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CLASSIFICATION OF AF-ALGEBRAS AND THEIR DIMENSION GROUPS**

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Abstract

This paper gives the intrinsic character of the classification for AF-algebras defined by J. Cuntz and G. K. Pedersen in terms of their dimension groups.

Keywords AF-algebra, Demension group, Classification.
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§0. Introduction

The classification problem of C^*-algebras is one of the most important and difficult problems in the study of C^*-algebras. For the complexity of its internal structure, the study on this problem did not make any great progress until now. But in the last two-decades, with the founding and developing of K-theory, Extension theory and KK-theory, one has paid more and more attention to this problem again.[1,2,3,4,5,6].

Usually, one divides C^*-algebras into liminary, postliminary and antiliminary types in terms of their representations. There are many works on these types of C^*-algebras[7,8]. In [9], similar to the case of von Neumann algebras, J. Cuntz and G. K. Pedersen have introduced another classification for C^*-algebras. In this paper, we shall study this classification for a special class of C^*-algebras, i. e., AF-algrbras.

This paper is organized as follows. In section one, we first recall the definition of the classification given by J. Cuntz and G. K. Pedersen, and then list some necessary facts about AF-algebras. Section two studies the relation between the traces on AF-algebras and the functionals on their dimension groups; in that section, we introduce the concept of generalized functionals which is correspondent to the traces on AF-algebras. The third section is the main part of this paper. We give the intrinsic characters of the classification for AF-algebras in terms of their dimension groups. We introduce some new concepts in dimension groups, as like as finite scale and archimedean elements. Our main results of this paper are Theorems 3.2, 3.5, 3.6, 3.7.

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§1. Preliminaries

1.1 The classification of C^*-algebras

In this section, we first recall the classification of C^*-algebras defined by J. Cuntz and G. K. Pedersen[9].

Let A denote a C^*-algebra, A_+ be the positive part of it. In A_+, we define the following equivalence relation:

If $x,\ y \in A_+$ and there exists a sequence of elements $\{z_n\}_{n=1}^{\infty}$ in A with

$$x = \sum_{n=1}^{\infty} z_n z_n^*, \quad y = \sum_{n=1}^{\infty} z_n^* z_n, \tag{1.1}$$

we say that x is equivalent to y, denoted by $x \sim y$, where the sum is in the norm sense. If $x,\ y \in A_+$ and there is a $z \in A_+$ with $x \sim z \leq y$, we denote it by $x \prec y$. By [10], we know that the relation "$\prec$" is transitive, and "$\sim$" is an equivalence relation, which is countably summable.

In the same way as the comparison of projections in von Neumann algebras, J. Cuntz and J. K. Pedersen have given the following definition in [9].

Definition 1.1. a) *an element x in A_+ is called finite, if for any $y \in A_+$ with $y \leq x$ and $y \prec x$, we have $x = y$;*

b) *If any element in A_+ is finite, we say that A is finite;*

c) *If for any nonzero element x in A_+, there always exists a nonzero finite element y in A_+ such that $y \leq x$, then we say that A is semi-finite.*

d) *If there exists no nonzero finite element in A_+, A is called pure-infinite* (III-*type C^*-algebras in* [9]).

For the separable C^*-algebras, the following result holds[9]:

Theorem A. *Suppose that A is a separable C^*-algebra. Then*

a) *A is a finite C^*-algebra iff there exists a finite faithful trace on A;*

b) *A is a semi-finite C^*-algebra iff there is a faithful, semi-finite, lower-semicontinous trace on A.*

A trace τ on A is said to be semi-finite, if for any nonzero $y \in A_+$ there is an $x \in A$ with $0 < x \leq y$ such that $\tau(x) < +\infty$.

1.2. Dimension groups

In this paper, we shall study the classification for AF-algebras defined above. For this purpose, we need list some elementary facts about AF-algebras and their dimension groups. For the details, one can consult the reference [11].

For a given AF-algebra A, there exists a dimension group $(G(A), G(A)_+)$ and a scaled dimension group $(G(A), G(A)_+, \Gamma(A))$ with it. A dimension group is a countably ordered group (G, G_+) with the Riesz interpolation property, i. e., for any $a_i, b_i \in G_+$ $(i = 1, 2)$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there is a $c \in G_+$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$.

For an ordered group (G, G_+), there are some properties which are equivalent to the Riesz interpolation property.

Theorem B.[12] *Let (G, G_+) be an ordered group. Then the following are equivalent:*

(1) *Given $a_i \leq b_j$ $(i, j = 1, 2)$, there exists a $c \in G$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$;*

(2) *Given* $a_i \leq b_j$ $(i = 1,2,\cdots,r;\ j = 1,2,\cdots,s)$, *there exists a* $c \in G$ *such that* $a_i \leq c \leq b_j$ *for all* $1 \leq i \leq r, 1 \leq j \leq s$;

(3) *If* $0 \leq a \leq b_1 + b_2 + \cdots + b_s$, $b_i \geq 0$, *then there exist* $a_i \in G$ *with* $0 \leq a_i \leq b_i$ *such that* $a = a_1 + a_2 + \cdots + a_s$;

(4) *If* $\sum\limits_{i=1}^{r} a_i = \sum\limits_{j=1}^{s} b_j, a_i, b_j \geq 0$, *then there exist* $c_{ij} \in G_+$ *such that*

$$a_i = \sum_{j=1}^{s} c_{ij}, \quad b_j = \sum_{i=1}^{r} c_{ij}.$$

A scale for a dimension group $(G,\ G_+)$ is a subset Γ of G_+ with the following three properties;

Γ_1: (generating) For any $a \in G_+$, there exist $a_1,\ a_2, \ldots a_r \in \Gamma$ with $a = \sum\limits_{i=1}^{r} a_i$;

Γ_2: (hereditary) If $a \in \Gamma, b \in G_+$ and $b \leq a$, then $b \in \Gamma$;

Γ_3: (directed) For any $a, b \in \Gamma$, there exists a $c \in \Gamma$ with $a, b \leq c$.

In fact, the scaled dimension group of an AF-algebra A is exactly the scaled preordered K_0-group[13,11] $(K_0(A), K_0(A)_+, \Gamma_0(A))$, where $K_0(A)_+$, $\Gamma_0(A)$ are respectively the images of $\mathrm{Proj}(M_\infty(A))$, $\mathrm{Proj}(A)$ in $K_0(A)$. Therefore, we can denote the element in $\Gamma(A)$ by $[p]$, where $[p]$ denotes the equivalence class of the projection p.

We call a group homomorphism φ from a dimension group (G, G_+) to R (the real numbers) with $\varphi(G_+) \subseteq R_+$ (positive real numbers) a functional on G.

For the traces on A and the functionals on G, we have

Theorem C.[14] *Let A be an AF-algebra, $(G(A), G(A)_+)$ be its dimension group. Then the functional φ on $(G(A), G(A)_+)$ is one-to-one correspondent to the densely defined, lower-semicontinuous trace τ on A with $\tau(p) = \varphi([p])$.*

§2. The Classification of AF-Algebras and the Functionals on Their Dimension Groups

In this section, we shall study the relations between the types of AF-algebras and the functionals on their dimension groups.

Definition 2.1. *Let (G, G_+, Γ) be a scaled dimension group, f be a functional on G. If f satisfies*

$$\sup\{f(a) |\ a \in \Gamma\} < +\infty, \tag{2.1}$$

it is called a bounded functional on G.

By Theorem C, we obtain immediately the following lemma.

Lemma 2.1. *Let A be an AF-algebra, $(G(A), G(A)_+, \Gamma(A))$ be its scaled dimension group. Then the bounded functionals on $G(A)$ are one-to-one correspondent to the finite traces on A.*

Lemma 2.2. *Let A be an AF-algebra, τ be a densely defined, lower-semicontinuous trace on A. If τ_* is its correspondent functional on $G(A)$, then τ is faithful iff τ_* is faithful on $G(A)$.*

Proof. the necessity is obvious, we only need to prove the sufficiency.

If τ is not faithful on A, set $J = \{x \in A : \tau(x^*x) = 0\}$, then $J \neq \{0\}$. Since τ is lower-semicontinuous, J is a closed two-sided ideal of A, it is an AF-algebra too. Therefore, there is a nonzero projection $p \in J \subseteq A$, and $\tau_*([p]) = \tau(p) = 0$. Since τ_* is faithful on $G(A)$, it follows that $[p] = 0$, i.e., $p = 0$, a contradiction.

By Theorem A and the above lemmas, we have

Proposition 2.1. *Let A be an AF-algebra. Then A is finite iff there exists a faithful bounded functional on $G(A)$.*

Definition 2.2. *Suppose that (G, G_+) is a dimension group, J is a subgroup of G. If $J = J_+ - J_+$, where $J_+ = J \bigcap G_+$, and for any $a \in G_+, b \in J_+$ with $a \leq b$, we have $a \in J_+$, then we call J an (ordered) ideal of G.*

Lemma 2.3. *Suppose that (G, G_+) is a dimension group, J_1, J_2 are two ideals of G. Then $J_1 + J_2$ is an ideal of G, and $(J_1 + J_2)_+ = J_{1+} + J_{2+}$. Moreover, if $J_{1+} \bigcap J_{2+} = \{0\}$ (in this case, we say that J_1 is perpendicular to J_2 or J_1 is disjoint from J_2), then every element in $J_1 + J_2$ has only a unique decomposition form.*

Proof. $J = J_1 + J_2$ is obviously a subgroup of G. Set $J_+ = (J_1 + J_2) \bigcap G_+$, then it is obvious that $J_{1+} + J_{2+} \subseteq J_+$.

Let $a \in J_+$. Then $a = a_1 + a_2$, with $a_i \in J_i$ $(i = 1, 2)$. Since J_i are ideals of G, there exist $a_i' \in J_{i+}$ such that $a_i' \geq a_i$. Then $a = a_1 + a_2 \leq a_1' + a_2'$. By Theorem B, there exist $a_i'' \in G_+$ such that $a_i'' \leq a_i'$, and $a = a_1'' + a_2''$. By the property Γ_2, we have $a_i'' \in J_{i+}$, i.e., $a = a_1'' + a_2'' \in J_{1+} + J_{2+}$. So we have proved that

$$J_+ = J_{1+} + J_{2+}. \tag{2.2}$$

Then it follows evidently that $J = J_+ - J_+$. If $b \in G_+$, and $b \leq a$ with $a \in J_+$, by the above equality we can decompose a as $a = a_1 + a_2$ with $a_i \in J_{i+}$. Therefore, $b = b_1 + b_2, 0 \leq b_i \leq a_i \in J_{i+}$, so $b_i \in J_{i+}$, i.e., $b \in J_+$. Thus J is an ideal of G.

If J_1 is perpendicular to J_2, for $a \in J_1 \bigcap J_2$ there exist $a_i^+, a_i^- \in J_{i+}$ such that $a = a_i^+ - a_i^-$ $(i = 1, 2)$, therefore $a = a_1^+ - a_1^- = a_2^+ - a_2^-$, i.e., $a_1^+ + a_2^- = a_1^- + a_2^+$. By Theorem B, there exist $c_{ij} (i, j = 1, 2) \in G_+$ such that

$$\begin{aligned} a_1^+ &= c_{11} + c_{12}, \qquad & a_1^- &= c_{11} + c_{21}, \\ a_2^- &= c_{21} + c_{22}, \qquad & a_2^+ &= c_{12} + c_{22}. \end{aligned} \tag{2.3}$$

Thus $c_{12} \leq a_1^+ \in J_{1+}$, and $c_{12} \leq a_2^+ \in J_{2+}$. It follows that $c_{12} = 0$ by the assumption $J_{1+} \cap J_{2+} = \{0\}$. Similarly, we have $c_{21} = 0$. Consequently, $a_2^- = c_{22} = a_2^+, a_1^+ = c_{11} = a_1^-$, so $a = 0$, i.e., $J_1 \cap J_2 = \{0\}$.

Now suppose that $a \in J_1 + J_2$, and it has the decompositions $a = a_1 + a_2 = a_1' + a_2', a_i, a_i' \in J_i$. Then $a_1 - a_1' = a_2' - a_2 \in J_1 \cap J_2$. By the above argument, we have $a_1 = a_1', a_2 = a_2'$.

This lemma can be generalized to the countable case.

Lemma 2.4. *Suppose that (G, G_+) is a dimension group, $\{J_i\}_{i=1}^N$ (N may be equal to ∞) are ideals of G. Then $J_1 + J_2 + \cdots + J_N$ (algebraical sum) is also an ideal of G, and*

$$J_+ = (J_1 + J_2 + \cdots + J_N)_+ = J_{1+} + J_{2+} + \cdots + J_{N+}.$$

Moreover, if J_i are pairwise disjoint ideals, then every element in $J_1 + J_2 + \cdots + J_N$ has only a unique decomposition form.

The proof is the same as the above lemma, we omit the details.

Definition 2.3. *Let (G, G_+) be a dimension group. If φ is a map from G_+ into $R_+ \cup \{+\infty\}$ and it satisfies $\varphi(a+b) = \varphi(a) + \varphi(b)$, $\forall a,\ b \in G_+$, it is called a generalized functional on G. Moreover, if for any $b \in G_+ \setminus \{0\}$ there always exists an $a \in G$ with $0 < a \leq b$ such that $\varphi(a) < +\infty$, then φ is called a semi-finite generalized functional.*

Let φ be a generalized functional on G, set $\text{Dom}^+\varphi = \{a \in G_+ |\ \varphi(a) < +\infty\}$. Then $J = \text{Dom}^+\varphi - \text{Dom}^+\varphi$ is an ideal of G, and φ can be extended to a functional on J.

If τ is a lower-semicontinuous trace on A_+, then $\text{Dom}\ (\tau) = \{x |\ \tau(x^*x) < +\infty\}$ is a two-sided ideal of A (non-closed). Obviously, we can extend τ to a densely defined, lower-semicontinuous trace on $\overline{\text{Dom}\,(\tau)} = I$. Thus a functional τ'_* on $J = G(I)$ can be induced, where $J = G(I)$ is an ideal of $G(A)$. If we set

$$J_+^{\perp} = \{a \in G(A)_+ :\ \text{there is no nonzero element } b \text{ in } J_+ \text{ with } b \leq a\}. \tag{2.4}$$

Then $J^{\perp} = J_+^{\perp} - J_+^{\perp}$ is an ideal of $G(A)$, and it is perpendicular to J.

By Lemma 2.3, we can extend the functional τ'_* to $J + J^{\perp}$ by the following equality, which we still denote by τ'_*,

$$\tau'_*(a+b) = \begin{cases} +\infty, & \text{if } b \neq 0, \\ \tau'_*(a), & \text{if } b = 0, \end{cases} \tag{2.5}$$

where $a + b \in (J + J^{\perp})_+$, $a \in J_+, b \in J_+^{\perp}$. It is apparent that $J + J^{\perp}$ is an essential ideal of $G(A)$, i. e., any nonzero ideal of $G(A)$ has a nontrivial intersection with $J + J^{\perp}$. Thus, τ'_* may be extended to a generalized functional τ_* on $G(A)$ by the following equality

$$\tau_*(a) = \sup\{\tau'_*(b) \mid 0 \leq b \leq a,\ b \in (J + J^{\perp})_+\}, \quad \forall a \in G(A)_+. \tag{2.6}$$

It is easy to verify that τ_* is a generalized functional on $G(A)$, and by the construction we know that if the trace τ is faithful on A, then τ_* is faithful on $G(A)$.

Conversely, if a faithful generalized functional φ on $G(A)$ has been given, we will construct a faithful lower-semicontinuous trace on A. We have already known that $J = \text{Dom}\varphi$ is an ideal of $G(A)$, where $\text{Dom}^+\varphi = \{a \in G(A)_+ |\ \varphi(a) < \infty\}$. Because there exists a one-to-one correspondence relation between the ideals of AF-algebras and the ideals of the dimension groups, there exists an ideal I of A such that $G(I) = J$. By Theorem C, we have a densely defined, faithful and lower-semicontinuous trace τ' on I, such that $\tau'_* = \tau|_J$. It is easy to verify that $\text{Dom}(\tau') = \{x \in I :\ \tau'(x^*x) < +\infty\}$ is not only a two-sided ideal of I, but also an ideal of A.

Define a function $s :\ \text{Dom}(\tau') \times \text{Dom}(\tau') \to \mathbb{C}$ by

$$s(x,\ y) = \tau'(y^*x), \quad \forall x,\ y \in \text{Dom}(\tau'). \tag{2.7}$$

Obviously, it satisfies

1) s is linear in the first variable, and conjugated linear in the second variable, moreover, $s(x,\ x) \geq 0$. For this reason, $\text{Dom}(\tau')$ can be considered as an inner product space;

2) $s(y,\ x) = s(x^*,\ y^*)$;

3) $s(zx,\ y) = s(x,\ z^*y)$, $\quad \forall z \in A$;

4) $\forall z \in A$, the map $x \to zx$ defined on $\text{Dom}(\tau')$ is continuous, where $\text{Dom}(\tau')$ is considered as an inner product space.

5) $\{xy :\ x \in \text{Dom}(\tau'), y \in \text{Dom}(\tau')\}$ is dense in the inner product space $\text{Dom}(\tau')$.

In fact, if there is $z \in \mathrm{Dom}(\tau')$, and $z \perp \{xy : x,\ y \in \mathrm{Dom}(\tau')\}$, then

$$s(z,\ xy) = \tau'(y^*x^*z) = 0, \quad \forall x,\ y \in \mathrm{Dom}(\tau'). \tag{2.8}$$

Since τ' is faithful on $\mathrm{Dom}(\tau')$, letting $y = x^*z$, we have $x^*z = 0$, $\forall x \in \mathrm{Dom}(\tau')$, so $z = 0$.

All these facts state that s is a bitrace on A[7]. Therefore, there exists a lower-semi-continuous trace τ'' on A, which is an extension of τ'.

Let $I^\perp = \{x \in A : xy = 0, \forall y \in I\}$. Then $I^\perp$ is a two-sided ideal of A, $I + I^\perp$ is an essential ideal of A, and every element in $I+I^\perp$ has only a unique decomposition. Moreover $(I+I^\perp)_+ = I_+ + I_+^\perp$. Now, we give a modification for the trace τ'' on A as follows:

$$\tau(x) = \begin{cases} +\infty, & \text{if there is a } y \in I_+^\perp \setminus \{0\} \text{ with } y \le x, \\ \tau''(x), & \text{if there is no nonzero } y \in I_+^\perp \text{ such that } y \le x, \end{cases} \tag{2.9}$$

where $x \in A_+$. It is easy to verify that τ is homogeneous (multiplies with positive number) and additive. In the following, we shall prove the equality

$$\tau(xx^*) = \tau(x^*x), \qquad \forall x \in A. \tag{2.10}$$

If there exists a $z \le x^*x$, $z \in I_+^\perp \setminus \{0\}$, setting $y = xz^{1/2}$, since $z^{1/2}zz^{1/2} \le z^{1/2}x^*xz^{1/2} = y^*y$, we have $y \ne 0$ and $y \in I^\perp$. Noticing $yy^* = xz^{1/2}z^{1/2}x^* \le \|z\|xx^*$, we see that there exists a nonzero $z' \in I_+^\perp$ such that $z' \le xx^*$. And the converse is also true. From these statements we have proved the equality (2.10), it implies that τ is a trace on A. What remains to be proved is that τ is lower semi-continuous.

Suppose that $x_n \in A_+$ such that $\tau(x_n) \le 1$ and $x_n \to x$. We need to prove that $\tau(x) \le 1$.

Since $\tau(x_n) \le 1$, there exists no nonzero $y \in I_+^\perp$, $\forall n \in N$ such that $y \le x$. So we have $\tau(x_n) = \tau''(x_n) \le 1$. Because τ'' is lower semicontinuous, we have $\tau''(x) \le 1$.

By $x \in A_+$, there is $y = y_1 + y_2 \in (I+I^\perp)_+$, where $y_1 \in I_+, y_2 \in I_+^\perp$ such that $y = y_1 + y_2 \le x$. Since $x_n \to x$, $\forall \varepsilon > 0$ there is an $n_0 \in N$ such that, when $n \ge n_0$, we have $x \le x_n + \varepsilon$. Then

$$y_1 + y_2 \le x_n + \varepsilon, \qquad \forall n \ge n_0. \tag{2.11}$$

By [8], there exist $z_{ij}^n, i, j = 1, 2$, such that

$$\begin{aligned} y_1 &= z_{11}^{n\,*} z_{11}^n + z_{12}^{n\,*} z_{12}^n, & x_n &\ge z_{11}^n z_{11}^{n\,*} + z_{21}^n z_{21}^{n\,*}, \\ y_2 &= z_{21}^{n\,*} z_{21}^n + z_{22}^{n\,*} z_{22}^n, & \varepsilon &\ge z_{12}^n z_{12}^{n\,*} + z_{22}^n z_{22}^{n\,*}, \end{aligned} \tag{2.12}$$

so $z_{21}^{n\,*} z_{21}^n \le y_2$. It follows that $z_{21}^n \in I^\perp$, and therefore $z_{21}^n z_{21}^{n\,*} \in I_+^\perp$. Notice that $z_{21}^n z_{21}^{n\,*} \le x_n$, but there exists no nonzero element in $I_+^\perp$ smaller than x_n, so we have $z_{21}^n = 0$. Thus

$$\|y_2\| = \|z_{22}^{n\,*} z_{22}^n\| = \|z_{22}^n z_{22}^{n\,*}\| \le \varepsilon, \tag{2.13}$$

it implies that $y_2 = 0$. Consequently, there exists no nonzero $y \in I_+^\perp$ such that $y \le x$. By the definition of τ, we have

$$\tau(x) = \tau''(x) \le 1. \tag{2.14}$$

Combining all the above statements, we have proved the following theorem.

Theorem 2.1. *Suppose that A is an AF-algebra. Then*

(1) *If τ is a faithful, lower-semicontinuous trace on A, then there exists a faithful generalized functional φ.*

(2) *If φ is a faithful, generalized functional on $G(A)$, then there is a faithful, lower-semicontinuous trace τ such that $(\tau|_{\mathrm{Dom}(\tau)})_* = \varphi|_{\mathrm{Dom}\varphi}$.*

Theorem 2.2. *Let A be an AF-algebra. Then*

(1) *A is a semi-finite AF-algebra iff there exists a faithful, semi-finite generalized functional on $G(A)$.*

(2) *A is a pure-infinite AF-algebra iff there are not any nontrivial, faithful generalized functionals on $G(A)$.*

Proof. (1) Necessity: If A is semi-finite, there exists by Theorem A a faithful, semi-finite and lower-semicontinuous trace τ on A. According to Theorem 2.1 a generalized functional φ on $G(A)$ can be induced by the trace τ. It is faithful and semi-finite by its construction.

Sufficiency: If there exists a faithful, semi-finite generalized functional φ on $G(A)$, it has a faithful, lower-semicontinuous trace τ on A by Theorem 2.1, which is correspondent to φ. Since φ is semi-finite, Domφ is an essential ideal of $G(A)$, thus $I^{\perp} = \{0\}$. Therefore, $\tau = \tau''$ is semi-finite.

(2) Necessity: If there exists a nontrivial faithful generalized functional φ on $G(A)$, then there exists a nontrivial faithful, lower-semicontinuous trace τ on A by Theorem 2.1. It is obvious that every nonzero element in $\mathrm{Dom}^+(\tau)$ $(\neq 0)$ is a finite element of A, which contradicts the assumption.

Sufficiency: If there exists a nonzero finite element $x \in A_+$, then the hereditary C^*-subalgebra B of A generated by x is a finite AF-algebra[9]. It implies that there exists a faithful finite trace τ on B by Theorem A, and it can be extended to a lower-semicontinuous trace τ' on A by [9]. By the construction method in the proof of Theorem 2.1, we can construct a faithful lower-semicontinuous trace on A which is an extension of τ'. It follows by Theorem 2.1 that there is a nontrivial faithful generalized functional on $G(A)$, a contradiction.

§3. The Classification of the Dimension Groups

In this section, we shall give the character of the classification for AF-algebras in terms of their dimension groups.

Definition 3.1. *Let (G, G_+, Γ) be a scaled dimension group.*

(1) *If $a \in \Gamma$, and for any $n \in N$, we always have $na \in \Gamma$, then a is called a stable element of Γ. Otherwise, if there exists an $n_0 \in N$ such that $n_0 a \notin \Gamma$, a is called a finite element of Γ.*

(2) *If every nonzero element in Γ is finite, we say that Γ is finite; if every element in Γ is stable, Γ is called stable.*

Let Γ_s denote the set of all stable elements in Γ. We have

Proposition 3.1. *Γ_s is a hereditary cone in Γ, i. e., $J_s = \Gamma_s - \Gamma_s$ is an ideal of G.*

The proof is easy. We omit the details.

Corollary 3.1. *Γ is a stable scale iff $\Gamma = G_+$.*

Corollary 3.2. *If A is an AF-algebra, then A is stable (i. e., $A \cong A \otimes K$ where K is the set of all compact operators in a separable Hilbert space) iff the scale $\Gamma(A)$ is stable.*

In the following, we will give the character of the finite AF-algebras. For this purpose, we need the Hahn-Banach's type theorem of K. Goodearl and D. Handelman[15].

Let G be a dimension group, $u \in G_+$ be an order unit of G (i.e., $\forall a \in G_+$, there exists $n \in N$ such that $a \leq nu$. In this case, the set $[u] = \{a : 0 \leq a \leq u\}$ is a scale of G, which is called a unit scale). If $a \in G_+$, set

$$f_*(a) = \sup\{h/m : h \geq 0, m > 0, hu \leq ma\}, \tag{3.1}$$

$$f^*(a) = \inf\{k/n : k > 0, n > 0, na \leq ku\}. \tag{3.2}$$

Theorem 3.1.[15] *Let G be a dimension group, u be an order unit of G and a be an element of G_+. Then*

(1) $0 \leq f_*(a) \leq f^*(a) \leq +\infty$;

(2) *If f is a functional on G with $f(u) = 1$, then $f_*(a) \leq f(a) \leq f^*(a)$;*

(3) *Given a number r with $f_*(a) \leq r \leq f^*(a)$, then there exists a functional f on G such that $f(u) = 1$, $f(a) = r$.*

Lemma 3.1. *Suppose that (G, u) is a dimension group with an order unit u. If $a \in G_+$ is a finite element in the unit scale $[u]$, then there exists a functional f on G such that $f(u) = 1, f(a) > 0$.*

Proof. By Theorem 3.1, it is enough to prove that $f^*(a) > 0$.

In fact, if $f^*(a) = 0$, then for all $m \in N$ there exist $k > 0$, $n > 0$ such that $k/n < 1/m$ and $na \leq ku$ by the equality (3.2). Therefore,

$$mna \leq mku \leq nu, \qquad \forall m \in N. \tag{3.3}$$

Thus $ma \leq u$, $\forall m \in N$; this implies that a is a stable element in $[u]$, which contradicts the assumption.

Proposition 3.2. *Let (G, G_+, Γ) be a scaled dimension group. Then for every finite element $a_0 \in \Gamma$, there exists a bounded functional φ on G such that $\varphi(a_0) > 0$.*

Proof. By [11, Proposition 7.2], we can embed the dimension group (G, G_+, Γ) into a unital dimension group $G^1 = G \oplus \mathbb{Z}$, the scale of G^1 is given by the following

$$\Gamma(G^1) = \{(a, 0) : a \in \Gamma\} \bigcup \{(-a, 1) : a \in \Gamma\} \tag{3.4}$$

and G^1_+ is generated by $\Gamma(G^1)$. It is easy to verify that $u = (0, 1)$ is an order unit of G^1 and $\Gamma(G^1) = [u]$. The embedding map of G into G^1 is given by

$$i : a \to (a, 0) \qquad \forall a \in G. \tag{3.5}$$

It is evident that an element in Γ is finite iff its image is finite in $\Gamma(G^1)$. This implies that $(a_0, 0)$ is a finite element in $\Gamma(G^1)$. By the above lemma, we obtain a functional f on G^1 such that $f(u) = 1$ and $f((a_0, 0)) > 0$.

Let $\varphi(a) = f(i(a)) = f((a, 0))$. Then φ is a bounded functional on G and $\varphi(a_0) > 0$, which completes the proof.

Theorem 3.2. *Let (G, G_+, Γ) be a scaled dimension group. Then the following statements are equivalent:*

(1) *Γ is a finite scale;*

(2) *For every $a \in \Gamma \setminus \{0\}$, there exists a bounded functional f on G such that $f(a) > 0$;*

(3) *There exists a faithful bounded functional φ on G.*

Proof. 1) $\Longrightarrow$ 2) follows immediately from the above proposition.

2) $\Longrightarrow$ 3) Since Γ is a countable set, it can be written as $\Gamma \setminus \{0\} = \{a_1, a_2, \cdots\}$. By the assumption, for every $a_n \in \Gamma \setminus \{0\}$, there exists a bounded functional f_n such that $f_n(a_n) > 0$ and $||f_n|| = \sup\{f_n(a) : a \in \Gamma\} = 1$. Set

$$\varphi(a) = \sum_{n=1}^{\infty} \frac{f_n(a)}{2^n}. \tag{3.6}$$

It is apparent that φ is a faithful bounded functional on G.

3) $\Longrightarrow$ 1) If Γ is not finite, then there is an $a \in \Gamma \setminus \{0\}$ such that $na \in \Gamma, \forall n \in N$. So

$$n\varphi(a) = \varphi(na) \le ||\varphi|| = \sup\{\varphi(b) : b \in \Gamma\} < +\infty, \qquad \forall n \in N. \tag{3.7}$$

It follows that $\varphi(a) = 0$, but φ is faithful, we have $a = 0$, a contradiction.

Combining the above theorem and Proposition 2. 1, we can get

Theorem 3.3. *Let A be an AF-algebra. Then A is finite iff the scale $\Gamma(A)$ of its scaled dimension group $(G(A), G(A)_+, \Gamma(A))$ is finite.*

This result obviously generalizes the corresponding results in [2, 9, 16].

Definition 3.2. *Suppose that (G, G_+) is a dimension group.*

(1) *If $b \in G_+$, and for every nonzero element a in G_+ there always exists a positive integer number n_0 such that the inequality $n_0 a \le b$ does not hold, then b is called an archimedean element of G;*

(2) *If $b \in G_+ \setminus \{0\}$, and for every $a \in G_+ \setminus \{0\}$ with $a \le b$ there always exists $c \in G_+ \setminus \{0\}$ such that $nc \le a$, $\forall n \in N$, then b is called pure-infinite;*

(3) *If all elements in G_+ are archimedean, then G is called archimedean; if there are no nonzero archimedean elements in G_+, then G is called pure-infinite, in this case, every nonzero element in G_+ is pure-infinite.*

Remark. These concepts and the concept of finite scale in Definition 3. 1 are first introduced in this paper, where the concept of the archimedean elements is transferred from the theory of Riesz space[17], but it is different from the above.

Let P_+ denote the set of all pure-infinite elements in G_+. Then

Proposition 3.3. *P_+ is a hereditary cone of G_+, i. e, $P = P_+ - P_+$ is an ideal of G.*

The proof is simple, we omit the details.

Proposition 3.4 *If the dimension group (G, G_+) has an order unit u, then the scale $[u]$ generated by u is finite iff G is archimedean.*

Corollary 3.3. *If the dimension group (G, G_+) is finitely generated and G is archimedean, then G has a finite scale.*

By Theorem 3. 2, we know that if a dimension group has a finite scale, it must be archimedean, since in this case it has a faithful functional.

Lemma 3.2. *Let (G, G_+, Γ) be a scaled dimension group. If every element in Γ is archimedean, then G is itself archimedean.*

Proof. Let $a, b \in G_+$, and $na \le b$, $\forall n \in N$.

Since $b \in G_+$, there exist $b_1, b_2, \ldots, b_r \in \Gamma$ such that $b = \sum_{i=1}^{r} b_i$. By the directed property of the scale (Γ_3), there exists $b_0 \in \Gamma$ such that $b_i \le b_0$, $1 \le i \le r$. Therefore

$$b = b_1 + b_2 + \cdots + b_r \le rb_0 \tag{3.8}$$

and $nra \leq b \leq rb_0$. It follows that $na \leq b_0$, $\forall n \in N$. By the assumption, we have $a = 0$. Consequently, b is archimedean, which completes the proof.

Theorem 3.4. *If A is a liminal AF-algebra, then $(G(A), G(A)_+)$ is archimedean.*

Proof. By the above lemma, it is enough to prove that every element in $\Gamma(A)$ is archimedean.

Since A is liminal, for every irreducible representation (π, H) of A we have $\pi(A) = C(H)$, where H is a separable Hilbert space and $C(H)$ is the set of all compact operators on H. Moreover, for every $p \in \mathrm{Proj}(A) \setminus \{0\}$, there exists an irreducible representation (π_p, H_p) of A such that $\pi_p(p) \neq 0$.

If $\Gamma(A)$ is not archimedean, then there are $a, b \in \Gamma(A), a \neq 0$, such that $na \leq b\ \forall n \in N$. Since $a \in \Gamma(A)$, there exists $p \in \mathrm{Proj}(A)$ such that $[p] = a$. Take an irreducible representation (π, H) of A such that $\pi(p) \neq 0$. Notice that the representation $\pi: A \longrightarrow C(H)$ may induce a homomorphism π_* between the K_0-groups

$$\pi_*: K_0(A) \longrightarrow \boldsymbol{Z}. \tag{3.9}$$

It is apparent that $\pi_*([p]) = [\pi(p)] \neq 0$, in contrast we have

$$\pi_*(n[p]) = n\pi_*([p]) = n\pi_*(a) \leq \pi_*(b), \qquad \forall n \in N. \tag{3.10}$$

Since $(\boldsymbol{Z}, \boldsymbol{Z}_+)$ is archimedean, we have $\pi_*(a) = 0$, a contradiction.

We have proved that an AF-algebra A is semi-finite iff there is a faithful, semi-finite generalized functional φ on G(A). If we set $\mathrm{Dom}^+\varphi = \{a \in G(A)_+ : \varphi(a) < \infty\}$, $\mathrm{Dom}\varphi = \mathrm{Dom}^+\varphi - \mathrm{Dom}^+\varphi$, then $\mathrm{Dom}\varphi$ is an essential ideal of G and φ can be extended to a functional on $\mathrm{Dom}\varphi$. Since φ is faithful on $\mathrm{Dom}\varphi$, $\mathrm{Dom}\varphi$ is archimedean. Thus, if A is semi-finite, then $G(A)$ has an archimedean essential ideal. In the following, we shall prove that the converse of this statement is also true.

Lemma 3.3. *Let (G, G_+) be a dimension group, $a_1, a_2 \in G_+$. If the ideals J_{a_1}, J_{a_2} generated by a_1, a_2 are disjoint, then we call a_1 disjoint from a_2 and denote it by $a_1 \perp a_2$. Set $[a] = \{x \in G_+ : x \leq a\}$. Then $a_1 \perp a_2$ iff $[a_1] \cap [a_2] = \{0\}$.*

Proof. The necessity is obvious.

Sufficiency. If there exists a $c \in J_{a_1+} \bigcap J_{a_2+}$ with $c \neq 0$, then there are n_1, $n_2 \in N$ such that $c \leq n_1 a_1$, $n_2 a_2$. Let $n = \max\{n_1, n_2\}$. Then we have $c \leq na_1$, na_2 and there exist by Theorem B c_i^1, $c_i^2 \in G_+$, $i = 1, 2, \ldots, n$ such that

$$c = \sum_{i=1}^{n} c_i^1 = \sum_{i=1}^{n} c_i^2, \quad \text{and } c_i^1 \leq a_1,\ c_i^2 \leq a_2,\ i = 1, 2, \cdots, n. \tag{3.11}$$

By Theorem B again, we get $c_{ij} \in G_+$ such that

$$c_i^1 = \sum_{j=1}^{n} c_{ij}, \quad c_j^2 = \sum_{i=1}^{n} c_{ij}, \qquad i, j = 1, 2, \cdots n. \tag{3.12}$$

Since $c \neq 0$, there is at least a pair subscripts i_0, j_0, $1 \leq i_0$, $j_0 \leq n$ such that $c_{i_0 j_0} \neq 0$.

But

$$c_{i_0j_0} \le \sum_{j=1}^{n} c_{i_0j} = c^1_{i_0} \le a_1, \tag{3.13}$$

$$c_{i_0j_0} \le \sum_{i=1}^{n} c_{ij_0} = c^2_{j_0} \le a_2, \tag{3.14}$$

which is impossible by the assumption.

Theorem 3.5. *If A is an AF-algebra, then the following statements are equivalent:*

(1) *A is semi-finite;*

(2) *There exists a faithful, semi-finite generalized functional on $G(A)$;*

(3) *$G(A)$ contains an essential archimedean ideal;*

(4) *There is an essential ideal with a finite scale in $G(A)$.*

Proof. 1)⇔ 2)⇒ 3) have been proved above.

3)⇒ 4) Let J be the essential archimedean ideal of $G(A)$. Notice the fact that if J' is an essential ideal of J, then J' itself is an essential ideal of $G(A)$. Thus we can immediately suppose that $G(A)$ is an archimedean dimension group.

Since $G(A)_+$ is countable, we can write $G(A)_+ \setminus \{0\}$ as

$$G(A)_+ \setminus \{0\} = \{a_1,\ a_2,\ \dots\ a_n, \dots\}. \tag{3.15}$$

Let $b_1 = a_1$. If the other elements in $G(A)_+ \setminus \{0\}$ are joint with b_1, then we let the set m be equal to $\{b_1\}$. Otherwise, we take the first element a_{n_1} in $G(A)_+ \setminus \{0\}$ which is disjoint from b_1, denote it by b_2. Following this method, we can construct a subset m which is composed of pairwise disjoint elements in $G(A)_+ \setminus \{0\}$, and any element in $G(A)_+ \setminus \{0\}$ is joint with m. Let $m = \{b_1,\ b_2,\ \dots\}$. By Lemma 3. 3, we know that the ideals of G generated by b_i are pairwise disjoint, J_{b_i} denoted by J_i. Construct the algebraical sum $J = \sum J_i$, then J is an essential ideal of G. By Lemma 2. 4, we know that every element in J has only a unique decomposition form. From these facts, it is easy to verify that the set

$$\Gamma = \{\sum c_i :\ c_i \le b_i,\ \ \text{and it has only finitely many } c_i \ne 0\} \tag{3.16}$$

is a finite scale of J.

4)⇒ 2) Let J be an essential ideal of G, which has a finite scale. Then there exists a faithful functional $\tilde{\varphi}$ on J. Define

$$\varphi(a) = \sup\{\tilde{\varphi}(b) :\ b \in J_+,\ b \le a\}, \qquad \forall a \in G(A)_+. \tag{3.17}$$

Then it is easy to prove that φ is a faithful, semi-finite generalized functional on $G(A)$.

Definition 3.3. *Let (G, G_+) be a dimension group, $a \in G_+ \setminus \{0\}$. If there is $b \in G_+ \setminus \{0\}$ such that $b \le a$, and b is archimedean, then a is called a semi-finite element of G. If all nonzero elements in G_+ are semi-finite, we say that G is semi-finite.*

Theorem 3.6. *If A is an AF-algebra, then A is semi-finite iff $G(A)$ is a semi-finite dimension group.*

Proof. By Theorem 3. 5, the necessity is evident.

Sufficiency: If $G(A)$ is semi-finite, let AG_+ denote the set of all archimedean elements in $G(A)_+$, and write it as $AG_+ = \{b_1,\ b_2,\ \dots b_n, \dots\}$.

As in the proof of Theorem 3.5, we can construct a maximal subset m of AG_+ which is composed of pairwise disjoint elements, and furthermore we can construct an essential ideal J of G with a finite scale from m. By Theorem 3.5, A is semi-finite.

Theorem 3.7. *Let A be an AF-algebra, then A is pure-infinite iff $G(A)$ is pure-infinite.*

Proof. Sufficiency: If A were not pure-infinite, then there would be a nontrivial faithful generalized functional φ on $G(A)$ by Theorem 2.2. Thus there would be an $a_0 \in G(A)_+$ such that $0 < \varphi(a_0) < +\infty$. Since $G(A)$ is pure-infinite, there is a $b \in G(A)_+ \setminus \{0\}$ such that $nb \leq a_0$, $\forall n \in N$. Therefore

$$n\varphi(b) \leq \varphi(a_0) < +\infty \qquad \forall n \in N. \tag{3.18}$$

It follows that $\varphi(b) = 0$, but φ is faithful, so $b = 0$, a contradiction.

Necessity: If $G(A)$ were not pure-infinite, i.e., $G(A)_+$ contained a nonzero archimedean element a, then the ideal J_a of G generated by a would be archimedean. Thus there would be a faithful functional $\tilde{\varphi}$ on J_a, and similar to the proof of Theorem 2.1 we could extend $\tilde{\varphi}$ to a faithful generalized functional φ on G, which contradicts Theorem 2.2.

Obviously, $G(A)$ is semi-finite iff every element in $\Gamma(A)$ is semi-finite and it is the same that $G(A)$ is pure-infinite iff every element in $\Gamma(A)$ is pure-infinite. Thus combining the preceding results, we can obtain the following theorem.

Theorem 3.8. *Let A be an AF-algebra. Then A is finite, or semi-finite or pure-infinite iff the scale $\Gamma(A)$ of A is respectively finite, semi-finite or pure-infinite.*

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