curve with two g_3^1's.

Example 2. The notations are the same as in Example 1. We take global sections

$$q(x,u) = x_1x_2 + u(x_1^2 + x_3^2) \in H^0(\mathcal{O}_P(2) \otimes \pi^*\mathcal{O}_C(p)),$$
$$t(x,u) = x_1x_4^2 + u(x_1^2x_2 + x_2^3 + x_4^3) \in H^0(\mathcal{O}_P(3) \otimes \pi^*\mathcal{O}_C(p)),$$
$$h(x) = x_2(x_1^2x_2 + x_2^3 + x_4^3) - x_4^2(x_1^2 + x_3^2) \in H^0(\mathcal{O}_P(4)).$$

These global sections define a subvariety $\Sigma \subseteq P$. The only singular line of Σ is $u = x_2 = x_4 = 0$. After desingularization, a non-hyperelliptic fibration of genus 4 $f : S \longrightarrow C$ is obtained and the relative 1-canonical image of f is Σ. The singular fibre S_p looks like



where

$$K_S\Gamma_1 = 4, \quad \Gamma_1^2 = -4, \quad p_a(\Gamma_1) = 1,$$
$$K_S\Gamma_2 = 2, \quad \Gamma_2^2 = -4, \quad p_a(\Gamma_2) = 0,$$
$$K_S\Gamma_3 = 0, \quad \Gamma_3^2 = -2, \quad p_a(\Gamma_3) = 0.$$

Note that the general fibre of f is a non-hyperelliptic curve with one g_3^1.

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