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COMPARISON THEOREMS TO BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

A unified approach is given to several classes of nonlinear boundary value problems, and some previous existence theorems are extended.

Keywords Boundary value problem, Existence theorem, Brouwer degree, Schauder's fixed point theorem.
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§1. Introduction

Consider the following boundary value problems

$$x' = f(t, x), \tag{1.1}$$

$$B(x) = r \tag{1.2}$$

and

$$y' = g(t, y), \tag{1.3}$$

$$B(y) = r, \tag{1.4}$$

where $f, g: [a, b] \times R^n \to R^n$ and B is a mapping from the space of continuous functions $C([a, b], R^n)$ into R^n.

One interesting problem is the following:

Under what conditions can the existence of solutions to BVP (1.1), (1.2) imply the existence of solutions to BVP (1.3),(1.4) ?

The aim of this paper is to prove some comparison theorems which provide some affirmative answers to the above question. Here one special result is the following

Theorem 1.1. *Suppose that*

(i) $f, g: [a, b] \times R^n \to R^n$ *are continuous and the solution of the Cauchy problem to Equation* (1.1) *is unique,*

(ii) *there is an* $l > 0$ *such that for all* $p \in R^n$, *and any possible solution* $x(t)$ *of Equation* (1.1) *and* $y(t)$ *of Equation* (1.2) *with the same initial data* p *at* $t = t_0 \in [a, b]$, *one has*

$$|x(t) - y(t)| \le l, \quad \textit{for} \ \ t \in [a, b],$$

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(iii) $B: C([a,b], R^n) \to R^n$ *is continuous and satisfies*

$$|B(x+h) - B(x)| \le M, \quad \textit{for } x, h \in C([a,b], R^n) \textit{ and } \|h\| < l + d,$$

for suitable constants $M, d > 0$,

(iv) *for every* $r \in R^n$, *BVP* (1.1)(1.2) *has a unique solution.*

Then BVP (1.3), (1.4) *has at least one solution.*

With these basic results, we can generalize many results of [1, 4, 5, 7–12].

Our consideration is motivated from Lakshmikantham's paper [3] on the existence of periodic solutions and the arguments are based on the continuity of the Brouwer degree.

We shall first prove Theorem 1.1 in section 2 and then provide a more general result in section 3. Some applications will be given in section 4. Finally in section 5, we shall present general comparison theorems along the line of the nice paper of Lasota and Opial[5].

§2. Proof of Theorem 1.1

We first prove the theorem under the additional assumption that for every $p \in R^n$, there exists a unique solution $y(t, t_0, p)$ of Equation (1.3) such that $y(t_0) = p$.

Define a homotopy $H(p, \lambda)$ on $R^n \times [0,1]$ by

$$H(p,\lambda) = B(x(\cdot, t_0, p)) + \lambda[B(y(\cdot, t_0, p)) - B(x(\cdot, t_0, p))] - r.$$

Take $m > M$ and set

$$S(r,m) = \{p \in R^n : |p - r| < m\},$$

and

$$D = \{p \in R^n : p = B_0^{-1}(q), q \in S(r,m)\},$$

where $B_0(p) \equiv B(x(\cdot, t_0, p))$. From (iv) it follows that $D \subset R^n$ is open, and from (ii) we see that for every $p \in \overline{D}$ the solution $y(t, t_0, p)$ exists on $[a,b]$, since $x(\cdot, t_0, p)$ exists on $[a,b]$.

From the choice of m and (iii) we get

$$H(p,\lambda) \ne 0, \text{ for } (p,\lambda) \in \partial D \times [0,1]. \tag{2.1}$$

Obviously $H(p,\lambda)$ is continuous on $\overline{D} \times [0,1]$, since $x(t, t_0, p)$ and $y(t, t_0, p)$ are continuous in p. Hence $\deg(H(\cdot,\lambda), D, 0)$ is well defined. From (2.1) and the continuity of Brouwer degree we have $\deg(H(\cdot,1), D, 0) = \deg(H(\cdot,0), D, 0)$. According to Theorem 3.3.3 of Lloyd [6], we get from (iv), $\deg(H(\cdot,0), D, 0) = \pm 1$ and then $\deg(H(\cdot,1), D, 0) = \pm 1$. Therefore, there is $p_0 \in D$ such that $B(y(\cdot, t_0, p_0)) = r$, i.e., BVP (1.3), (1.4) has at least one solution.

To prove the general case, we use an approximation procedure.

Set

$$E = \bigcup_{t \in [a,b]} \{z \in R^n : z = x(t, t_0, p), p \in \overline{D}\}$$

and

$$S(E,l) = \{p \in R^n : \operatorname{dist}(p, E) < l\}.$$

By virtue of the Weierstrass theorem, there is a sequence $\{g_k(t,x)\}$ of C^1 functions such that $g_k(t,x) \to g(t,x)$ uniformly on $[a,b] \times \bar{S}(E,l)$. According to a well known theorem (see

[2], Chapter II, Theorem 3. 2) and (ii) , we assume that for each k the following holds:

$$|x(t, t_0, p) - y_k(t, t_0, p)| \le l + d,$$

where $y_k(t, t_0, p)$ is a solution of $y' = g_k(t, y)$ with $y(t_0) = p$. By previous arguments, for each k the BVP $y' = g_k(t, y)$, $B(y) = r$ has at least one solution $y_k(t)$ with $y_k(t_0) \in D$ and $y_k(t) \in \bar{S}(E, l)$, for $t \in [a, b]$. Applying the Arzela-Ascoli theorem to the sequence $\{y_k(t)\}$, we can conclude the existence of solutions to BVP (1.3), (1.4), which complets the proof.

§3. A General Theorem

In this section, we present a general comparison theorem.

Theorem 3.1. *Suppose that the conditions* (i)–(iii) *of Theorem* 1.1 *hold, and for some* $m > M$, $D(r) \equiv \{p \in R^n : p \in B_0^{-1}(q), q \in S(r, m)\}$ *is bounded and* $\deg(B_0 - r, D(r), 0) \neq 0$, *where* $B_0(p) \equiv B(x(\cdot, t_0, p))$ *and* $S(r, m) = \{p \in R^n : |p - r| < m\}$. *Then BVP* (1.3), (1.4) *has at least one solution.*

Proof. Similar to that of Theorem 1.1.

Remark. In Theorem 1.1, the unique solvability of BVP (1.1), (1.2) is assumed. Hence the condition $\deg(B_0 - r, D(r), 0) \neq 0$ is more generel.

§4. Some Applications

In this section, using Theorem 1.1, we give a unified approach to several classes of problems and extend some previous existence theorems.

We first extend the results of Opial[7] and Vidossich[10] to the following

Theorem 4.1. *Let* $f : [a, b] \times R^n \to R^n$, $B : C([a, b], R^n) \to R^n$ *be continuous and satisfy* (iii) *in Theorem* 1.1. *Suppose that*

(i) *for a suitable* $K > 0$, *one has*

$$|f(t, x) - f(t, y)| \le K|x - y|, \text{ for } x, y \in R^n \text{ and } t \in [a, b],$$

(ii) *BVP* (1.1), (1.2) *has a unique solution for evrey* $r \in R^n$.

Then for every bounded continuous function $e : [a, b] \times R^n \to R^n$ *and every* $r \in R^n$, *BVP* (1.3)(1.4) *has at least one solution for* $g(t, x) = f(t, x) + e(t, x)$.

Proof. Let $x(t)$ and $y(t)$ be any solution of Equation (1.1) and Equation (1.2) respectively, with the same initial data which exist on $[a, b]$. Then

$$\begin{aligned} |x(t) - y(t)| &= \Big| \int_{t_0}^{t} [f(s, x) - f(s, y) - e(s, y)] ds \Big| \\ &\le \Big| \int_{t_0}^{t} (K|x(s) - y(s)| + N) ds \Big|, \quad \text{for } t \in [a, b], \end{aligned}$$

where $N = \sup\{|e(t, y)| : t \in [a, b], y \in R^n\}$. From this and the Gronwall lemma it follows that (ii) in Theorem 1.1 is satisfied. Here the remaining conditions in Theorem 1 also hold. Therefore we complete the proof of the theorem by applying Theorem 1.1.

Remark. In [7] and [10], $B(x)$ is a linear operator. Here, $B(x)$ is not necessarily. This makes us present a new argument.

Secondly we prove an existence theorem of periodic solutions.

Theorem 4.2. *Suppose that* $f: R \times R^n \to R^n$ *is continuous, and for some constant* $T > 0, f(t+T,x) \equiv f(t,x)$, *for all* $(t,x) \in R \times R^n$. *Suppose that there is a continuous function* $\alpha: [0,T] \to R$ *with* $\int_0^T \alpha(s)ds = \alpha_0 < 0$ *such that*

$$(x-y)\cdot[f(t,x)-f(t,y)] \le \alpha(t)|x-y|^2, \quad \textit{for } (t,x) \in [0,T]\times R^n \tag{4.1}$$

where "·" denotes the usual inner product. Moreover suppose that Equation (1.1) *has at least one solution* $x_0(t)$ *existing on [0,T]. Then for every bounded continuous function* $e: R \times R^n \to R^n$ *with* $e(t+T,x) \equiv e(t,x)$ *for all* $(t,x) \in R \times R^n$, *the equation*

$$x' = f(t,x) + e(t,x) \tag{4.2}$$

has at least one T-periodic solution.

Proof. We claim that for every r the following BVP

$$x' = f(t,x), \quad x(T) - x(0) = r \tag{4.3}$$

has a unique solution. In fact, we define a mapping $F: R^n \to R^n$ by $F(p) = x(T,0,p) - r$, and then prove that F is a contract mapping. From (4.1) we get

$$\begin{aligned}\frac{d}{dt}|x_0(t) - x(t,0,p)|^2 &= 2[x_0(t) - x(t,0,p)]\cdot[f(t,x_0(t)) - f(t,x(t,0,p))]\\ &\le 2\alpha(t)|x_0(t) - x(t,0,p)|^2, \text{ for } t \in [0,T],\end{aligned}$$

which implies

$$|x_0(t) - x(t,o,p)|^2 \le |x_0(0) - p|^2 \exp[2\int_0^t \alpha(s)ds], \text{ for } t \in [0,T].$$

Hence F is well defined on R^n and is contract by substituting $x(t,0,q)$ for $x_0(t)$ above, since $\alpha_0 < 0$. Therefore, according to the Banach contract mapping principle, F has a unique fixed point, i.e., BVP (4.3) has a unique solution.

By similar arguments as above, (ii) in Theorem 1.1 also holds. Consequently by virtue of Theorem 1.1, we get a solution $x(t)$ of Equation (4.2) with $x(0) = x(T)$. Set

$$x_1(t) = x(t+kT), \quad \text{for} \quad t \in [kT,(k+1)T] \text{ and } k = 0, \pm 1, \cdots.$$

Then $x_1(t)$ is the desired sloution of Equation (4.3). The proof is complete.

Next we prove an existence theorem for boundary value problems of kth-order ordinary differential equations.

Consider the following BVP

$$x^{(k)} = f(t,x,\dots,x^{(k-1)}), \tag{4.4}$$

$$B(x) = r. \tag{4.5}$$

We have

Theorem 4.3. *Let* $B: C^{k-1}([a,b],R^n) \to R^{kn}$ *be continuous and satisfy that for every* $l > 0$, *there is an* $M(l) > 0$ *such that*

$$|B(x+h) - B(x)| \le M(l), \quad \text{for } x,\ h \in C^{k-1}([a,b],R^n), \quad \text{and } \|h\| \le l.$$

Let $z(t,c_1,\dots,c_k)$ *be the solution of the Cauchy problem*

$$x^{(k)} = 0, \quad x^{(i)}(a) = c_{i+1}, \quad i = 0,\cdots,k-1.$$

If the equation $B(z(\cdot, c_1, \ldots, c_k)) = r$ *has a unique solution* $(c_1, \ldots, c_k)$, *for every* $r \in R^{kn}$, *then for every bounded continuous function* $f : [a, b] \times R^{kn} \to R^n$, *BVP* (4.4), (4.5) *has at least one solution.*

Obviously this theorem generalizes the classical Scorza-Dragoni theorem on two point boundary value problems for second order equations.

Proof of Theorem 4.3. Rewrite Equation (4.4) in the following form

$$X' = \begin{pmatrix} O & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ & & & O \end{pmatrix} X + \begin{pmatrix} 0 \\ \vdots \\ f(t, x) \end{pmatrix} \equiv AX + F(t, X), X = \begin{pmatrix} x \\ \vdots \\ x^{(k-1)} \end{pmatrix}.$$

From this and the assumptions we see easily that conditions in Theorem 1.1 are all satisfied. So the conclusion follows from Theorem 1.1.

§5. Generalization of the Lasota-Opial Theorem

In this final section, we extend a well known theorem of Lasota and Opial[5].

Consider the following BVP

$$x^{(n)} = f(t, x, \cdots, x^{(n-1)}), \tag{5.1}$$

$$B(x) = r. \tag{5.2}$$

The first extension is the following

Theorem 5.1. *Suppose that* $f : [a, b] \times R^n \to R$ *and* $B : C^{(n-1)}([a, b], R) \to R^n$ *are continuous. Suppose that there are continuous functions* $p_i : [a, b] \to R^+ (i = 0, \cdots, n)$ *such that for every function* $a_i \in L^2[a, b], i = 0, 1, \cdots, n$ *with* $|a_i(t)| \le p_i(t)$ *for* $t \in [a, b]$ *and* $i = 0, \cdots, n$ *and for every* $r \in R^n$, *the following BVP*

$$x^{(n)} = \sum_{i=0}^{n-1} a_i(t) x^{(i)} + a_n(t), B(x) = r \tag{5.3}$$

has a unique solution and f *satisfies*

$$|f(t, x_0, \cdots, x_{n-1})| \le \sum_{i=0}^{n-1} p_i(t)|x_i| + p_n(t) \text{ for } t \in [a, b]$$
$$\text{and } \ x_i \in R (i = 0, \cdots, n-1). \tag{5.4}$$

Then BVP (5.1), (5.2) *has at least one solution for every* $r \in R^n$.

To prove this theorem, we need the following

Lemma 5.1. *Let* $D \subset R^n$ *be open and* E *a closed subset of topological space* X, *such that any sequence in* E *admits a converging subsequence. Suppose that* $F : D \times E \to R^n$ *is continuous and for each* $\lambda \in E, F(\cdot, \lambda) : D \to R^n$ *is injective. If the sequence* $\{S_\lambda\}$ *with* $S_\lambda \subset$ *Range* $(F(\cdot, \lambda))$ *are uniformly bounded in* $\lambda \in E$, *then so is* $\{F^{-1}(S_\lambda)\}$.

Proof. If not, there were sequences $\lambda_k \in E, x_k \in F^{-1}(S_{\lambda_k})$ such that

$$y_k = F(x_k, \lambda_k) \to y_0, \lambda_k \to \lambda_0 \text{ and } |x_k| \to \infty (k \to \infty), \tag{5.5}$$

for suitable $\lambda_0 \in E$ and $y_0 \in S_{\lambda_0}$. Let $x_0 = F^{-1}(y_0, \lambda_0)$. From the invariance of domain theorem (see [13], p.705, Theorem 16. C) it follows that $F(S(x_0, \delta), \lambda_0)$ is open, where $\delta > 0$

is suitably small. Hence $m = \text{dist}(y_0, \partial F(s(x_0,\delta),\lambda_0)) > 0$. Obviously,

$$F(x,\lambda) \to F(x,\lambda_0)(\lambda \to \lambda_0)$$

uniformly on $x \in \partial S(x_0,\delta)$. By this and (5.5), there is a $k_0 > 0$ such that

$$|y_k - y_0| < \frac{m}{4}, \quad \text{dist}(y_0, \partial F(S(x_0,\delta),\lambda_k)) \geq \frac{m}{2},$$

for every $k \geq k_0$, which implies $y_k \in F(S(x_0,\delta),\lambda_k)$ for all $k \geq k_0$. Since for fixed $\lambda \in E$, $F(\cdot,\lambda)$ is injective, we get $x_k = F^{-1}(y_k) \in S(x_0,\delta)$, which contradicts $|x_k| \to \infty (k \to \infty)$. The proof is complete.

Proof of Theorem 5.1. Set

$$c(t,x_0,\cdots,x_{n-1}) = a_n(t) + \sum_{i=0}^{n-1} p_i|x_i|,$$

$$b_i(t,x_0,\cdots,x_{n-1}) = p_i(t) f(t,x_0,\cdots,x_{n-1})\text{sgn}x_i / c(t,x_0,\cdots,x_{n-1}) \ (i = 1,\cdots,n-1),$$

$$b_n(t,x_0,\cdots,x_{n-1}) = p_n(t) f(t,x_0,\cdots,x_{n-1}) / c(t,x_0,\cdots,x_{n-1}).$$

Then

$$f(t,x_0,\cdots,x_{n-1}) = \sum_{i=0}^{n-1} b_i(t,x_0,\cdots,x_{n-1})x_i + b_n(t,x_0,\cdots,x_{n-1}).$$

Rewrite Equation (5.1) in the following form

$$\begin{aligned} X' &= \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & & \\ & & & 0 & 1 \\ b_{n-1} & \cdots & & b_1 & b_0 \end{pmatrix} X + \begin{pmatrix} 0 \\ \vdots \\ b_n \end{pmatrix} \\ &= A(t,X) + h(t,X), \left[X = \begin{pmatrix} x \\ \vdots \\ x^{(n-1)} \end{pmatrix} \right]. \end{aligned} \tag{5.6}$$

Set

$$Q = \{A(t,u) : u \in C([a,b],R^n)\}, \qquad H = \{h(t,u) : u \in C([a,b],R^n)\},$$

and

$$\begin{aligned} Q_1 = \{e(t) :\ & \text{there is a sequence } \{A_k(t)\} \subset CO(H) \text{ such that } A_k \\ & \text{is weakly convergent to } e \text{ in } L([a,b],R^{n\times n})\}, \end{aligned}$$

where $CO(H)$ denotes the convex hull of H. Similar to Q_1, we get the set H_1 to H. Since each $b \in CO(H)$ has the following form

$$b(t) = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ b_{n-1} & \cdots & \cdots & b_1 & b_n \end{pmatrix}$$

with $|b_i(t)| \leq p_i(t)$, for $t \in [a,b]$, $i = 0,\cdots,n-1$, by the Hahn-Banach theorem it follows that $Q_1 \subset L^2([a,b],R^{n\times n})$ is bounded, convex and closed. Similarly so is H_1. Define a mapping $F_\lambda : R^n \to R^n$ by

$$F_\lambda(X_0) = B(X(\cdot, t, X(t, a, X_0, b, c))),$$

where $\lambda = (t, b, c), b \in Q_1, c \in H_1$ and $X(t, a, X_0, b, c)$ denotes the solution of the Cauchy problem:

$$X' = b(t)X + c(t), \quad X(a) = X_0.$$

It is easily seen that $X(t, a, X_0, b, c)$ is continuous with respect to $(t, b, c) \in [a, b] \times Q_1 \times H \equiv E$ by applying the Gronwall lemma. By the assumption (5.3) for each λ, F_λ is injective. Hence $\{F_\lambda^{-1}(\bar{S}(r, 1)\}$ is uniformly bounded in $\lambda \in E$, by virtue of Lemma 5.1. Therefore there is an $l > 0$ such that for every $u \in C([a, b], R^n)$ together with solution $X_u(t)$ of the following BVP

$$X' = A(t, u)X + h(t, u), \tag{5.7}$$

$$B(x) = r, \tag{5.8}$$

we have

$$|X_u(t)| \le l, \text{ for } t \in [a, b]. \tag{5.9}$$

Define an operator $N : C([a, b], R^n) \to C([a, b], R^n)$ by $N(u) = X_u(t)$, which is well defined by (5.3), (5.4). Since $X_u(t)$ is a solution of Equation (5.7), from the Arzela-Ascoli theorem it follows that N is compact. It is easily seen that N is continuous by the uniqueness of $X_u(t)$ and the continuity of solutions with respect to the initial data and parameter. By (5.9), $N(C([a, b], R^n))$ is bounded. Therefore according to the Schauder theorem, N has at least one fixed point u_0. Then $u_0(t)$ is the desired solution. The proof is complete.

Remark. In Lasota and Opial's result, $B(x) = \big(x(t_1), \cdots, x(t_n)\big)$, where $a < t_1 < \cdots < t_n < b$. Here, in Theorems 5.1 and 5.2, $B(x)$ may be a general linear oprator, even nonlinear one.

The following theorem is a direct extension of the Lasota and Opial's theorem[5].

Theorem 5.2. *Suppose that* $f : [a, b] \times R^n \to R$ *is continuous and that* $B : C([a, b], R^n) \to R^n$ *is continuous and satisfies the following:*

For every $m > 0$, *there is an* $M(m) > 0$ *such that*

$$|B(x + h) - B(x)| \le M(m) \quad \text{for } x,\ h \in C([a, b], R^n) \text{ and } \|h\| \le m. \tag{5.10}$$

Suppose moreover that there are continuous function $p_i : [a, b] \to R^+ (i = 0, \cdots, n)$ *and* $\varepsilon_0 > 0$ *such that for every function* $a_i \in L^2[a, b], i = 0, \cdots, n-1$ *with* $|a_i(t)| \le p_i(t) + \varepsilon_0$, *for* $t \in [a, b]$ *and* $i = 0, \cdots, n-1$, *and for every* $r \in R^n$, *the following BVP*

$$x^{(n)} = \sum_{i=0}^{n-1} a_i(t) x^{(i)}, B(x) = r$$

has a unique solution and f *satisfies*

$$|f(t, x_0, \cdots, x_{n-1})| \le \sum_{i=0}^{n-1} p_i(t)|x_i| + p_n(t) \quad \text{for } t \in [a, b],\ x_i \in R\ (i = 0, \cdots, n-1).$$

Then BVP (5.1), (5.2) *has at least one solution.*

Here we point out that in Lasota and Opial's theorem, $B(x) \equiv (x(t_1), \ldots, x(t_n)) = (r_1, \ldots, r_n)$, where $a < t_1 < \cdots < t_n < b$. Hence the above theorem generalizes their result.

Proof of Theorem 5.2. Along the line of the proof of Theorem 5.1, we have (5.9), where $H_1 = \{0\}$. By (5.7) and the assumption, there is an $m_0 > 0$ such that

$$|X(t, a, X_0, \mu A(\cdot, u), h(\cdot, u)) - X(t, a, X_0, \mu A(\cdot, u), 0)| < m_0,$$
$$\text{for } \ t \in [a, b], \ x_0 \in R^n, \ u \in C([a, b], R^n) \ \text{ and } \ \mu \in [0, 1]. \tag{5.11}$$

Set

$$D = \{p \in R^n : p = B_{0u\mu}^{-1}(q), \ q \in S(r, M(m_0))\},$$

where $B_{0u\mu}(p) = B(X(\cdot, a, p, \mu A(\cdot, u), 0))$. The set $D \subset R^n$ is bounded, open and satisfies $D \supset D_u$ for every $u \in C([a, b], R^n)$.

By (5.10), (5.11) and the definition of D, any possible solution $X_\mu(t)$ of the following BVP

$$X' = \mu A(t, X)X + \mu h(t, X), \tag*{$(5.12)_\mu$}$$
$$B(X) = r \tag{5.13}$$

must satisfy $X_\mu(a) \notin \partial D$, for $\mu \in [0, 1]$. Now we prove the existence of solution to BVP $(5.12)_\mu$, (5.13). Using an approximation procedure similar to the proof of Theorem 1.1, we may assume that $h(t, X)$ is continuously differentiable in $X \in R^n$. Recalling the proof of Theorem 1.1, we get $\deg(B_0 - r, D, 0) = \pm 1$, where $B_0(p) \equiv B(X(\cdot, a, p, 0, 0))$. From this and the continuity of Brouwer degree it follows that

$$\deg(B_1 - r, D, 0) = \deg(B_0 - r, D, 0) \neq 0,$$

where $B_\mu(p) = B(X(\cdot, a, p, \mu))$ and $X(t, a, p, \mu)$ denotes the solution of Equation $(5.12)_\mu$ with initial data $X(a) = p$. Therefore B_1 has at least one fixed point $p_0 \in D$, and then $X(t, a, p_0, 1)$ is the desired solution. The proof is completed.

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