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ON THE THEOREM OF ARROW-BARANKIN-BLACKWELL FOR WEAKLY COMPACT CONVEX SET

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Abstract

This paper studies the known density theorem of Arrow-Barankin-Blackwell. The following main result is obtained: If X is a Hausdorff locally convex topological space and $C \subset X$ is a closed convex cone with bounded base, then for every nonempty weakly compact convex subset A, the set of positive proper efficient points of A is dense in the set of efficient points of A.

Keywords Positive proper efficient point, Efficient point, Convex cone, Hausdorff space.

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In 1953, in reference [1], Arrow, Barankin and Blackwell proved the following conclusion: In R^n with nonnegative coordinate cone R^n_+, for every compact convex set $A \subset R^n$, the set of its positive proper efficient points is dense in the set of the efficient points. Afterwards, in [3], Bitran and Magnanti proved that if R^n is equipped with arbitrary closed convex pointed cone C, the above result holds still. In 1980, Borwein extended the theorem of Arrow-Barankin-Blackwellto real normed space partially ordered by a closed convex cone with weakly compact base. Recently, Jahn[8] proved that in a real normed space if the ordered cone is a Bishop-Phelps cone, then the theorem of Arrow-Barankin-Blackwellholds. In 1990, Petschke noticed that Jahn's proof is effective for real normed space, when the ordered cone is a convex cone with a bounded base. Therefore, Jahn's result[8] shows that the theorem of Arrow-Barankin-Blackwellstill holds in real normed spaces partially ordered by a closed convex cone with a bounded base. In reference [12], the author has proved the following conclusion: In a real normed space, for any compact convex set, the set of the positive proper efficient points is dense in the set of the efficeint points if and only if the ordered cone is equipped with a base. Thus it can be seen that the work to extend the theorem of Arrow-Barankin-Blackwellto real normed spaces has concluded. Soon, the author extended this result to local convex spaces. In view of application, Jahn's result[8] is better. He has proved that in a real normed space if the ordered cone is equipped with a bounded base, then for arbitrary weakly compact convex set, the set of the positive proper efficient points is dense in the set of the efficient points. Therefore, he extended the theorem

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of Arrow-Barankin-Blackwellto the case of weakly compact sets. Now we ask whether Jahn's result[8] can be extended to locally convex spaces. The purpose of this paper is to show a positive answer for that question.

For convenience, we first introduce several elementary concepts and definitions.

Let X be a Hausdorff locally convex space, let its dual space be X^*, $C \subset X$ is a closed convex pointed cone, $A \subset X$ is a subset. We say that $x^* \in A$ is a efficient point if

$$A \cap (x^* - C) = \{x^*\}.$$

$E(A, C)$ denotes the set of efficient points. Let

$$C^* = \{f \in X^* : f(x) \geq 0, \forall x \in C\},$$
$$C^{\#} = \{f \in X^* : f(x) > 0, \forall x \in C/\{0\}\},$$

where every function in $C^{\#}$ is said to be a strictly positive function. A point x^* is said to be a positive proper efficient point of A, if there exists a strictly positive function $f \in C^{\#}$ such that

$$f(x^*) \leq f(x), \quad \forall x \in A,$$

that is,

$$f(x^*) = \min\{f(x) : x \in A\}.$$

We use $PS(A)$ to denote the set of positive proper efficient points of A.

Let $M \subset X$ be an arbitrary subset, the symbol $\mathrm{cl}(M)$ denotes the closure of M, $\mathrm{int}(M)$ denotes the interier. Secondly, we use $\mathrm{cone}(M)$ to denote the smallest convex cone which contains M. When M is a convex set, (see reference [13])

$$\mathrm{cone}(M) = \{\lambda x : \lambda \geq 0, x \in M\}.$$

We say that a cone C is equipped with base B, if B is a convex set of C and satisfies

$$\begin{cases} 0 \notin \mathrm{cl}(B), \\ C = \mathrm{cone}(B) = \{\lambda x : \lambda \geq 0, x \in B\}. \end{cases} \tag{1}$$

A cone with base must be a pointed cone, i.e., $C \cap (-C) = \{0\}$. And a convex cone C is equipped with a base if and only if

$$C^{\#} \neq \emptyset.$$

Theorem 1. *Let X be a Hausdorff locally convex space, $C \subset X$ be a closed convex cone. Then C is equipped with bounded base if and only if there exists $0 \neq f \in X^*$ such that for any continuous seminorm p, there exists $C_p > 0$ such that*

$$f(x) \geq C_p p(x), \quad \forall x \in C. \tag{2}$$

Proof. Let B is a bounded base of C.

Since $0 \notin \mathrm{cl}(B)$, applying the separate theorem, we see that there exists $0 \neq f \in X^*$ such that

$$f(b) \geq 1, \quad b \in B. \tag{3}$$

Assume $p : x \to R$ to be an arbitrary continuous seminorm. Then

$$U_p = \{x \in X : p(x) \leq 1\}$$

is a neighbourhood of zero. By the boundedness of base B, we know that there exists $r_p > 0$ such that

$$B \subset r_p U_p, \quad \text{or} \quad p(x) \le r_p, \quad \forall x \in B. \tag{4}$$

Then, since B is a base of C, we know that for any $x \in C/\{0\}$ there exists $\lambda_x > 0$, $b \in B$, such that $x = \lambda_x b$. By (4), we have

$$p(x) = \lambda_x p(b) \le \lambda_x r_p, \quad \text{or} \quad r_p \ge \frac{1}{\lambda_x} p(x).$$

Noticing (3) again, we have

$$f(x) = \lambda_x f(b) \ge \lambda_x \ge \frac{1}{r_p} p(x).$$

Conversely, assume that inequality (2) holds, i.e., there exists $0 \ne f \in X^*$ such that

$$f(x) \ge C_p p(x), \quad \forall x \in C.$$

Let

$$B = \{x \in C : f(x) = 1\}.$$

It is obvious that B is a convex subset and $0 \notin \mathrm{cl}(B)$. In the following we will prove the boundedness of B.

Let U be a balanced convex open neighbourhood of zero, p_U be a Minkowski functional of U

$$p_U(x) = \inf\{|\lambda| : x \in \lambda U\}.$$

Then p_U is a continuous seminorm. By the supposition, there exists a $C_U > 0$ such that

$$f(x) \ge C_U p(x), \quad \forall x \in C.$$

According to the properties of Minkowski fucntional, we have

$$U_p = \{x \in X : p(x) < 1\} \subset U.$$

But for any $x \in B$,

$$1 = f(x) \ge C_U p(x), \text{ or } p(x) \le \frac{1}{C_U}.$$

So

$$B \subset \frac{1}{C_U} U_p \subset \frac{1}{C_U} U.$$

Therefore B is bounded.

Lastly, we prove $C = \{\lambda x : \lambda \ge 0, x \in B\}$. In order to do this, we explain first that for any $x \in C/\{0\}$, $f(x) > 0$, i.e., $f \in C^{\#}$. Otherwise, for $x_0 \in C/\{0\}$ we get $f(x_0) \le 0$. Then for every continuous seminorm p,

$$0 \ge f(x_0) \ge C_p p(x_0), \text{ or } p(x_0) \le 0.$$

Note that for arbitrary seminorm p,

$$p(x_0) \ge 0.$$

Therefore

$$p(x_0) = 0 \quad \text{for any continuous seminorm } p. \tag{5}$$

Since X is a Hausdorff locally convex space, the set of continuous seminorms separates the points of X, i.e., for any $0 \neq x \in X$ there exists a continuous seminorm p such that $p(x) > 0$. From (5), we obtain $x_0 = 0$, it deduces contradiction.

Now, we prove $C = \{\lambda x : \lambda \geq 0, x \in B\}$, since for any $x \in C/\{0\}$, $f(x) > 0$. Let $y = x/f(x)$. Then

$$y \in B \quad \text{or} \quad x = f(x)y \in \{\lambda x : \lambda \geq 0, x \in B\},$$

i.e., $C \subset \{\lambda x : \lambda \geq 0, x \in B\}$. The converse containing is clear.

In the following, the symbol "$\stackrel{w}{\rightharpoonup}$" denotes weakly convergence, and the symbol "$\to$" denotes strongly convergence.

Lemma 1. *Let the closed convex cone C be equipped with a bounded base. Then for any net $\{x_n\} \subset C$, $x_n \to 0 \iff x_n \stackrel{w}{\rightharpoonup} 0$.*

Proof. Let $x_n \stackrel{w}{\rightharpoonup} 0$, and assume x_n does not converge to 0. Then there exists a convex open neighbourhood U of zero such that

$$x_n \notin U \quad \text{for any } n, \tag{6}$$

which implies $x_n \neq 0$.

Since B is bounded, there exists a $t > 0$ such that

$$tB \subset U.$$

By $0 \notin \mathrm{cl}(B)$ and the separate theorem, there exists $0 \neq f \in X^*$ such that

$$f(x) \geq 1 \quad \text{for any } x \in B. \tag{7}$$

Notice that the set $C_1 = \{x = \lambda b : b \in B, 0 \leq \lambda \leq t\}$ is a convex set and obviously $C_1 \subset U$ (because $tB \subset U$ and U is convex). Since $x_n = t_n b_n$, $t_n > 0$, $b_n \in B$, we have $t_n > t$ (if it is not true, let $t_n \leq t$, then $x_n = t_n b_n \in C_1 \subset U$, this contradicts (6)). By (7), we obtain

$$f(x_n) = t_n f(b_n) \geq t_n > t.$$

This is contradictory to $x_n \stackrel{w}{\rightharpoonup} 0$, i.e., $f(x_n) \to 0$.

Below, we construct its "expansion" cone for the cone C with base.

Now let B be a base of C, according to $0 \notin \mathrm{cl}(B)$ there exists a balanced convex open neighbourhood U^* of zero such that

$$U^* \cap B = \emptyset. \tag{8}$$

Let

$$N(0) = \{U \subset U^* : U \text{ is a balanced convex open neighbourhood of zero}\}. \tag{9}$$

For every $U \in N(0)$, notice that $B + U$ is convex, and then let

$$C_U = \mathrm{cl}(\mathrm{cone}(B + U)) = \mathrm{cl}\{\lambda x : \lambda \geq 0, x \in B + U\}. \tag{10}$$

It is clear that C_U is a closed convex cone.

Lemma 2. *There holds*

$$C/\{0\} \subset \mathrm{int}(C_U),$$

and C_U is a pointed cone when $U - U \subset U^*$.

Proof. Let $x \in C/\{0\}$. Then there exist $b \in B$, $\lambda > 0$, such that $x = \lambda b$. Thus

$$x + \lambda U = \lambda b + \lambda U \subset \lambda(B+U) \subset \mathrm{cone}(B+U) \subset C_U.$$

Therefore $x \in \mathrm{int}(C_U)$.

Now we prove that C_U is pointed. If it is not true, then there would exist an $x \neq 0$ such that $x \in C_U$ and $-x \in C_U$, and there would exist net

$$\lambda_\tau(b_\tau^1 + u_\tau^1) \to x, \quad \mu_\tau(b_\tau^2 + u_\tau^2) \to -x,$$

where b_τ^1, $b_\tau^2 \in B$, and u_τ^1, $u_\tau^2 \in U$, λ_τ, $\mu_\tau > 0$. So we obtain

$$\lambda_\tau(b_\tau^1 + u_\tau^1) + \mu_\tau(b_\tau^2 + u_\tau^2) \to 0.$$

Since

$$\begin{aligned}&\lambda_\tau(b_\tau^1 + u_\tau^2) + \mu_\tau(b_\tau^2 + u_\tau^2)\\ =&(\lambda_\tau + \mu_\tau)\left[\Big(\frac{\lambda_\tau}{\lambda_\tau+\mu_\tau}b_\tau^1 + \frac{\mu_\tau}{\lambda_\tau+\mu_\tau}b_\tau^2\Big) + \Big(\frac{\lambda_\tau}{\lambda_\tau+\mu_\tau}u_\tau^1 + \frac{\mu_\tau}{\lambda_\tau+\mu_\tau}u_\tau^2\Big)\right],\end{aligned}$$

we write

$$b_\tau = \frac{\lambda_\tau}{\lambda_\tau+\mu_\tau}b_\tau^1 + \frac{\mu_\tau}{\lambda_\tau+\mu_\tau}b_\tau^2 \in B,$$
$$u_\tau = \frac{\lambda_\tau}{\lambda_\tau+\mu_\tau}u_\tau^1 + \frac{\mu_\tau}{\lambda_\tau+\mu_\tau}u_\tau^2 \in U,$$

then $b_\tau + u_\tau \to 0$.

Since U is a neighbourhood of zero, there exists a τ_0 such that $b_{\tau_0} + u_{\tau_0} \in U$, or

$$b_{\tau_0} \in U - u_{\tau_0} \subset U - U \subset U^*.$$

Notice that $b_{\tau_0} \in B$, then $b_{\tau_0} \in B \cap U^*$; this is contradictory to (8).

Lemma 3. *Let the closed convex cone C be equipped with bounded base B and net $\{z_U \in C_U : U \in N(0)\}$. Then*

(i) $z_U \xrightarrow{w} 0$ *implies* $z_U \to 0$*;*

(ii) $z_U \xrightarrow{w} z \neq 0$ *implies* $z \in C$.

Proof. Since B is a bounded base of C, and $0 \notin \mathrm{cl}(B)$, by applying the separate theorem it is obtained that there exists $0 \neq f \in C^{\#}$ such that

$$f(b) \geq 1, \quad \forall b \in B. \tag{3}$$

For any $U \in N(0)$, by $z_U \in C_U$, there exists net $\lambda_\tau(b_\tau + u_\tau) \to z_U$, where $\lambda_\tau \geq 0$, $b_\tau \in B$, $u_\tau \in U$. Notice that $z_U + U$ is a neighbourhood of z_U. Then there exists index τ_U such that

$$\lambda_{\tau_U}(b_{\tau_U} + u_{\tau_U}) \in z_U + U \quad \text{for any } U \in N(0),$$

or there exists $w_U \in U$ such that

$$\lambda_{\tau_U}(b_{\tau_U} + u_{\tau_U}) = z_U + w_U \quad \text{for any } U \in N(0), \tag{11}$$

where $w_U \in U$.

It is obvious that two nets $\{u_{\tau_U} : U \in N(0)\}$ and $\{w_U : U \in N(0)\}$ all converge to 0.

Moreover, if net $\{z_U\}$ weakly converges, then net $\{z_U\}$ is bounded, and the number net $\{\lambda_{\tau_U} : U \in N(0)\}$ given by (11) is bounded. (If it is not true, without loss of generality supposing $\lambda_{\tau_U} \to \infty$, by (11) we obtain

$$b_{\tau_U} + u_{\tau_U} = \frac{z_U + w_U}{\lambda_{\tau_U}}.$$

Let $f \in C^{\#}$ be decided by (3). Then

$$f(b_{\tau_U}) = f(b_{\tau_U} + u_{\tau_U}) - f(u_{\tau_U}) = \frac{1}{\lambda_{\tau_U}}(f(z_U) + f(w_U)) - f(u_{\tau_U}).$$

Since $\{z_U\}$ is bounded, $\{f(z_U)\}$ is bounded and $f(w_U)$, $f(u_{\tau_U}) \to 0$. So

$$f(b_{\tau_U}) \to 0.$$

But on the other side, by (3) we obtain $f(b_{\tau_U}) \geq 1$, this is contradictory).

In the following we will prove respectively (i) and (ii).

(i) Since $\{\lambda_{\tau_U} : U \in N(0)\}$ is bounded, let $\lambda_{\tau_U} \to \lambda \geq 0$.

Assume $\lambda > 0$. Then according to (11), and $f(z_U) \to 0$ (notice $z_U \xrightarrow{w} 0$), $f(w_U)$, $f(u_{\tau_U}) \to 0$ ($\lambda_{\tau_U} \to \lambda > 0$), we obtain

$$f(b_{\tau_U}) \to 0;$$

this is contradictory to $f(b_{\tau_U}) \geq 1$.

So $\lambda = 0$, i.e., $\lambda_{\tau_U} \to 0$. Notice that $\{b_{\tau_U}\} \subset B$ is bounded and u_{τ_U}, $w_U \to 0$, by (11) we obtain

$$z_U = \lambda_{\tau_U}(b_{\tau_U} + u_{\tau_U}) - w_U \to 0.$$

(ii) Notice that $\lambda_{\tau_U} \to \lambda \geq 0$ when $z_U \xrightarrow{w} z \neq 0$, which implies certainly $\lambda > 0$. Otherwise if $\lambda = 0$, i.e., $\lambda_{\tau_U} \to 0$, then with the same proof as above, by (11) we obtain $z_U \to 0$, so $z_U \xrightarrow{w} 0$; this is contradictory to $z_U \xrightarrow{w} z \neq 0$.

Therefore $\lambda > 0$. Thus

$$b_{\tau_U} = \frac{z_U + w_U}{\lambda_{\tau_U}} - u_{\tau_U} \xrightarrow{w} \frac{z}{\lambda}, \text{ let } \frac{z}{\lambda} = b.$$

Since $\{b_{\tau_U}\} \subset B \subset C$ and C is a closed convex cone, C is a weakly closed convex cone too. So weakly limit $b = z/\lambda \in C$, or

$$z = \lambda b \in C.$$

Now we will prove the main theorem in this paper.

Theorem 2. *Let X be a Hausdorff locally convex space, $C \subset X$ be a closed convex cone with bounded base. Then for every nonempty weakly compact convex subset A, the set of positive proper efficient points is dense in the set of efficient points of A, i.e.,*

$$E(A, C) \subset \mathrm{cl}(PS(A)). \tag{12}$$

Proof. Let $A \neq \emptyset$ be a weakly compact convex set, $\bar{x} \in E(A, C)$, let

$$M = A - \bar{x}.$$

Then $O \in E(M, C)$, i.e., $(O - C) \cap M = \{0\}$, and M is still a weakly compact convex set. Moreover, let the closed convex cone C_U be defined by (10), i.e.,

$$C_U = \mathrm{cl}(\mathrm{cone}(B + U)), \quad \text{for any } U \in N(0).$$

For any $U \in N(0)$, let

$$M_U = (O - C_U) \cap M.$$

Then M_U is a weakly compact convex set (notice that C_U is weakly closed). By the existence theorem of efficient point (see [13]), there exists an efficient point in M_U. Let $X_U \in E(M_U, C_U)$. Since M_U is a section of A at 0, and the efficient point of the section M_U is the efficient point of M, we have $x_U \in E(M, C_U)$. Notice that C_U is a pointed cone. Then

$$(x_U - C_U) \cap M = \{x_U\} \quad \text{for any } U \in N(0).$$

Since M is a weakly compact set and net $\{x_U : U \in N(0)\} \subset M$, without loss of generality we can assume $x_U \xrightarrow{w} x_0 \in M$.

By $x_U \in M_U = (-C_U) \cap M$, i.e.,

$$x_U \in -C_U, \quad \text{or} \quad -x_U \in C_U, \quad \text{and} \quad -x_U \xrightarrow{w} -x_0.$$

According to Lemma 3 we obtain $-x_0 \in C$ or $x_0 \in -C$. As $x_0 \in M$, we have

$$x_0 \in (-C) \cap M = (O - C) \cap M = \{0\}.$$

Thus $x_0 = 0$, and we get

$$x_U \xrightarrow{w} 0, \text{ or } -x_U \xrightarrow{w} 0, \quad -x_U \in C_U.$$

Apply Lemma 3 (i) again to obtain

$$-x_U \to 0, \quad \text{i.e.,} \quad x_U \to 0.$$

Now we prove

$$x_U \in PS(M).$$

In fact, by Lemma 2 we have $\text{int}(C_U) \neq \emptyset$.

Since $x_U \in E(M, C_U)$, x_U is a weakly efficient point of M (see [13]). Then

$$(x_U - \text{int}(C_U)) \cap M = \emptyset.$$

According to the separate theorem of convex set, there exist $0 \neq f_U \in X^*$ and $t \in R$ such that

$$\begin{cases} f_U(x_U - C_U) \leq t \leq \inf f_U(M), \\ f_U(x_U - \text{int}(C_U)) < t \leq \inf f_U(M). \end{cases} \tag{13}$$

Since $x_U \in M$, by the second inequality of (13) we deduce that

$$f_U(x_U - \text{int}(C_U)) < t \leq \inf f_U(M) \leq f_U(x_U),$$

or

$$f_U(-\text{int}(C_U)) < 0, \text{ and } f_U(\text{int}(C_U)) > 0.$$

According to Lemma 2, $C/\{0\} \subset \text{int}(C_U)$. Then

$$f_U(C/\{0\}) > 0, \text{ i.e., } f_U \in C^{\#}.$$

By $O \in C_U$, from the first inequality of (13), we have

$$f_U(x_U) \leq \inf f_U(M),$$

i.e., $f_U(x_U) = \min\{f_U(x) : x \in M\}$. So x_U is a positive efficient point of M, i.e., $x_U \in PS(M)$.

At last, since $x_U \in M$, we have $x_U = y_U - \bar{x}$, $y_U \in A$. So from

$$f_U(x_U) \le f_U(x), \quad \text{for } x \in M = A - \bar{x},$$

we immediately deduce that

$$f_U(y_U) \le f_U(y), \quad \text{for any } y \in A,$$

i.e., y_U is a positive proper efficient point of A, $y_U \in PS(A)$. Secondly, by $x_U = y_U - \bar{x} \to 0$ we obtain $y_U \to \bar{x}$. So

$$\bar{x} \in \mathrm{cl}(PSA).$$

The proof is finished.

When X is a real normed space, the above Theorem 2 implies immediately the following theorem.

Theorem 3. *Let $(X, \|\cdot\|)$ be a real normed space, $C \subset X$ be a closed convex cone with bounded base. Then for every weakly compact convex set A in X, the set of positive proper efficient points of A is dense in the set of efficient points of A.*

The Theorem 3 was first obtained by Jahn (see [8]) for Bishop-Phelps cone. Soon, Petschke noticed the relation between the cone with bounded base and Bishop-Phelps cone, and he proved the Theorem 3 for the cone with bounded base.

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