Chin. Ann. of Math.
15B: 2(1994),205-216.

HOMOCLINIC BIFURCATION WITH CODIMENSION 3**

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Abstract

First it is proved that both the integral of the divergence and the Melnikov function are invariants of the C^2 transformation. Then, the problem of the planar homoclinic bifurcation with codimension 3 is considered. It is proved that, in a small neighborhood of the origin in the parameter space of a C^r ($r \geq 5$) system, there exist exactly two C^{r-1} semi-stable-limit-cycle branching surfaces, and their common boundary is a unique C^{r-1} three-multiple-limit-cycle branching curve. The bifurcation pictures and the asymptotic expansions of the bifurcation functions are given. The stability criterion for the homoclinic loop is also obtained when the integral of the divergence is zero. The proof of the auxiliary theorems will be presented in [16].

Keywords Homoclinic bifurcation, Codimension, Semi-stable-limit-cycle branch, Three-multiple-limit-cycle branch.
1991 MR Subject Classification 34C05, 34C37.

§0. Introduction and Main Results

In recent years, a great number of papers considered the problem of the homoclinic loop bifuraction (see [1, 2, 4, 6-11, 13-15] and the references of [2]). Paper [2] solved the problem of the homoclinic loop bifurcation in high dimension with codimension 2, i.e., with the resonant eigenvalues and an additional condition (for the exact meaning see [2] or the following several paragraphs). In this paper, we are interested in the planar homoclinic loop bifurcation with codimension 3.

Consider the system

$$\begin{aligned}\dot{x} &= F(x,y,\alpha),\\ \dot{y} &= G(x,y,\alpha)\end{aligned} \tag{0.1$_\alpha$}$$

with the hypotheses: $F, G \in C^r$, $r \geq 5$, $\alpha \in R^n$ is a multi-parameter, $\mathrm{div}(F,G) = 0$ at point $(x,y,\alpha) = (0,0,0)$, and there exists a homoclinic loop Γ_0 passing through the saddle $O(0,0)$ when $\alpha = 0$.

Choosing new parameters if necessary, it follows (see [6,7,10]) that, for any given m satisfying $0 \leq m \leq [\frac{r-1}{2}]$, there exists a C^r transformation T such that, in a neighborhood

Manuscript received November 26, 1991. Revised April 24, 1992.
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**Project supported by the National Natural Science Foundation of China

U of the origin, system $(0.1)_\alpha$ takes the form (rescale the time if necessary)

$$\begin{aligned}\dot{x} &= -\lambda x - \sum_{i=1}^{m} a_i(\alpha)x(xy)^i + x(xy)^m R_1(x,y,\alpha),\\ \dot{y} &= y + \sum_{i=1}^{m} b_i(\alpha)y(xy)^i + y(xy)^m R_2(x,y,\alpha),\end{aligned} \tag{0.2$_m$}$$

where R_1, $R_2 \in C^r$, R_1, $R_2 = o((x^2+y^2)^{(r-2m-1)/2})$, the sign $[x]$ denotes the integral part of x, $\lambda = 1+\mu$, $\alpha = (\alpha_1, \cdots, \alpha_{n-1}, \mu)$ and T is linear outside some neighborhood $U_1 \supset U$.

Let $v_i = a_i - b_i$. If $\mu = v_1 = \cdots = v_{i-1} = 0$, $v_i \neq 0$, then the origin O is called a fine saddle of order i, and v_i is called the i-th saddle value.

Throughout the paper, we assume $v_1(\alpha) \neq 0$. In other words, O is a fine saddle of order 1 when $\mu = 0$.

Denote $I(\alpha) = \int_{\Gamma_0} \operatorname{div}(F(x,y,\alpha), G(x,y,\alpha))dt$ and call it the integral of the divergence. In the case of dimension 2, paper [2] considered the homoclinic loop bifurcation with a fine saddle of order 1 and $I(0) \neq 0$.

In the following, we always assume

$$I(0) = 0, \quad v_1 > 0. \tag{0.3}$$

The sign of v_1 is not essential. In fact, if $v_1 < 0$, the transformation $(x,y,t) \to (y,x,-t)$ changes v_1 into $-v_1$.

By using suitable variable to define the Poincaré map, we will actually prove that the homoclinic loop bifurcation is uniquely determined by μ, $I(\alpha)$ and the Melnikov function when $v_1 \neq 0$, although, strictly speaking, the third one should be replaced by the distance $d(\alpha)$ between the stable manifold and the unstable manifold. For convenience' sake (but without loss of generality), instead of α, we regard $\theta(\alpha^*) = \exp(I(\alpha^*)) - 1$, μ and $d(\alpha)$ as the bifurcation parameters, where $\alpha^* = (\alpha_1, \cdots, \alpha_{n-1}, 0)$.

The main purpose of this paper is to show the existence of exactly two semi-stable-limit-cycle branching (SCB) surfaces, and the uniqueness both of the three-multiple-limit-cycle branching (TCB) curve and of the intersection curves of these two surfaces with any section running parallel to the parameter coordinate plane. We also obtain the asymptotic expansions of the bifurcation curves. As a by-product, we get the stability criterion for the homoclinic loop Γ when the integral of the divergence along Γ is zero and $v_1 \neq 0$. A direct consequence of our results is that the number of limit cycles produced in the homoclinic loop bifurcation when $v_1 \neq 0$ is at most 3, which is already known in [7,10].

In section 2, similar to [2,3], we use the Sil'nikov variable to define the Poincaré map. Thus we should transform system $(0.1)_\alpha$ into the normal form $(0.2)_2$ in some neighborhood of the origin. But it is essential to prove firstly that both $I(\alpha)$ and the Melnikov function are invariants of the C^r transformation. In [6], we have shown that $I(\alpha)$ is an invariant of the linear transformation. In section 1, we obtain the following sharper result: The integral of the divergence, along a periodic orbit or any simple closed path consisting of orbits and singular points, is an invariant of the C^2 coordinate transformation. The same conclusion is true for the Melnikov function up to a positive constant factor.

The main result of the paper is the following theorem.

Main Theorem. *Suppose that $v_1(0) > 0$, $I(0) = 0$, $r \geq 5$. In a small neighborhood of the origin of the parameter space $\{p(\theta, \mu, d)\}$, there exist exactly two continuous SCB surfaces Σ_1, Σ_2 (see Figure 1). They are C^{r-1} for $\mu \neq 0$, and their common boundary is a unique C^{r-1} TCB curve S. System $(0.1)_\alpha$ has exactly three (resp. a unique and stable) limit cycles in the neighborhood of Γ_0 when P is above the $\theta - \mu$ plane and below $\Sigma_1 \cup \Sigma_2$ (resp. at somewhere else), and exactly two (resp. none) limit cycles when P is below the $\theta - \mu$ plane and above Σ_2 (resp. at somewhere else).*

Since it is clear that a homoclinic loop Γ_α ($\Gamma_\alpha \to \Gamma_0$ as $\alpha \to 0$) exists if and only if P is situated in the $\theta - \mu$ plane, i.e., $d = 0$, we do not state this fact in the Main Theorem and the following theorems.

The Main Theorem is an immediately consequence of the following six theorems where we always assume that (0.3) holds.

Theorem 0.1. *In a neighborhood of the origin on the $\theta - d$ plane, there exists a unique SCB curve (see Figure 2) $d = d_1(\theta)$, where*

$$d_1(\theta) = (4\delta v_1)^{-1}\theta^2 \ln^{-1}\theta + o(\theta^2 \ln^{-1}\theta), \quad 0 < \theta << 1,$$

$d_1(0) = 0$, d_1 is C^{r-1} when $\theta > 0$ and C^2 at $\theta = 0$, and $\delta > 0$ is a constant.

Moreover, when $\theta > 0$, the system $(0.1)_\alpha$ corresponding to $d \geq 0$, $d_1(\theta) < d < 0$ or $d < d_1(\theta)$ has exactly one (stable), two or none limit cycle near Γ_0 respectively, and when $\theta \leq 0$, the system has a unique and stable (resp. none) limit cycle near Γ_0 corresponding to $d > 0$ (resp. ≤ 0).

Corollary 0.1. *Suppose that O is a fine saddle with order 1 of $(0.2)_m$, Γ is a homoclinic loop connecting O, and the integral of the divergence along Γ is zero. Then Γ is inner stable (resp. inner unstable) if $v_1 > 0$ (resp. < 0).*

In [8], we have shown that the order of a fine saddle combined with the sign of its saddle value can not be a universal criterion for the stability of a homoclinic loop. And we learn from [9] that the stability of Γ is determined by the sign of the integral of the divergence whenever it is not zero. By now, but, we can easily obtain these conclusions together with Corollary 0.1 from the formula (2.14) of the successive function

$$P_3(s, \theta, d) = d + \delta\theta s + \delta^3 v_1 s^2 \ln s + O(s^2) + O(\theta s^2 \ln s),$$

where $s \geq 0$. It is very similar to the following formula given in [10]

$$P(h) = c_1 + c_2 h \ln h + c_3 h + c_4 h^2 \ln h + \cdots, \tag{0.4}$$

which is the first order approximation with respect to ε of the successive function near a homoclinic loop of the system $x = -H_y - \varepsilon f$, $y = H_x + \varepsilon g$, where the Hamiltonian function $H(x, y) = h$. Without too much difficulty, it can be shown that c_1 is a constant Melnikov function, $c_2 = \mu$ and $c_3 = \varepsilon^{-1} I(\varepsilon)$. From (2.14) and (0.4), we see the dominant function played by d, μ, θ and v_1 in the homoclinic bifurcation.

Corollary 0.2. *Suppose that the perturbation keeps O as a fine saddle with order 1. Then, Γ_0 can bifurcate at most one limit cycle if $I(0) \neq 0$, while there exist six possibilities shown in Figure 2 if $I(0) \neq 0$.*

The above result corresponding to $I(0) \neq 0$ can be easily obtained from Figure 2 and is firstly proved in [8].

Theorem 0.2. *In a neighborhood of the origin on the* $\theta - \mu$ *plane, there exists a unique and continuous SCB curve* (*see Figure* 3) $\theta = \theta_1(\mu)$, *where*

$$\theta_1(\mu) = -\mu \ln \mu + h.o.t., \quad \mu \geq 0,$$

and θ_1 *is* C^{r-1} *when* $\mu > 0$ *and has the inverse function* $\mu_1(\theta) = -\theta \ln^{-1} \theta + h.o.t.$.

Moreover, the corresponding system $(0.1)_\alpha$ *has exactly two* (*resp. none*) *limit cycles near* Γ_0 *when* $\mu > 0$, $\theta > \theta_1(\mu)$ (*resp.* $< \theta_1(\mu)$), *one* (*resp. none*) *limit cycle when* $\mu = 0$, $\theta > 0$ (*resp.* ≤ 0), *and one and stable limit cycle when* $\mu < 0$.

Theorem 0.3. *In a neighborhood of the origin on the* $\mu - d$ *plane, there exists a unique* C^{r-1} *SCB curve* (*see Figure* 4) $d = d_2(\mu)$, *where*

$$d_2(\mu) = -(2T+1)(4\delta v_1)^{-1}\mu^2 + o(\mu^2), \quad 0 < -\mu << 1,$$

$d_2(0) = 0$ *and* d_2 *is* C^2 *at* $\mu = 0$.

Moreover, system $(0.1)_\alpha$ *has exactly one and stable* (*resp. none*) *limit cycle near* Γ_0 *when* $\mu \geq 0$ *and* $d > 0$ (*resp.* < 0), *and one, two or none limit cycle when* $\mu < 0$ *and* $d > 0$, $d_2(\mu) < d < 0$ *or* $d < d_2(\mu)$ *respectively.*

Theorem 0.4. (i) *Assume* $0 < \theta_0 << 1$. *Then there exists a* C^{r-1} *function* $\mu_2(\theta_0) > \mu_1(\theta_0)$ *such that, in a neighborhood of the origin on the section* $\theta = \theta_0$, *there exist exactly two SCB curves* (*see Figure* 5) $d = d_3(\theta_0, \mu)$ *for* $0 \leq \mu \leq \mu_2(\theta_0)$ *and* $d = d_4(\theta_0, \mu)$ *for* $-1 << \mu \leq \mu_2(\theta_0)$, *and a unique TCB point* $(\mu_2, d_3(\theta_0, \mu_2))$, *where* d_3, d_4 *are* C^{r-1} *when* $\mu \neq 0$, *and* d_3 (*resp.* d_4) *is* C^1 (*resp. continuous*) *at* $\mu = 0$, $d_3(\theta_0, 0) = d_3(0, \mu) = 0$, $d_4(\theta_0, 0) = d_1(\theta_0)$, $d_4(\theta_0, \mu_1(\theta_0)) = 0$, $d_4(0, \mu) = d_2(\mu)$, $d_3(\theta_0, \mu_2(\theta_0)) = d_4(\theta_0, \mu_2(\theta_0))$, $\frac{\partial}{\partial \mu} d_i(\theta_0, \mu) > 0$ *for* $\mu \neq 0$ *and* $i = 3, 4$, $\frac{\partial}{\partial \mu} d_3(\theta_0, 0) = 0$, *and*

$$\lim_{\mu \to 0} \mu^{-1} d_3(\theta_0, \mu)(1+\theta_0)^{1/\mu} = \delta \exp(-1 + 2T/(1+\theta_0)).$$

Moreover, system $(0.1)_\alpha$ *has exactly one* (*stable*), *three, two or none limit cycle near* Γ_0 *corresponding to* $\{d > d_3,\ \mu \leq \mu_2\} \cup \{0 < d < d_4,\ \mu_1 < \mu \leq \mu_2\} \cup \{d > 0,\ \mu \leq 0$ *or* $\mu > \mu_2\}$, $\{\max\{0, d_4\} < d < d_3,\ 0 < \mu < \mu_2\}$, $\{d_4 < d < 0,\ \mu < \mu_1\}$ *or* $\{d < d_4,\ \mu \leq \mu_1\} \cup \{d < 0,\ \mu > \mu_1\}$ *respectively.*

(ii) *Assume* $0 < -\theta_0 << 1$. *Then, in a neighborhood of the origin on the section* $\theta = \theta_0$, *there exists a unique* C^1 *SCB curve* (*see Figure* 6) $d = d_4(\theta_0, \mu)$ *for* $\mu \leq 0$ *satisfying*

$$d_4(0, \mu) = d_2(\mu), \quad d_4(\theta_0, 0) = \frac{\partial}{\partial \mu} d_4(\theta_0, \mu) = 0, \quad \frac{\partial}{\partial \mu} d_4(\theta_0, \mu) > 0 \quad \textit{for } \mu < 0,$$

$$\lim_{\mu \to 0} \mu^{-1} d_4(\theta_0, \mu)(1-\theta_0)^{-1/\mu} = \delta \exp(-1 + 2T/(1-\theta_0)),$$

and d_4 *is* C^{r-1} *when* $0 < -\mu << 1$.

Moreover, if define $d_4(\theta_0, \mu) = 0$ *for* $\mu > 0$, *then system* $(0.1)_\alpha$ *has exactly one* (*stable*), *two or none limit cycle near* Γ_0 *when* $d > 0$, $d_4 < d < 0$ *or* $d < d_4$ *respectively for* $|\mu| << 1$.

Theorem 0.5. (i) *If* $0 < \mu_0 << 1$, *then, in a neighborhood of the origin on the section* $\mu = \mu_0$, *there exist exactly two* C^{r-1} *SCB curves* (*see Figure* 7) $L_1 : d = d_3(\theta, \mu_0)$ *and* $L_2 :$

$d = d_4(\theta, \mu_0)$ when $\theta \geq \theta_2(\mu_0)$, and a unique TCB point $(\theta_2, d_3(\theta_2, \mu_0))$, where $\theta_2 \in C^{r-1}$ is an inverse function of μ_2,

$$0 < \theta_2(\mu_0) < \theta_1(\mu_0), \quad d_3(\theta_2(\mu_0), \mu_0) = d_4(\theta_2(\mu_0), \mu_0), \quad d_4(\theta_1(\mu_0), \mu_0) = 0.$$

Moreover, system $(0.1)_\alpha$ has exactly three (resp. two) limit cycles near Γ_0 when $\theta \geq \theta_2(\mu)$, $\max\{d_4, 0\} < d < d_3$ (resp. $\theta \geq \theta_1(\mu_0)$, $d_4 < d < 0$), and one (resp. none) limit cycle near Γ_0 when (θ, d) on the left of $L_1 \cup L_2$ and $d > 0$ (resp. $d < 0$).

(ii) *If $0 < -\mu_0 << 1$, then, in a neighborhood of the origin on the section $\mu = \mu_0$, there exists a unique C^{r-1} SCB curve (see Figure 8) $d = d_4(\theta, \mu_0)$ satisfying $d_4(\theta, 0) = d_1(\theta)$ when $\theta \geq 0$ and $d_4(\theta, 0) = 0$ when $\theta < 0$. Moreover, system $(0.1)_\alpha$ has exactly one (stable), two or none limit cycle near Γ_0 corresponding to $d \geq 0$, $d_4(\theta, \mu_0) < d < 0$ or $d < d_4$ respectively.*

Theorem 0.6. (i) *If $0 < d_0 << 1$, then, in a neighborhood of the origin on the section $d = d_0$, there exist exactly two C^{r-1} SCB curves $L_1 : \mu = \mu_3(d_0, \theta)$ and $L_2 : \mu = \mu_4(d_0, \theta)$ when $\theta \geq \theta^*$, and a unique TCB point (θ^*, μ^*), where $\mu_3(d_0, \theta^*) = \mu_4(d_0, \theta^*) = \mu^* = \mu_2(\theta^*)$, $\mu_3(d_0, \theta) < \mu_4(d_0, \theta)$ for $\theta > \theta^*$, $d_0 = d_3(\theta, \mu_3) = d_4(\theta, \mu_4)$, and $\mu_3(d_0, \theta) \to 0$, $\mu_4(d_0, \theta) \to \mu_1(\theta)$ as $d_0 \to 0$. System $(0.1)_\alpha$ has a unique and stable limit cycle (resp. exactly three limit cycles) near Γ_0 when (θ, μ) is in the side of $L = L_1 \cup L_2$ which contains (resp. does not contain) the origin.*

(ii) *If $0 < -d_0 << 1$, then, in a neighborhood of the origin on the section $d = d_0$, there exists a unique and continuous SCB curve L_2 (see Figure 10) with limits $\theta = \theta_1(\mu)$ and the negative θ-axis as $d_0 \to 0$, and L_2 is C^{r-1} when $\mu \neq 0$. System $(0.1)_\alpha$ has exactly two (resp. none) limit cycles near Γ_0 if (θ, μ) is in the right (resp. left) side of L_2.*

The proof of Theorems 0.1-0.6 will be given in [16].

§1. Invariance of the Divergence Integral and Melnikov Function

In this section, we show that the integral of the divergence and the Melnikov function are invariants of the C^2 coordinate transformation.

Let $L : x = x_i(t)$, $y = y_i(t)$ for $i = 1, \cdots, m$ and $-\frac{1}{2}T_i \leq t \leq \frac{1}{2}T_i$ be a periodic orbit (corresponding to $i = 1$ and T_1 finite) or a simple closed path consisting of m singular points and m pieces of orbits (corresponding to $T_i = \infty$) of the following C^r system

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned} \tag{1.1}$$

If $r_1 \leq r$, then a C^{r_1} transformation T

$$\begin{aligned} u &= u(x, y), \\ v &= v(x, y) \end{aligned} \tag{1.2}$$

transforms (1.1) into a $C^{r_1 - 1}$ system

$$\begin{aligned} \dot{u} &= f(u, v), \\ \dot{v} &= g(u, v), \end{aligned} \tag{1.3}$$

where $\begin{pmatrix} f \\ g \end{pmatrix} = A \begin{pmatrix} P \\ Q \end{pmatrix}$, $A = \frac{\partial(u,v)}{\partial(x,y)}$, and L becomes

$$L' : u = u_i(x(t), y(t)), \quad v = v_i(x(t), y(t)) \quad \text{for } i = 1, \cdots, m.$$

Theorem 1.1. *If $2 \le r_1 \le r$, then $\int_{L'}(f_u+g_v)dt = \int_L (P_x+Q_y)dt$. In other words, the integral of the divergence of a C^r $(r \ge 2)$ system along a periodic orbit or a simple closed path consisting of singular points and orbits is an invariant of the C^2 coordinate transformation.*

Proof. Denote $D = \det A = u_x v_y - u_y v_x$. Then $D \in C^{r_1-1} \subset C^1$, $D \neq 0$, and $x_u D = v_y$, $x_v D = -u_y$, $y_u D = -v_x$, $y_v D = u_x$. By a careful calculation, we obtain

$$\begin{aligned} f_u + g_v &= \mathrm{tr}(\frac{\partial(f,g)}{\partial(u,v)}) \\ &= \mathrm{tr}(A\frac{\partial(P,Q)}{\partial(x,y)}A^{-1}) + D^{-1}(P,Q)\begin{pmatrix} v_y u_{xx} - v_x u_{xy} - u_y v_{xx} + u_x v_{xy} \\ v_y u_{xy} - v_x u_{yy} - u_y v_{xy} + u_x v_{yy} \end{pmatrix} \\ &= \mathrm{tr}(\frac{\partial(P,Q)}{\partial(x,y)}) + D^{-1}(D_x P + D_y Q). \end{aligned}$$

The last equality holds since the trace is an invariant of the similarity transformation. Now it follows that

$$\begin{aligned} \int_{L'}(f_u+g_v)dt &= \int_L (P_x+Q_y)dt + \int_L D^{-1}(D_x dx + D_y dy) \\ &= \int_L (P_x+Q_y)dt + \int_L d(\ln|D|) \\ &= \int_L (P_x+Q_y)dt. \end{aligned}$$

Next we prove that The Melnikov function is an invariant of the C^2 coordinate transformation up to a positive constant factor.

Consider the C^r perturbation system

$$\begin{aligned} \dot{x} &= P(x,y) + \varepsilon P_1(x,y,\varepsilon,\alpha), \\ \dot{y} &= Q(x,y) + \varepsilon Q_1(x,y,\varepsilon,\alpha), \end{aligned} \tag{1.4$_{\varepsilon\alpha}$}$$

where P, Q, P_1, $Q_1 \in C^r$, $r \ge 2$, $\varepsilon \in R$, $\alpha \in R^n$ are parameters.

Let $L : x = x(t)$, $y = y(t)$ be either a periodic orbit with period T or a homoclinic orbit or a heteroclinic orbit of $(1.4)_{0\alpha}$. Take point $p \in L$ arbitrarily and time t such that the coordinate of p is $(x(0), y(0))$. Let Σ be a section of an orthogonal orbit of system $(1.4)_{\varepsilon\alpha}$ passing through p, $L_{\varepsilon\alpha}$ an orbit of $(1.4)_{\varepsilon\alpha}$ running through p when L is a periodic orbit, $p_1 \in \Sigma$ the first returning point of p along $L_{\varepsilon\alpha}$ with the increasing of t, $d(\varepsilon,\alpha)$ the directed distance between p and p_1. When L is either a homoclinic orbit or a heteroclinic orbit, we denote by W^s (resp. W^u) the stable (resp. unstable) manifold of a singular point sufficiently near L with W^s, $W^u \to L$ as $\varepsilon \to 0$, p_1^s and p_1^u the first intersection points of W^s and W^u with Σ respectively, $d(\varepsilon,\alpha)$ the directed length of the vector $p_1^s - p_1^u$. If L is either a periodic orbit or a homoclinic orbit, then, as usual, the outer direction is designed as the positive direction.

It follows from [5, 6, 12] that

$$d(\varepsilon,\alpha) = \begin{cases} \varepsilon\Phi(\alpha) + O(\varepsilon) & \text{when } L \text{ is a periodic orbit,} \\ \varepsilon B_0^{-1} M(\alpha) + O(\varepsilon) & \text{when } L \text{ is either a homoclinic orbit or a heteroclinic orbit,} \end{cases}$$

where $B_0 = (P^2(x(0), y(0)) + Q^2(x(0), y(0)))^{\frac{1}{2}}$, $\Phi(\alpha) = B_0^{-1}M(\alpha)$, and

$$M(\alpha) = \int_L \exp\Big(-\int_0^t (P_x + Q_y)dt\Big)(PQ_1(x, y, 0, \alpha) - QP_1(x, y, 0, \alpha))dt. \tag{1.5}$$

$M(\alpha)$ is usually called the Melnikov function, and it plays a very important part in the research of the homoclinic and heteroclinic bifurcation problems, whereas the function $\Phi(\alpha)$ plays an essential role in the bifurcation problem concerned with a family of closed orbits (see [5, 6, 12]).

Under the C^{r_1} transformation (1.2), system $(1.4)_{\varepsilon\alpha}$ takes the form

$$\begin{aligned} \dot{u} &= f(u, v) + \varepsilon f_1(u, v, \varepsilon, \alpha), \\ \dot{v} &= g(u, v) + \varepsilon g_1(u, v, \varepsilon, \alpha), \end{aligned} \tag{1.6}$$

where $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = A \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}$. Now, L, Φ and M become L', Φ' and M' respectively.

Let $D_0 = D(x(0), y(0))$, $B_{10} = (f^2(u_0, v_0) + g^2(u_0, v_0))^{\frac{1}{2}}$, where $u_0 = u(x(0), y(0))$, $v_0 = v(x(0), y(0))$.

Theorem 1.2. *If $2 \le r_1 \le r$, then $\Phi'(\alpha) = D_0B_0B_{10}^{-1}\Phi(\alpha)$ and $M'(\alpha) = D_0M(\alpha)$, i.e., neglecting a positive constant factor, the functions $\Phi(\alpha)$ and $M(\alpha)$ are invariants under a C^2 transformation.*

Proof. Since the proof is similar, we only show the invariance of $M(\alpha)$.

Denote $D(t) = D(x(t), y(t))$. Then from the proof of Theorem 1.1 we have the following equality

$$\int_0^t (f_u + g_v)\big|_{L'} dt = \int_0^t (P_x + Q_y)\big|_L dt + \ln|D(t)| - \ln|D_0|.$$

Using the relation $fg_1(x, y, 0, \alpha) - gf_1(x, y, 0, \alpha) = D(PQ_1(x, y, 0, \alpha) - QP_1(x, y, 0, \alpha))$, we obtain

$$\begin{aligned} M'(\alpha) &= \int_{L'} \exp(-\int_0^t (f_u + g_v)dt)(fg_1(x, y, 0, \alpha) - gf_1(x, y, 0, \alpha))dt \\ &= D_0 \int_L \exp(-\int_0^t (P_x + Q_y)dt)(PQ_1(x, y, 0, \alpha) - QP_1(x, y, 0, \alpha))dt \\ &= D_0M(\alpha). \end{aligned}$$

Remark 1.1. Let $\varepsilon = |\alpha|$, $P(x, y) = F(x, y, 0)$, $Q(x, y) = G(x, y, 0)$, $\varepsilon P_1(x, y, \alpha) = F(x, y, \alpha) - P(x, y)$, $\varepsilon Q_1(x, y, \alpha) = G(x, y, \alpha) - Q(x, y)$. Then system $(0.1)_\alpha$ has the form of $(1.4)_{\varepsilon\alpha}$.

§2. Poincaré Map

Consider the C^r system

$$\begin{aligned} \dot{x} &= F(x, y, \alpha), \\ \dot{y} &= G(x, y, \alpha), \end{aligned} \tag*{$(2.1)_\alpha$}$$

where $r \ge 5$, $\alpha \in R^n$. Assume that O is a fine saddle of system $(2.1)_0$ with order 1, Γ_0 is a homoclinic loop of $(2.1)_0$ passing through O, the first saddle value $v_1(0) > 0$, and the divergence integral $I(0) = 0$.

Take a C^r transformation T such that system $(2.1)_\alpha$ is changed into the following C^r system (rescale the time t if necessary)

$$\begin{aligned}\dot{x} &= -\lambda x + P(x,y) + f(x,y,\alpha),\\ \dot{y} &= y + Q(x,y) + g(x,y,\alpha),\end{aligned} \tag{2.2}$$

where $P,\ Q,\ f,\ g \in C^r,\ f(x,y,0) = g(x,y,0) = 0,\ \lambda = 1+\mu,\ \alpha = (\alpha_1, \cdots, \alpha_{n-1}, \mu)$, functions P, Q, f, g and their first order derivatives all vanish at the point $(x,y) = (0,0)$. Moreover, we can choose T such that, in some neighborhood U of the origin, system (2.2) has the form

$$\begin{aligned}\dot{x} &= -\lambda x - a_1(\alpha)x^2y + x^3y^2R_1(x,y,\alpha),\\ \dot{y} &= y + b_1(\alpha)xy^2 + x^2y^3R_2(x,y,\alpha),\end{aligned} \tag*{$(2.2)_1$}$$

where $R_1,\ R_2 \in C^r,\ R_1(0,0,\alpha) = a_2(\alpha),\ R_2(0,0,\alpha) = b_2(\alpha)$, for the meaning of a_2 and b_2 see $(0.2)_m$. For simplicity, we still use Γ_0 to denote the image of Γ_0 under the transformation T.

Using the Sil'nikov variable, now we establish the Poincaré map defined by the orbits in the neighborhood of Γ_0.

Take $\delta > 0$ sufficiently small such that the segments $L_x = \{(x,y) : 0 \le x \le \delta, y = \delta\}$ and $L_y = \{(x,y) : x = \delta, 0 \le |y| \le \delta\}$ are completely situated in U. For any $0 < y_0 < \delta$, there exists a unique time τ such that the orbit starting from (δ, y_0) at $t = 0$ firstly intersects L_x at $t = \tau$. Equivalently, the second component of the solution $(x(t,\delta,y_0), y(t,\delta,y_0))$ for the initial value problem of (2.2) satisfies $y(\tau,\delta,y_0) = \delta$.

1) The case $\mu > 0$.

Let $x_1 = x(\tau,\delta,y_0)$. When $0 \le t \le \tau$, it follows from the constant variation formula that

$$\begin{aligned}x(t) &= e^{-\lambda t}\Big(\delta - \int_0^t e^{\lambda s}(a_1x^2y + x^3y^2R_1)ds\Big),\\ y(t) &= e^t\Big(y_0 + \int_0^t e^{-s}(b_1xy^2 + x^2y^3R_2)ds\Big).\end{aligned} \tag{2.3}$$

Since the hyperbolicity guarantees the validity of the C^1 linearization theorem (see [3]) in the case of 2 dimension, we can use

$$x(t) = \delta e^{-\lambda t}, \quad y(t) = y_0e^t, \quad y_0 = \delta e^{-\tau}$$

as the first approximation in the right hand of (2.3), and get

$$\begin{aligned}x_1 &= \delta s^{1+\mu} - \mu^{-1}a_1\delta^3s^{2+\mu} + \mu^{-1}a_1\delta^3s^{2+2\mu} + O(\mu^{-1}s^{3+\mu}),\\ y_0 &= \delta s - \mu^{-1}b_1\delta^3s^2 + \mu^{-1}b_1\delta^3s^{2+\mu} + O(\mu^{-1}s^3),\end{aligned} \tag{2.4}$$

where $s = e^{-\tau}$. s is called the Sil'nikov time, and $s = 0$ is corresponding to $\tau = +\infty$.

Define the Poincaré map

$$\pi_1 : L_x \to L_y, \quad x_1 \mapsto \pi_1(x_1),$$

where π_1 is induced by the orbits. Selecting a suitable system of orthogonal curvilinear coordinates, we can obtain (see [12] Chapters 2 and 4)

$$\pi_1(x_1) = d(\alpha) + a(\alpha)x_1 + O(x_1^2), \tag{2.5}$$

where $d(\alpha)$ is the ordinate of the first intersection point of L_y with the unstable manifold of saddle O of the perturbation system (2.2), and

$$a(\alpha) = \exp\Big(\int_{T_1}^{T_2} (-\mu + P_x + Q_y + f_x + g_y)dt - \frac{1}{2}\int_{\Gamma_{01}} \frac{dB}{B}\Big), \tag{2.6}$$

where $B = (-x+P)^2+(y+Q)^2$, Γ_{01} is a piece of segment of Γ_0 starting at $(0,\delta)$ and ending at $(\delta,0)$, and T_1, T_2 are the times corresponding to points $(0,\delta)$ and $(\delta,0)$ respectively.

From the special form of $(2.2)_1$, it is easy to see that

$$\begin{aligned}\int_{T_1}^{T_2} (P_x + Q_y + f_x + g_y)dt &= \int_{-\infty}^{+\infty} (P_x + Q_y + f_x + g_y)dt \\ &= \int_{-\infty}^{+\infty} (f_x + g_y)dt. \end{aligned}\tag{2.7}$$

The last equality is valid simply because $I(0) = 0$.

Let L_y^+ be the upper part of L_y situated in the above of the x-axis. Then, still by the speciality of $(2.2)_1$, we get

$$\int_{\Gamma_{01}} \frac{dB}{B} = -\Big(\int_{L_y^+} + \int_{L_x} \frac{dB}{B}\Big) = 0. \tag{2.8}$$

Denote $\alpha^* = (\alpha_1, \cdots, \alpha_{n-1}, 0)$, $\exp(I(\alpha^*)) = 1+\theta$, $T_2 - T_1 = 2T$, $h_\theta = (1+\theta)(1-2\mu T)$, $h = h_0$, $E_\theta = a_1 h_\theta + b_1$, $E = E_0$, and

$$P_1(s,\theta,\mu,d) = \pi_1(x_1) - y_0. \tag{2.9}$$

Obviously,

$$I(\alpha^*) = \int_{-\infty}^{+\infty} (f_x + g_y)dt \tag{2.10}$$

is the divergence integral of the perturbation system (2.2) which keeps 0 as a fine saddle with order 1. Combining (2.4)-(2.10), we have

$$a(\alpha) = (1+\theta)e^{-2\mu T}, \tag{2.11}$$

$$\begin{aligned}P_1(s,\theta,\mu,d) &= d - \delta s + \delta h_\theta s^{1+\mu} + \mu^{-1} b_1 \delta^3 s^2 - \mu^{-1} E_\theta \delta^3 s^{2+\mu} \\ &\quad + \mu^{-1} a_1 \delta^3 h_\theta s^{2+2\mu} + r_1,\end{aligned}\tag{2.12}$$

where $r_1 = O(s^{2+2\mu}) + O(\mu^2 s^{1+\mu}) + O(\mu^{-1}s^3)$.

2) The case $\mu < 0$.

Let $t \to -t$. Then system $(2.2)_1$ becomes

$$\begin{aligned}\dot{x} &= \lambda x + a_1(\alpha)x^2 y - x^3 y^2 R_1(x,y,\alpha), \\ \dot{y} &= -y - b_1(\alpha)xy^2 - x^2 y^3 R_2(x,y,\alpha).\end{aligned}\tag*{$(2.2)_2$}$$

Define the Poincaré map $\pi_2 : L_y^+ \to L_x$, where we assume that L_x has been extended to the region $x < 0$. And denote $\theta_1 = (1+\theta)^{-1} - 1$, $g_{\theta_1} = (1+\theta_1)(1+2\mu T)$, $g = g_0$, $F_{\theta_1} = a_1 + b_1 g_{\theta_1}$, $F = F_0$, $s = e^{-\lambda\tau}$, $\mu_1 = (1+\mu)^{-1} - 1$, and $P_2(s,\theta,\mu,d) = \pi_2(y_1) - x_0$.

Then

$$
\begin{aligned}
P_2(s,\theta,\mu,d) &= d_1 + a^{-1}(\alpha)y_1 - x_0 + O(y_1^2) \\
&= d_1 - \delta s + \delta g_{\theta_1} s^{1+\mu_1} - \mu^{-1}a_1\delta^3 s^2 + \mu^{-1}\delta^3 F_{\theta_1} s^{2+\mu_1} \\
&\quad - \mu^{-1} b_1 \delta^3 g_{\theta_1} s^{2+2\mu_1} + r_2,
\end{aligned}
\tag{2.13}
$$

where $r_2 = O(s^{2+2\mu_1}) + O(\mu^{-1}s^3) + O(\mu^2 s^{1+\mu_1})$, d_1 and d have different signs, and $d_1 = 0$ iff $d = 0$. For simplicity, in the following, we always substitute $-d$ for d_1.

3) The case $\mu = 0$.

Using (2.3) and $s = e^{-\tau}$, we get

$$
\begin{aligned}
x_1 &= x(\tau) \\
&= \delta s + a_1\delta^3 s^2 \ln s + O(s^3 \ln s), \\
y_0 &= \delta s + b_1 \delta^3 s^2 \ln s + O(s^3 \ln s).
\end{aligned}
$$

Define $P_3(s,\theta,d) = \pi_1(x_1) - y_0$. Then

$$
\begin{aligned}
P_3(s,\theta,d) &= d + a(\alpha)x_1 - y_0 + O(x_1^2) \\
&= d + \delta\theta s + \delta^3 v_1 s^2 \ln s + r_3,
\end{aligned}
\tag{2.14}
$$

where $r_3 = O(s^2) + O(\theta s^2 \ln s)$.

Theorem 2.1. *P_1, P_2, P_3 defined by* (2.12)-(2.14) *respectively are C^r* (*resp. continuous*) *when $0 < s << 1$* (*resp. $0 \le s << 1$*) *and $|\theta|$, $|\mu|$, $|d| << 1$, and can be C^1 extended to the region $|s| << 1$. Moreover, $\lim\limits_{\mu\to 0^+}(P_1 - r_1) = P_3 - O(s^2)$, $r_1 = O(s^{2+2\mu}) + O(\mu^2 s^{1+\mu})$, and $r_2 = O(s^{2+2\mu_1}) + O(\mu^2 s^{1+\mu_1})$.*

Proof. Clearly, P_1 and P_3 are linear with respect to θ and d, whereas P_2 is linear with respect to θ_1 and d_1. And since system $(2.1)_\alpha$ is C^r, P_i is also C^r with respect to μ and $\tau = -\ln s$ (or $\tau = -\lambda^{-1}\ln s$ in case $i = 2$) for $i = 1,2,3$. Consequently, P_i is C^r when $s > 0$ for $i = 1,2,3$.

It is easy to see that, from the expression of P_i ($i = 1,2,3$), P_i is C^1 at $s = 0$, and can be extended to the region $s \le 0$.

Denote $s^\mu = \exp(\mu \ln s) = 1 + \mu\ln s + O(\mu^2)$. Then a simple calculation shows that $\lim\limits_{\mu\to 0}(P_1 - r_1) = P_3 - O(s^2)$. Comparing r_1 with r_3, we get $r_1 = O(s^{2+2\mu}) + O(\mu^2 s^{1+\mu})$.

Similarly, we can get the asymptotic expression of r_2.

Remark 2.1. In [10], it was established a general asymptotic expansion in any differentiable class k for the Poincaré map along a homoclinic loop Γ of any planar vector field unfolding x_α:

$$
\begin{aligned}
D_\alpha[x] - x &= \beta_0 + \alpha_1 x w[\cdots] + \beta_1 x[\cdots] + \cdots + \beta_k x^k[\cdots] \\
&\quad + \alpha_{k+1}x^{k+1}w[\cdots] + \phi_k,
\end{aligned}
\tag{2.15}
$$

where $w = \alpha_1^{-1}(x^{\alpha_1} - 1)$, ϕ_k is C^k for $x > 0$ and x is the parametrization of some transversal section.

Comparing (2.12)-(2.14) with (2.15), we easily see that the expansions (2.12)-(2.14) have the following three advantages.

1) The expansions of P_i $(i = 1, 2, 3)$ are more precise than that of $D_\alpha[x] - x$.

2) The first three coefficients in (2.12)-(2.14) have the already known meanings.

3) P_i $(i = 1, 2, 3)$ can be C^1 extended to the region $|s| << 1$ as claimed in Theorem 2.1, and this is essential to the proof of our results.

The proof of Theorems 0.1-0.6 will be presented in [16].

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Fig.1

Fig.2 **Fig.3** **Fig.4**

Fig.5 **Fig.6** **Fig.7**

Fig.8 **Fig.9** **Fig.10**