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WIDE SENSE STABILITY OF COMPLEX SYSTEMS OF DIFFERENTIAL EQUATIONS OF ARBITRARY DIMENSION

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Abstract

It is shown how the extended Routh array (ERA), which naturally extends the Routh array to the complex case, can handle the appearance of vanishing leading array elements, and how after minor modifications it can be used to test the stability in the wide sense of systems of differential equations with complex coefficients and of arbitrary dimension. The recent result advanced for the asymptotic or strict sense stability of such systems falls out as a special case.

Keywords Stability, Wide sence stability, Routh array, Extended Routh array.
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§1. Introduction

A fundamental problem in control theory deals with the stability of a given linear system of differential equations, which is guaranteed if and only if the eigenvalues of the system have negative real parts. Different methods of solutions of these problems are available in the litterature, but they are mainly restricted to systems with real coefficients (see for example [4], [6] and [9]). These problems are related to the classical Routh-Hurwitz criterion which witnessed a revival of interest in recent years (among many others, see [1], [6] and [10]).

The motivation for studying the stability of systems of differential equations having complex coefficients has been provided in [3]. Our previous work in this direction ([14] and [15]) focuses on the asymptotic stability of such systems. In [15], we introduced the extended Routh array (ERA) which is a sophisiticated algorithm for testing the asymptotic stability of systems of differential equations with complex coefficients and of arbitrary dimension. The ERA elegantly generalizes the famous Routh array which treats the real case. In [13], which is a special case of our present work, we considered the question of stability in a wide sense, which will be defined later, of systems with low dimensions. Tests of wide sense stability, as opposed to strict sense stability or asymptotic stability, are known to be more complicated, since they deal with the singularities that may appear in the stability tests. For example, a singularity arises when the left-most element of a certain row in the Routh array is zero. In this respect, a flurry of results have recently been reported (e. g. see [2], [5] and [12]).

In this present work, we show how the presence of singularities in the ERA does not affect its ability to test the stability of a system. More precisely, we prove that under a necessary condition on the characteristic polynomial of the system, the ERA can still serve

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as an algorithm for testing the stability in a wide sense of complex systems of differential equations of arbitrary dimension. The main results of [15] fall out as a special case.

This paper is structured as follows. In section 2 we introduce the basic definitions and recall some important lemmas. In section 3, we give a brief discussion and a geometric interpretation to the set of common roots to a polynomial f and its paraconjugate f^*. The main results of this paper and related discussions will be given in section 4.

§2. Preliminaries

Consider a system of differential equations $X' = AX$ where A is an $n \times n$ complex matrix and $X(t)$ is a column vector of the n dependent variables. The basic notions concerning the stability of this system are the same as in [15]. We only recall the necessary facts:

Definition 2.1. *A non-constant polynomial is a Hurwitz polynomial if all its roots have negative real parts.*

Definition 2.2. *If $g(z)$ is any rational function, its paraconjugate is defined by*

$$g^*(z) = \overline{g(-\bar{z})},$$

where $\bar{z}$ denotes the complex conjugate of z.

Definition 2.3. *A function $h(z)$ is said to be positive if* $\operatorname{Re} h(z) > 0$ *whenever* $\operatorname{Re} z > 0$.

It is easy to see that $\frac{1}{h}$ is positive if and only if h is positive.

Throughout the paper,

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-2} z^2 + a_{n-1} z + a_n$$

denotes the characteristic polynomial of the $n \times n$ complex system $X' = AX$, and its paraconjugate is given by:

$$f^*(z) = (-1)^n z^n + (-1)^{n-1} \bar{a}_1 z^{n-1} + \cdots + \bar{a}_{n-2} z^2 - \bar{a}_{n-1} z + \bar{a}_n.$$

As in [15], define

$$h = \begin{cases} \dfrac{f - f^*}{f + f^*} & \text{if } n \text{ odd}, \\ \dfrac{f + f^*}{f - f^*} & \text{if } n \text{ even}. \end{cases}$$

Then h may be written in the form

$$h(z) = \frac{z^n + i\operatorname{Im} a_1 z^{n-1} + \operatorname{Re} a_2 z^{n-2} + i\operatorname{Im} a_3 z^{n-3} + \operatorname{Re} a_4 z^{n-4} + \cdots}{\operatorname{Re} a_1 z^{n-1} + i\operatorname{Im} a_2 z^{n-2} + \operatorname{Re} a_3 z^{n-3} + i\operatorname{Im} a_4 z^{n-4} + \cdots}.$$

$\frac{1}{h}$ is sometimes referred to as the test fraction for the polynomial f (see [7]).

We recall that if f and f^* do not vanish simultaneously, then f is a Hurwitz polynomial if and only if h is positive ([11, Chapter 5, Theorem 5.1]). Since $h^* = -h$, the function $-h^*$ is positive if h is positive. Such a function h can be expanded in the following partial fraction form

$$h(z) = a + bz + \frac{b_1}{z - iw_1} + \frac{b_2}{z - iw_2} + \cdots + \frac{b_n}{z - iw_n},$$

where $\operatorname{Re} a = 0$, $b \geq 0$, $b_k \geq 0$ and w_k are distinct real numbers for $k = 1, \cdots n$ ([11, Chapter 5, Theorem 5.2]). Any function satisfying the conditions of this theorem is called positive para-odd. It follows that the above expansion of $h(z)$ is unique.

If we call $[h] = a + bz$ the integral part of h, then by [11], h is positive para-odd if and only if

(1) $[h] = a + bz$ satisfies $\operatorname{Re} a = 0$ and $b \geq 0$,

(2) the function $\tilde{h} = h - [h]$ is positive para-odd.

We now state the following three lemmas, the proofs of which may be found in [15].

Lemma 2.1. *If f is a Hurwitz polynomial, then f and f^* have no roots in common.*

Lemma 2.2. *Any common root to f and f^* is also a common root to $f + f^*$ and $f - f^*$ and vice versa.*

Lemma 2.3. *If h is positive para-odd, then the degrees of its numerator and denominator differ by 1 at most.*

§3. Common Roots to f and f^*

From the definition of f^*, we conclude that $|f^*(iz) = |f(iz)|$ for any real number z. It follows that any pure imaginary root of f is a root of f^* and vice versa. In the main theorem, we shall need the assumption that all common roots to f and f^* lie on the imaginary axits. Therefore the following geometrical interpretation may be helpful:

Suppose that z_1 is a common root to f and f^* and $\operatorname{Re} z_1 \neq 0$. Write $f(z)$ and $f^*(z)$ in the factored forms, $f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$, and

$$f^*(z) = (-1)^n (z + \bar{z}_1)(z + \bar{z}_2) \cdots (z + \bar{z}_n).$$

Then z_1 cannot equal $-\bar{z}_1$, for otherwise $\operatorname{Re} z_1 = 0$. Hence z_1 coincides with another root of f^* which we may assume to be $-\bar{z}_2$. But $z_1 = -\bar{z}_2$ implies $z_2 = -\bar{z}_1$. Then $f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$.

This analysis leads to the conclusion that the set S of common roots to f and f^* consists of two disjoint subsets S_1 and S_2 where

$$S_1 = \{z \in \mathbb{C} : f(z) = 0 \text{ and } \operatorname{Re} z = 0\}$$

and

$$S_2 = \{z \in \mathbb{C} : f(z) = f(-\bar{z}) = 0 \text{ and } \operatorname{Re} z \neq 0\}.$$

The elements of S_2 can be paired off in symmetric couples with respect to the imaginary axis. Fig.1 illustrates an element z_1 of §$_1$ and a pair of elements z_2 and$\bar{z}_2$ of S_2.

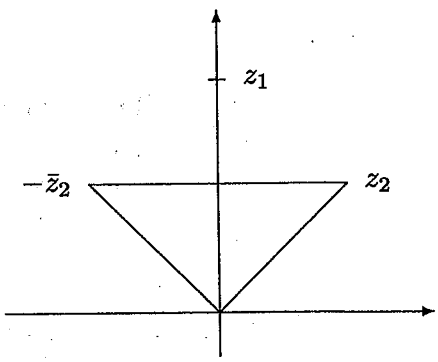

Fig.1

We summarize in the next lemma.

Lemma 3.1. *The following three propositions are equivalent:*

(1) *The set of common roots to f and f^* lies on the imaginary axis.*

(2) *f has no symmetric roots with respect to the y-axis outside the y-axis.*

(3) $S_2 = \emptyset$.

We note that propositions (2) and (3) are expressed solely in terms of f. In [13], we were able to express these propositions in more explicit forms, since we were dealing with low-dimensional systems.

§4. Wide Sense Stability

In [13], [14] and [15] we followed the definitions of asymptotic stability, uniform stability and stability of a system of differential equations $X' = AX$ as given in [8]. Since A is a constant matrix, stability and uniform stability coincide. Therefore for stability which may or may not be asymptotic, the wording wide sense stability seems to be more appropriate than uniform stability which we used in [13].

We now give a quick reminder of the way the ERA has been constructed. As defined before, let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-2}z^2 + a_{n-1}z + a_n$$

denote the characteristic polynomial of $X' = AX$, and

$$h(z) = \frac{z^n + i\mathrm{Im}\, a_1 z^{n-1} + \mathrm{Re}\, a_2 z^{n-2} + i\mathrm{Im}\, a_3 z^{n-3} + \mathrm{Re}\, a_4 z^{n-4} + \cdots}{\mathrm{Re}\, a_1 z^{n-1} + i\mathrm{Im}\, a_2 z^{n-2} + \mathrm{Re}\, a_3 z^{n-3} + i\mathrm{Im}\, a_4 z^{n-4} + \cdots}.$$

Let f_1 be the numerator and f_2 the denominator of h. If $\mathrm{Re}\, a_1 \neq 0$, call f_3 the remainder of the division of f_1 by f_2 which is a polynomial of degree $n-2$. If $n = 1$, we get $f_3 = 0$, and the process ends. So, we may assume $n \geq 2$. By induction we define the polynomial f_j to be the remainder of division of f_{j-2} by f_{j-1} for $j = 3, \cdots, n+1$. The ERA is the following array in which the j^{th} row represents the coefficients of f_j for $j = 1, 2, 3, \cdots, n+1$, and where each row is completed by zeros to the size of the first row.

$$\begin{array}{lllllllll}
1 & i\mathrm{Im}\, a_1 & \mathrm{Re}\, a_2 & i\mathrm{Im}\, a_3 & \mathrm{Re}\, a_4 & i\mathrm{Im}\, a_5 & \cdot & \cdot \\
\mathrm{Re}\, a_1 & i\mathrm{Im}\, a_2 & \mathrm{Re}\, a_3 & i\mathrm{Im}\, a_4 & \mathrm{Re}\, a_5 & \cdot & \cdot & \cdot \\
b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \cdot & \cdot & \cdot & \cdot \\
b_{4,1} & b_{4,2} & b_{4,3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n,1} & b_{n,2} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n+1,1} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

The main result of [15] states that the $n \times n$ complex system $X' = AX$ is asymptotic stable if and only if each term of the first cloumn of the ERA is positive.

For more details on the construction of the ERA with related comments and resutls, see [15].

In the process of generating the ERA, it is necessary that all elements of the first column of this array are non-zero except possibly the last one. For technical reasons, let the second row of the ERA be of the form:

$$b_{2,1} \quad b_{2,2} \quad b_{2,3} \quad b_{2,4} \quad b_{2,5} \quad \cdot \quad \cdot$$

where it is understood that

$$b_{2,1} = \operatorname{Re} a_1, \quad b_{2,2} = i\operatorname{Im} a_2, \quad b_{2,3} = \operatorname{Re} a_3 \quad \text{and so on.}$$

Also, let $b_{1,1} = 1$ be the first entry and add a zero-row at the bottom. Hence we obtain the following array which we still call the ERA:

$$\begin{array}{cccccccc}
b_{1,1} & i\operatorname{Im} a_1 & \operatorname{Re} a_2 & i\operatorname{Im} a_3 & \operatorname{Re} a_4 & i\operatorname{Im} a_5 & \cdot & \cdot \\
b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} & \cdot & \cdot & \cdot \\
b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \cdot & \cdot & \cdot & \cdot \\
b_{4,1} & b_{4,2} & b_{4,3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n,1} & b_{n,2} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n+1,1} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n+2,1} & b_{n+2,2} & b_{n+2,3} & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

where $b_{n+2,k} = 0$ for $k = 1, 2, \cdots, n+1$.

For the sake of simplicity, we mention that according to the way the ERA is constructed, in any j^{th} row the last $j-1$ elements are all zero.

Our main result may now be given.

Theorem 4.1. *$S_2 = \emptyset$, then the system $X' = AX$ where A has no repeated zero eigenvalue is stable in the wide sense if and only if for some integer m, $2 \le m \le n+2$ we have $b_{j,1} > 0$ for all j, $1 \le j < m$ and $b_{m,k} = 0$ for all k, $1 \le k \le n+1$, and where asymptotic stability occurs only when $m = n+2$.*

Proof. The system $X' = AX$ with no repeated zero eigenvalue is stable in the wide sense if and only if A has no eigenvalues with positive real parts ([8, Theorem 9.3]).

By [15], the system is asymptotically stable or stable in the strict sense if and only if $b_{j,1} > 0$ for all j, $1 \le j < n+2$. This is equivalent to letting $m = n+2$ in the statement of the theorem.

Now we discuss stability which is not asymptotic. Let p be the number of roots of f with negative real parts, and $n-p$ those with zero real parts. p satisfies $0 \le p < n$.

Write $f(z)$ in the form

$$f(z) = \begin{cases} \displaystyle\prod_{k=1}^{n}(z - z_k), & \text{where } \operatorname{Re} z_k = 0, \quad \text{if } p = 0, \\ \displaystyle\prod_{j=1}^{p}(z - z_j) \prod_{k=p+1}^{n}(z - z_k), & \text{where } \operatorname{Re} z_j < 0, \ \operatorname{Re} z_k = 0, \quad \text{if } 0 < p < n. \end{cases}$$

First consider the case when $p = 0$. Then

$$f(z) = \prod_{k=1}^{n}(z - z_k),$$

where $\operatorname{Re} z_k = 0$ for $k = 1, \cdots, n$ and at most one of these roots may be zero. Since

$\bar{z}_k = -z_k$, we get

$$f^*(z) = (-1)^n \prod_{k=1}^{n} (z + \bar{z}_k) = (-1)^n \prod_{k=1}^{n} (z - z_k).$$

Define the function k as follows:

$$k = \begin{cases} \dfrac{f + f^*}{f - f^*} & \text{if } n \text{ odd,} \\ \dfrac{f - f^*}{f + f^*} & \text{if } n \text{ even.} \end{cases}$$

It is evident that k is identically zero. With f and f^* as defined in section 2, we get

$$k(z) = \frac{\operatorname{Re} a_1 z^{n-1} + i\operatorname{Im} a_2 z^{n-2} + \operatorname{Re} a_3 z^{n-3} + i\operatorname{Im} a_4 z^{n-4} + \cdots}{z^n + i\operatorname{Im} a_1 z^{n-1} + \operatorname{Re} a_2 z^{n-2} + i\operatorname{Im} a_3 z^{n-3} + \operatorname{Re} a_4 z^{n-4} + \cdots}.$$

We conclude that

$$\operatorname{Re} a_1 = \operatorname{Im} a_2 = \operatorname{Re} a_3 = \operatorname{Im} a_4 = \cdots = 0$$

or equivalently $b_{2,k} = 0$ for all $k = 1, \cdots, n+1$. That corresponds to letting $m = 2$ in the statement.

Conversely, suppose $b_{2,k} = 0$ for $k = 1, \cdots, n+1$. Then the function k as defined above is identically zero. Therefore

$$f = \begin{cases} -f^* & \text{if } n \text{ odd,} \\ f^* & \text{if } n \text{ even.} \end{cases}$$

From this we conclude that f and f^* have the same set of roots. Our assumption on the common roots to f and f^* implies that all roots of f are pure imaginary. In other words $p = 0$.

Second, consider the case where $0 < p < n$, then

$$f(z) = \prod_{j=1}^{p} (z - z_j) \cdot \prod_{k=p+1}^{n} (z - z_k),$$

where $\operatorname{Re} z_j < 0$ for $j = 1, \cdots, p$, $\operatorname{Re} z_k = 0$ for $k = p+1, \cdots, n$ and at most one of the z_k's may be zero.

We claim that $b_{p+2,q} = 0$ for all $q = 1, \cdots .n+1$ and $b_{j,1} > 0$ for all $j = 1, \cdots, p+1$.

Define the function g in the following way:

$$g(z) = \frac{f(z)}{\prod\limits_{k=p+1}^{n} (z - z_k)} = \prod_{j=1}^{p} (z - z_j).$$

It is clear that g is a Hurwitz polynomial, and

$$g^*(z) = \frac{f^*(z)}{(-1)^{n-p} \prod\limits_{k=p+1}^{n} (z - z_k)} = (-1)^p (z + \bar{z}_j).$$

With h as defined in section 2, we have

$$h(z) = \frac{z^n + i\operatorname{Im} a_1 z^{n-1} + \operatorname{Re} a_2 z^{n-2} + i\operatorname{Im} a_3 z^{n-3} + \operatorname{Re} a_4 z^{n-4} + \cdots}{\operatorname{Re} a_1 z^{n-1} + i\operatorname{Im} a_2 z^{n-2} + \operatorname{Re} a_3 z^{n-3} + i\operatorname{Im} a_4 z^{n-4} + \cdots}. \tag{4.1}$$

Since

$$f - f^* = \prod_{k=p+1}^{n} (z - z_k) \cdot g(z) - (-1)^{n-p} \prod_{k=p+1}^{n} (z - z_k) g^*(z),$$

and

$$f + f^* = \prod_{k=p+1}^{n} (z - z_k) \cdot g(z) + (-1)^{n-p} \prod_{k=p+1}^{n} (z - z_k) g^*(z),$$

we can bring h to the form:

$$h = \begin{cases} \dfrac{g - g^*}{g + g^*} & \text{if } p \text{ odd,} \\ \dfrac{g + g^*}{g - g^*} & \text{if } p \text{ even.} \end{cases}$$

In both cases, we get

$$h(z) = \frac{\prod_{j=1}^{p} (z - z_j) + \prod_{j=1}^{p} (z + \bar{z}_j)}{\prod_{j=1}^{p} (z - z_j) - \prod_{j=1}^{p} (z + \bar{z}_j)}. \tag{4.2}$$

It is obvious that the numerator of h in (4.2) is a polynomial of degree p, and the leading term in the denominator is

$$-2\mathrm{Re}\,(z_1 + z_2 + \cdots + z_p) z^{p-1}.$$

Since $\mathrm{Re}\, z_j < 0$ for $j = 1, \cdots, p$, it follows that the denominator is a polynomial of degree $p-1$. Also, since g is a Hurwitz polynomial, it follows from section 2 that h is a positive function, and that no common roots exist between g and g^*, which is equivalent to say that $g - g^*$ and $g + g^*$ have no roots in common. Therefore h as in (4.2) is an irreducible rational function, i.e., no common factors exist between its numerator and denominator.

We note that we are dealing with two different forms of h, namely (4.1) and (4.2), and that form (4.1) can be reduced to form (4.2) after the $n-p$ common factors between numerator and denominator in form (4.1) of h are cancelled out.

If f_1 and f_2 are the numerator and denominator respectively of h in (4.1), we have proved that f_2 is identically zero if and only if $p = 0$. Therefore if $p > 0$, f_2 is not zero, and since h is positive para-odd, we conclude from Lemma 2.3 that $\mathrm{Re}\, a_1 \neq 0$. By executing a long division we get

$$h(z) = \frac{1}{\mathrm{Re}\, a_1} z + i r_1 + \frac{f_3}{f_2}$$

for some real r_1, where

$$\frac{1}{\mathrm{Re}\, a_1} z + i r_1$$

is the integer part of h. Hence $\mathrm{Re}\, a_1 > 0$ or $b_{2,1} > 0$. If $p = 1$, we conclude from (4.2) that h coincides with its integral part, and hence $f_3 = 0$. Therefore $b_{3,q} = 0$ for $q = 1, \cdots, n+1$. We also have $b_{1,1} > 0$ and $b_{2,1} > 0$.

Conversely suppose $b_{3,q} = 0$, for $q = 1, \cdots, n+1$ and $b_{2,1} > 0$. We claim that $p = 1$. In fact, since $f_3 = 0$,

$$h = \frac{f_1}{f_2} = \frac{1}{b_{2,1}} z + ir_1.$$

This form of h implies that there exist $n-1$ common roots between $f - f^*$ and $f + f^*$, and therefore between f and f^*. These roots must be pure imaginary. Write f in the form

$$f(z) = (z - z_1) \prod_{k=2}^{n} (z - z_k),$$

where $\operatorname{Re} z_k = 0$ for $k = 2, \cdots, n$. It is enough to show that $\operatorname{Re} z_1 < 0$.

If

$$g(z) = \frac{f(z)}{\prod_{k=2}^{n} (z - z_k)} = z - z_1,$$

then

$$g^*(z) = \frac{f^*(z)}{(-1)^{n-1} \prod_{k=2}^{n} (z - z_k)} = -(z + \bar{z}_1).$$

From the definition of h it follows that

$$h = \frac{g - g^*}{g + g^*}.$$

Hence

$$h(z) = -\frac{1}{\operatorname{Re} z_1} z + \frac{i \operatorname{Im} z_1}{\operatorname{Re} z_1}.$$

When we compare this to

$$h(z) = \frac{1}{b_{2,1}} z + ir_1,$$

where $b_{2,1} > 0$, we conclude that $\operatorname{Re} z_1 < 0$ and therefore $p = 1$.

It follows that if $p \geq 2$, then f_3 is not identically zero. We then go kack to

$$h(z) = \frac{1}{b_{2,1}} z + ir_1 + \frac{1}{\frac{f_2}{f_3}},$$

and carry on in exactly the same way with the positive para-odd $\frac{f_2}{f_3}$.

By induction, consider an integer k, $2 \leq k \leq p$, and suppose that $b_{j,1} > 0$ and the function $\frac{f_{j-1}}{f_j}$ is positive para-odd for all $j = 2, \cdots, k$. We claim that $b_{k+1,1} > 0$ and $\frac{f_k}{f_{k+1}}$ is positive para-odd.

From the construction of the ERA, it follows that

$$\frac{f_{k-1}}{f_k} = \frac{b_{k-1,1}}{b_{k,1}} z + ir_{k-1} + \frac{f_{k+1}}{f_k}$$

for some real r_{k-1}. Since $\frac{f_{k-1}}{f_k}$ is positive para-odd and

$$\frac{b_{k-1,1}}{b_{k,1}} > 0,$$

we conclude that $\frac{f_{k+1}}{f_k}$ is positive para-odd.

If f_{k+1} is identically zero, then h appears in the continued fraction form

$$h = \frac{b_{1,1}}{b_{2,1}}z + ir_1 + \cfrac{1}{\ddots + \cfrac{1}{\frac{b_{k-1,1}}{b_{k,1}}z + ir_{k-1}}} \tag{4.3}$$

where $b_{j,1} > 0$ for $j = 1, \cdots, k$.

When $k = 2$, it is understood that form (4.3) reduces to

$$h = \frac{b_{1,1}}{b_{2,1}}z + ir_1.$$

It is clear that (4.3) can be written as a polynomial of degree $k-1$ over a polynomial of degree $k-2$. Since $k \leq p$, then $k - 1 < p$. Therefore (4.3) violates form (4.2) of h. We conclude that f_{k+1} cannot be identically zero. By Lemma 2.3, $b_{k+1,1} \neq 0$. Now,

$$\frac{f_k}{f_{k+1}} = \frac{b_{k,1}}{b_{k+1,1}}z + ir_k + \frac{f_{k+2}}{f_{k+1}}$$

is positive para-odd, where r_k is a real number. Hence

$$\frac{b_{k,1}}{b_{k+1,1}} > 0,$$

leading to $b_{k+1,1} > 0$.

We just proved that $b_{j,1} > 0$ for all $j = 1, \cdots, p+1$. It remains to show that $b_{p+2,q} = 0$ for $q = 1, \cdots, n+1$.

From the above induction, it follows that

$$h = \frac{b_{1,1}}{b_{2,1}}z + ir_1 + \cfrac{1}{\frac{b_{2,1}}{b_{3,1}}z + ir_2 + \ddots + \cfrac{1}{\frac{b_{p,1}}{b_{p+1,1}}z + ir_p + \frac{f_{p+2}}{f_{p+1}}}}$$

where $b_{j,1} > 0$ for $j = 1, \cdots, p+1$.

Form (4.2) of h implies that f_{p+2} is identically zero. Hence $b_{p+2,q} = 0$ for all $q = 1, \cdots, n+1$.

Conversely, suppose that for some integer m, $2 < m \leq n+1$, we have $b_{j,1} > 0$ for all $j = 1, \cdots, m-1$ and $b_{m,k} = 0$ for $k = 1, \cdots, n+1$.

We claim that f has $m-2$ roots with negative real parts and $n-m+2$ pure imaginary roots. Since f_m is identically zero, it can be seen that

$$h(z) = \frac{b_{1,1}}{b_{2,1}}z + ir_1 + \cfrac{1}{\frac{b_{2,1}}{b_{3,1}}z + ir_2 + \ddots + \cfrac{1}{\frac{b_{m-2,1}}{b_{m-1,1}}z + ir_{m-2}}} \tag{4.4}$$

where $b_{j,1} > 0$ for $j = 1, \cdots, m-1$. When $m = 3$, it is understood that (4.4) reduces to

$$h(z) = \frac{b_{1,1}}{b_{2,1}}z + ir_1.$$

In both cases, it is clear that h appears as a polynomial of degree $m-2$ over a polynomial of degree $m-3$. Since the inverse of a positive function and the sum of two positive functions are positive, and the same applies on para-odd functions, we conclude that h is positive para-odd. Also, since

$$\frac{b_{j,1}}{b_{j+1,1}} > 0$$

for $j = 1, \cdots, m-2$, form (4.4) of h is thus irreducible, i.e., no common factors exist between its numerator and denominator. We conclude that there exist $n-m+2$ common roots between $f - f^*$ and $f + f^*$, hence $n-m+2$ common roots between f and f^* which must be pure imaginary. We get

$$f = \prod_{j=1}^{m-2} (z - z_j) \cdot \prod_{k=m-1}^{n} (z - z_k),$$

where $\operatorname{Re} z_j \neq 0$ for all $j = 1, \cdots, m-2$ and $\operatorname{Re} z_k = 0$ for $k = m-1, \cdots, n$.

Let

$$g = \frac{f}{\prod_{k=m-1}^{n} (z - z_k)} = \prod_{j=1}^{m-2} (z - z_j),$$

then

$$g^* = \frac{f^*}{(-1)^{n-m+2} \prod_{k=m-1}^{n} (z - z_k)} = (-1)^{m-2} \prod_{j=1}^{m-2} (z + \bar{z}_j).$$

It would be enough to show that g is a Hurwitz polynomial. From the definition of h it follows that

$$h = \begin{cases} \dfrac{g - g^*}{g + g^*} & \text{if } m \text{ odd}, \\[2mm] \dfrac{g + g^*}{g - g^*} & \text{if } m \text{ even}. \end{cases}$$

Since h is a positive function, we reach our claim if we show that no common roots exist between g and g^* ([11, Chapter 5, Theorem 5.1]). In fact, from

$$g = \prod_{j=1}^{m-2} (z - z_j)$$

and

$$g^* = (-1)^{m-2} \prod_{j=1}^{m-2} (z + \bar{z}_j),$$

we get

$$h(z) = \frac{\prod_{j=1}^{m-2} (z - z_j) + \prod_{j=1}^{m-2} (z + \bar{z}_j)}{\prod_{j=1}^{m-2} (z - z_j) - \prod_{j=1}^{m-2} (z + \bar{z}_j)}. \tag{4.5}$$

Call g_1 the numerator and g_2 the denominator of h in (4.5). It is clear that g_1 is a polynomial of degree $m-2$. Now g_2 cannot be identically zero, for if it were, we would have

$$\prod_{j=1}^{m-2}(z_1+\bar{z}_j)=0$$

which we obtain by making the substitution $z=z_1$ in g_2. Since $\operatorname{Re} z_1 \neq 0$, this last relation implies that $z_1+\bar{z}_j=0$ for some j, $2\leq j\leq m-2$, contradicting the fact that $S_2=\emptyset$. Now by Lemma 2.3, (4.5) is a polynomial of degree $m-2$ over a polynomial of degree $m-3$. When we compare (4.5) to (4.4), we conclude that h in (4.5) is irreducible. Therefore, no common roots exist between $g-g^*$ and $g+g^*$, hence none exists between g and g^*, and the proof of the theorem is complete.

We end up by making the following comments:

1. It is constructive to note that the above proof establishes a clear relationship between the integer m of the statement and the number of eigenvalues of the $n\times n$ complex system $X'=AX$ with negative or zero real parts. More precisely, we have

(a) $m=n+2$ if and only if all eigenvalues of A have negative real parts. This case is equivalent to stability in the strict sense ([15]).

(b) $m=2$ if and only if all eigenvalues of A are pure imaginary.

(c) $2<m<n+2$ if and only if A has $m-2$ eigenvalues with negative real parts and $n-m+2$ pure imaginary ones.

2. If the system $X'=AX$ is stable, then $S_2=\emptyset$, since A has no eigenvalues with positive real parts. In fact the condition that $S_2=\emptyset$ is necessary only in the opposite direction as shows the following simple example.

Let

$$f(z)=z^4-2iz^3-z^2-2iz-2,$$

which can be written in the factored form

$$f(z)=(z+1-i)(z-1-i)(z-i)(z+i).$$

Hence $S_2\neq\emptyset$ and the system is not stable. However the ERA takes the form:

$$\begin{array}{ccccc} 1 & -2i & -1 & -2i & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

3. All stability results established in this paper as well as in [15] hold true if we replace the system $X'=AX$ by the n^{th} order linear differential equation with complex coefficients:

$$y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-2}y''+a_{n-1}y'+a_ny=0,$$

whose characteristic polynomial is also given by

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-2}z^2+a_{n-1}z+a_n.$$

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