Chin. Ann. of Math.
15B: 2(1994),217-224.

TWO EXISTENCE THEOREMS OF PERIODIC SOLUTIONS FOR DIFFERENTIAL DELAY EQUATIONS**

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Abstract

This paper corrects and improves two theorems on the existence of non-trivial periodic solutions of differential delay equations published in this journal.

Keywords Differential delay equation, Periodic solution, Trajectory.
1991 MR Subject Classification 34C25, 34K10.

§1. Introduction

Since Kaplan and Yorke[1] provided a method of studying the existence of nontrivial periodic solutions of differential delay equations by use of ordinary differential equations, much work has been done to extend their results[2−4]. Instead of the equation $\dot{x}(t) = -f(x(t-1))$, paper [4] considers more general equations

$$\dot{x}(t) = -f(x(t), x(t-1)) \tag{1.1}$$

and

$$\dot{x}(t) = -F(x(t), x(t-1), \ldots, x(t-1)) \tag{1.2}$$

and gives two theorems on the existence of nontrivial periodic solutions. Although these theorems are interesting, there is some thing wrong in their proofs which makes the corresponding conclusions unacceptable. Our purpose is to correct and improve the results given there.

§2. Errors in Paper [4]

It is supposed in paper [4] that

1° $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $xf(y,x) > 0$ for $x \neq 0, y \in \mathbb{R}$;

2° $f(-y,x) = f(y,x), f(y,-x) = -f(y,x)$;

3° $\mid f(y,x) \mid \leq r(\mid x \mid)$, where $f(s) \geq 0$ is continuous in s with $r(0) = 0$ and $r(s) > 0$ for $s > 0$;

4° $\int_0^\infty f(y,x)dx = +\infty$ for any fixed $y \in \mathbb{R}$;

Manuscript received November 18, 1991. Revised August 18, 1993.
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**Project supported by the National Natural Science Foundation of China

5° there is a constant $M > 0$ such that

$$M \mid f(y_1, x) \mid \geq \mid f(y_2, x) \mid \quad \text{for} \quad \mid y_1 \mid \geq \mid y_2 \mid \geq 0;$$

6° $\alpha = \lim\limits_{x \to 0} f(y, x)/x$ and $\beta = \lim\limits_{x \to \infty} f(y, x)/x$ (see Lemma 4, [4]).

The conclusion in [4] is that for an integer $k > 0$ if

$$\alpha < \frac{\pi}{2}(4k+1) < \beta \quad \text{or} \quad \beta < \frac{\pi}{2}(4k+1) < \alpha,$$

then Equation (1.1) has a nontrivial periodic solution with period $4/(4k+1)$ (Theorem 1, [4]) or when

$$f(x, y) = \begin{cases} F(x, y, -x, -y, \cdots, (-1)^{\frac{n}{2}} x), & n = \text{ even}, \\ F(x, y, -x, -y, \cdots, (-1)^{\frac{n+1}{2}} x, (-1)^{\frac{n+1}{2}} y), & n = \text{ odd}, \end{cases}$$

Equation (1.2) has a nontrivial periodic solution with period $4/(4k+1)$ (Theorem 2, [4]).

Remark. Theorem 2 in [4] gives only the result for $k = 1$. It is not difficult to extend it to the case for $k > 1$ if the theorem is true.

One of the bases for the proof of the above results is Lemma 4 in [4]. But it is not correct. Let

$$J_\lambda = \frac{x(t.\lambda) f(y(t, \lambda), x(t, \lambda)) + y(t, \lambda) f(x(t, \lambda), y(t, \lambda))}{x^2(t, \lambda) + y^2(t, \lambda)},$$

where $(x(t, \lambda), y(t, \lambda)), \lambda > 0$, is a trajectory of the equations

$$\begin{cases} \dot{x} = -f(x, y), \\ \dot{y} = f(y, x) \end{cases} \tag{2.1}$$

passing through the point (λ, λ) in the x, y-plane. Paper [4] claims $\lim\limits_{\lambda \to 0} J_\lambda(t) = \alpha$ under conditions $1^0 - 6^0$. Unfortunately this claim is incorrect. Therefore Lemma 4 and, as a result, all main conclusions in [4] remain unproved.

We give a counterexample to the mentioned claim as follows. Let

$$f(y, x) = \begin{cases} \left[1 + \frac{|xy|}{x^2+y^2}\right] x, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0. \end{cases}$$

Obviously $f(x, y)$ is continuous on $\mathbb{R}^2, f(-y, x) = f(y, x), f(y, -x) = -f(y, x)$ and

$$\mid f(y_2, x) \mid \leq \frac{3}{2} \mid x \mid \leq \frac{3}{2}\Big[1 + \frac{\mid y_1 x \mid}{x^2 + y_1^2}\Big] \mid x \mid = \frac{3}{2} \mid f(y_1, x) \mid, \quad \text{for} \quad \mid y_2 \mid \leq \mid y_1 \mid$$

and

$$\alpha = \lim_{x \to 0} \frac{f(y, x)}{x} = \lim_{x \to 0}\Big[1 + \frac{\mid xy \mid}{x^2 + y^2}\Big] = 1.$$

No matter how small $\lambda > 0$ is, there are points on the trajectory $(x(t, \lambda), y(t, \lambda))$ such that $\mid x(t, \lambda) \mid = \mid y(t, \lambda) \mid$ since $(x(t, \lambda), y(t, \lambda))$ is a closed trajectory around the origin $(0, 0)$ (see Lemma 1, [4]). At such points $J_\lambda(t) = 3/2$. Therefore

$$\lim_{\lambda \to 0} J_\lambda(t) \neq 1 = \alpha.$$

The same problem arises when $\lambda \to \infty$. So Lemma 4 in [4] is not true since its validity rests on the claim.

Such errors come from the neglect of the difference between limits of functions of one variable and of multiple variables. Besides, we shall find that the conditions $3^\circ - 5^\circ$ are not necessary.

§3. Main Results

We make two groups of assumptions.

(H_1):

1) $f \in C^0(\mathbb{R}^2, \mathbb{R}), yf(x,y) > 0$ for $y \neq 0, f(-x,y) = f(x,y)$ and $f(x,-y) = -f(x,y)$;

2) For $x \in \mathbb{R}, f(x,y)/y$ tends uniformly to $\alpha \geq 0$ as $y \to 0$ and to $\beta \geq 0$ as $y \to \infty$, where α and β may be infinite;

3) For all $u \in \mathbb{R}, \lim\limits_{(x,y)\to(u,\infty)} | f(y,x)/f(x,y) | < \infty$ or for $x \in \mathbb{R}, | f(x,y) | \leq h(y) < \infty$, where $h(y)$ is continuous on $\mathbb{R}$.

(H_2):

1) The same as 1) in (H_1);

2) $\lim\limits_{x^2+y^2\to 0} f(x,y)/y = \alpha \geq 0$ and $\lim\limits_{x^2+y^2\to\infty} f(x,y)/y = \beta \geq 0$, where α and β may be infinite;

3) $\lim\limits_{(x,y)\to(u,\infty)} | f(y,x)/f(x,y) | < \infty$ for any $u \in \mathbb{R}$.

Clearly (H_1) and (H_2) do not imply each other.

Theorem 3.1. *Suppose that* (H_1) *or* (H_2) *holds and* $k \geq 0$ *is an integer. If* $\alpha < \frac{\pi}{2}(4k+1) < \beta$ *or* $\beta < \frac{\pi}{2}(4k+1) < \alpha$*, then Equation* (1.1) *has at least one nontrivial periodic solution with period* $4/(4k+1)$.

Corollary 3.1. *Suppose that* (H_1) *or* (H_2) *holds. If one of* α *and* β *is infinite, then Equation* (1.1) *has infinitely many nontrivial periodic solutions.*

Theorem 3.2. *Suppose that* (H_1) *or* (H_2) *holds and* $k \geq 0$ *is an integer. If* $\alpha < \frac{\pi}{2}(4k+1) < \beta$ *or* $\beta < \frac{\pi}{2}(4k+1) < \alpha$ *and*

$$f(x,y) = \begin{cases} F(x,y,-x,-y,\ldots,(-1)^{\frac{n}{2}}x), & n = \text{even}, \\ F(x,y,\ldots,(-1)^{\frac{n-1}{2}}x,(-1)^{\frac{n-1}{2}}y), & n = \text{odd}, \end{cases}$$

then Equation (1.2) *has at least one nontrivial periodic solution with period* $4/(4k+1)$.

§4. Proof of Main Results

Lemma 4.1. *Suppose that* (H_1) *holds and there is a numger* $m > 0$ *such that*

$$\varliminf_{y\to\infty} | f(x,y) | \geq m \quad \text{if} \quad \lim_{(x,y)\to(u,\infty)} | f(y,x)/f(x,y) | < \infty$$

does not hold. Then all the normal trajectories of Equations (2.1) *are closed curves around the origin in the* x,y*-plane.*

Proof. Under Hypothesis (H_1), it is easy to see that the trajectories of Equation (2.1) are all symmetric with respect to both x- and y-axes. Furthermore, they are also symmetric with repect to both the lines $x - y = 0$ and $x + y = 0$. So we need only to prove that any trajectory starting from point $(\lambda, \lambda), \lambda > 0$, will intersect the positive y-semiaxis. Otherwise the fact that $\dot{x}(t) < 0$ and $\dot{y}(t) > 0$ for $x(t) > 0,\ y(t) > 0$ implies that $\lim_{t\to\infty} x(t) = x_0 \geq 0$ and $\lim_{t\to\infty} y(t) = +\infty$. Therefore

$$\lim_{t\to\infty} \frac{dy}{dx} = \lim_{(x,y)\to(x_0,\infty)} \left[-\frac{f(y,x)}{f(x,y)}\right] = \infty.$$

If $\overline{\lim}_{(x,y)\to(u,\infty)} \mid f(y,x)/f(x,y) \mid < \infty$ for any $u \in \mathbb{R}$, then

$$\lim_{(x,y)\to(x_0,\infty)} \frac{f(y,x)}{f(x,y)} \neq \infty,$$

a contradiction. If $\mid f(x,y) \mid \leq h(y) < \infty$ and $\underline{\lim}_{y\to\infty} \mid f(x,y) \mid \geq m$ for any $x \in \mathbb{R}$, then

$$\lim_{(x,y)\to(x_0,\infty)} \left|\frac{f(y,x)}{f(x,y)}\right| \leq \frac{h(x_0)}{m} < \infty,$$

also a contradiction. This lemma is now proved.

Similarly we have

Lemma 4.2. *Suppose* (H_2) *holds. Then all the normal trajectories of Equation* (2.1) *are closed curves around the origin in the* x, y*-plane.*

Lemma 4.3. *If* $(x(t), y(t))$ *is a* 4ω*-periodic solution of Equation* (2.1), $\omega > 0$, *then* $x(t)$ *is a* 4ω*-periodic solution of*

$$\dot{x}(t) = -f(x(t), x(t-\omega)) \tag{4.1}$$

with $x(t-2\omega) = -x(t)$.

Proof. The proof of Lemma 3 of paper [4] shows $y(t) = x(t-\omega)$ and $x(t-2\omega) = -x(t)$. Then the first equation in Equation (2.1) implies our conclusion.

Lemma 4.4. *When* $\omega = 1/(4k+1)$, *any* $4/(4k+1)$*-periodic solution of Equation* (4.1) *is also a periodic solution of Equation* (1.1) *with the same period.*

For the proof see [1], [3] or [4].

Let $(x(t,\lambda), y(t,\lambda))$ be the periodic solution of Equation (2.1) passing through the point (λ, λ), $\lambda > 0$, and T_λ its period.

Lemma 4.5. *Under the conditions of Theorem* 3.1, *Equation* (2.1) *has at least one periodic solution with period* $4/(4k+1)$.

Proof. Without loss of generality we assume that

$$\beta < \frac{\pi}{2}(4k+1) < \alpha.$$

Let $\theta(t,\lambda) = \arctan[y(t,\lambda)/x(t,\lambda)]$. Then $\dot{\theta}(t,\lambda) = J_\lambda(t)$, where

$$\begin{aligned} J_\lambda(t) &= \frac{x(t,\lambda)f(y,x) + y(t,\lambda)f(x,y)}{x^2(t,\lambda) + y^2(t,\lambda)} \\ &= [x^2(t,\lambda) + y^2(t,\lambda)]^{-1}\Big[x^2(t,\lambda)\frac{f(y,x)}{x(t,\lambda)} + y^2(t,\lambda)\frac{f(x,y)}{y(t,\lambda)}\Big]. \end{aligned}$$

Under the conditions of Theorem 3.1 and the additional assumption

$$\underline{\lim}_{y\to+\infty} \mid f(x,y) \mid \geq m > 0, \quad \mid f(x,y) \mid \leq h(y) < \infty,$$

we have

$$(x(t,\lambda), y(t,\lambda)) \to (0,0) \text{ uniformly as } \lambda \to 0.$$

Otherwise there will be a singular trajectory other than $(0,0)$, a contradiction to Lemma 4.1.

a) Suppose (H_2) holds. If $\alpha < \infty$, then

$$\frac{f(y,x)}{x(t,\lambda)} = \alpha + o(1), \qquad \frac{f(x,y)}{y(t,\lambda)} = \alpha + o(1)$$

as $\lambda \to 0$. This implies $J_\lambda(t) = \alpha + o(1)$. Therefore when λ is small enough,

$$2\pi = \int_0^{2\pi} d\theta = \int_0^{T_\lambda} J_\lambda(t)dt = [\alpha + o(1)]T_\lambda$$

and hence

$$T = \frac{2\pi}{\alpha + o(1)} < \frac{4}{4k+1}.$$

If $\alpha = +\infty$, then for a number $M > \frac{\pi}{2}(4k+1)$ there is a $\delta > 0$ such that

$$\frac{f(y,x)}{x(t,\lambda)} > M, \qquad \frac{f(x,y)}{y(t,\lambda)} > M$$

and hence $J_\lambda(t) > M$ when $\lambda < \delta$. Therefore $T_\lambda =< 2\pi/M < 4/(4k+1)$.

Similarly we have $T_\lambda > 4/(4k+1)$ when λ is large enough, no matter whether $\beta > 0$ or $\beta = 0$. So there is at least one $\lambda_0 > 0$ such that $T_{\lambda_0} = 4/(4k+1)$.

b) Suppose (H_1) holds. If $\lim\limits_{(x,y)\to(u,\infty)} \mid f(y,x)/f(x,y) \mid < \infty$ holds for all $u \in \mathbb{R}$ or $\mid f(x,y) \mid \le h(y)$ and $\underline{\lim}_{y\to\infty} \mid f(x,y) \mid \ge m > 0$ hold for all $x \in \mathbb{R}$, then it follows from Lemma 4.1 that all the trajectories of Equation (2.1) are periodic around the origin.

The condition that $f(x,y)/y \to \alpha$ uniformly as $y \to 0$ implies

$$\frac{f(x,y)}{y(t,\lambda)} = \alpha + o(1) \qquad \text{when } \alpha < \infty$$

or

$$\frac{f(x,y)}{y(t,\lambda)} > M > \frac{\pi}{2}(4k+1) \qquad \text{when } \alpha = \infty.$$

A similar argument as above leads to the inequality.

$$T_\lambda < 4/(4k+1) \qquad \text{for small } \lambda. \tag{4.2}$$

On the other hand, the condition $f(x,y)/y \to 0$ uniformly as $y \to \infty$ implies that for a number $b \in (\beta, \frac{\pi}{2}(4k+1))$ there is a $G > 0$ such that

$$\frac{f(x,y)}{y(t,\lambda)} < b < \frac{\pi}{2}(4k+1) \qquad \text{for } \mid y(t,\lambda) \mid > G,$$

$$\frac{f(y,x)}{x(t,\lambda)} < b < \frac{\pi}{2}(4k+1) \qquad \text{for } \mid x(t,\lambda) \mid > G.$$

Let $K = \max\limits_{|y|\le G} h(y)$ and take $M > 0$ so large that

$$M > \sqrt{\frac{GK}{[\frac{\pi}{2}(4k+1) - b]}}.$$

Consider

$$D_1 = \{(x,y) \mid \mid x \mid, \mid y \mid > G\},$$
$$D_2 = \{(x,y) \mid \mid x \mid \le G, \mid y \mid > M\},$$
$$D_3 = \{(x,y) \mid \mid y \mid \le G, \mid x \mid > M\}.$$

Obviously $D_1 \cap D_2 = D_2 \cap D_3 = D_3 \cap D_1 = \phi$. Since

$$\lim_{\lambda\to\infty} [x^2(t,\lambda) + y^2(t,\lambda)] = \infty,$$

we have $(x(t,\lambda), y(t,\lambda)) \in D_1 \cup D_2 \cup D_3$, for λ large enough.

When $(x(t,\lambda), y(t,\lambda)) \in D_1$,

$$J_\lambda(t) = \frac{1}{x^2(t,\lambda)+y^2(t,\lambda)}\Big[x^2(t,\lambda)\frac{f(y,x)}{x(t,\lambda)} + y^2(t,\lambda)\frac{f(x,y)}{y(t,\lambda)}\Big]$$
$$< b < \frac{\pi}{2}(4k+1).$$

And when $(x(t,\lambda), y(t,\lambda)) \in D_2$,

$$J_\lambda(t) = \frac{1}{x^2(t,\lambda)+y^2(t,\lambda)}\Big[x(t,\lambda)f(y,x) + y^2(t,\lambda)\frac{f(x,y)}{y(t,\lambda)}\Big]$$
$$\le \frac{GK}{M_2} + b < \frac{\pi}{2}(4k+1).$$

Similarly $J_\lambda(t) < \frac{\pi}{2}(4k+1)$ when $(x(t,\lambda), y(t,\lambda)) \in D_3$. Therefore

$$2\pi = \int_0^{2\pi} d\theta = \int_0^{T_\lambda} J_\lambda(t)dt < \frac{\pi}{2}(4k+1)T_\lambda.$$

It follows that

$$T_\lambda > 4/(4k+1) \qquad \text{for } \lambda \quad \text{large enough .} \tag{4.3}$$

The inequalities (4.2) and (4.3) imply that there is at least one $\lambda_0 > 0$ such that

$$T_{\lambda_0} = 4/(4k+1).$$

c) The case which remains unproved is that (H_1) holds with $\mid f(x,y) \mid \le h(y) < \infty$ while $\underline{\lim}_{y\to\infty} \mid f(x,y) \mid = 0$. Since $f(x,y)/y \to \beta \ge 0$ uniformly as $y \to \infty$, such a case occurs only when $\beta = 0$. So there is an $M_1 > 0$ such that $f(x,y)/y < 1$ and hence $\mid f(x,y) \mid < \mid y \mid$ when $\mid y \mid \ge M_1$. Let

$$M_2 = \max_{|y|\le M_1} h(y) \le \sup_{|y|\le M_1} \mid f(x,y) \mid \ \text{ and } \ M = 1 + \max\{M_1, M_2\}.$$

We define $F(x,y)$ as follows:

$$F(x,y) = \begin{cases} f(x,y), & \mid y \mid \le M, \\ f(x, M\ \mathrm{sgn}(y)) + (y - M\ \mathrm{sgn}(y)), & \mid y \mid > M. \end{cases} \tag{4.4}$$

Obviously $F(x,y)$ satisfies all the requirements of (H_1) with

$$\bar\beta = 1 < \frac{\pi}{2}(4k+1) < \alpha = \bar\alpha$$

and

$$\bar h(y) = \begin{cases} h(y), & \mid y \mid \le M, \\ h(M\ \mathrm{sgn}(y)) + \mid y - M\ \mathrm{sgn}(y) \mid, & \mid y \mid > M. \end{cases}$$

The condition that $F(x,y)/y \to \bar{\beta} = 1$ uniformly as $y \to \infty$ implies

$$\varliminf_{y\to\infty} \mid F(x,y) \mid = \infty.$$

It follows from the established result in b) that

$$\dot{x}(t) = -F(x(t), x(t-1)) \tag{4.5}$$

has at least one $4/(4k+1)$-periodic solution $x(t)$.

We prove now that

$$m = \max_{t\in\mathbb{R}} \mid x(t) \mid = \max_{t\in\mathbb{R}} \mid x(t-1) \mid \le M.$$

Otherwise $m > M$. Let $x(t_0) = m$. Then $\dot{x}(t_0) = 0$ and thus $x(t_0 - 1) = 0$. Let E_1, E_2 and E_3 be the sets

$$\{t \in [t_0-1, t_0] \mid x(t-1) \mid \le M_1\}, \quad \{t \in [t_0-1, t_0] \mid M_1 <\mid x(t-1) \mid M_2\}$$

and $[t_0-1, t_0] - (E_1 + E_2)$ respectively. Then $\mu(E_1) + \mu(E_2) + \mu(E_3) = 1$, here $\mu(E_i)$ is the Lebesque measure of set $E_i,\ i = 1,2,3$. Now

$$\begin{aligned} x(t_0) &= -\int_{t_0-1}^{t_0} F(x(t), x(t-1))dt \\ &\le \Big(\int_{E_1} + \int_{E_2} + \int_{E_3}\Big) \mid F(x(t), x(t-1)) \mid dt \\ &\le M_2\mu(E_1) + M\mu(E_2) + (M + (m-M))\mu(E_3) \\ &< m, \end{aligned}$$

a contradiction. Therefore $\mid x(t) \mid, \mid x(t-1) \mid \le M$.

Since $x(t)$ is a $4/(4k+1)$-periodic solution of Equation (4.5) with $\mid x(t-1) \mid \le M$, we know from the relation $F(x,y) = f(x,y)$ for $\mid y \mid \le M$ that $x(t)$ is also a $4/(4k+1)$-periodic solution of Equation (1.1).

Lemma 4.5 is now proved.

It is obvious that Theorem 3.1 is a direct deduction of Lemmas 4.4 and 4.5.

Proof of Corollary 3.1. If $\alpha = \infty$, then we take a periodic solution $(x^*(t), y^*(t))$ of Equation (2.1) with trajectory Γ^* around the origin. Assume that its period is T^*. Take an integer $k_0 > 0$ such that $4/(4k_0+1) < T^*$. For any integer $k \ge k_0$ there is a periodic solution $(\bar{x}(t), \bar{y}(t))$ with the trajectory $\bar{\Gamma}$ enclosed by Γ^* and period less than $4/(4k+1)$ since $\alpha = \infty$. So Equation (2.1) has at least one nontrivial $4/(4k+1)$-periodic solution $(x_k(t), y_k(t))$ and thus $x_k(t)$ is a nontrivial $4/(4k+1)$-periodic solution of Equation (1.1).

If $\beta = \infty$, then $\varliminf_{y\to\infty} \mid f(x,y) \mid = \infty$ and hence the conditions of Lemma 4.1 are satisfied. A similar argument can show that Equation (1.1) has a nontrivial $4/(4k+1)$-periodic solution. The fact that $k \ge k_0$ is an arbitrary integer implies the expected result.

As to Theorem 3.2, we need only to notice that for a $4/(4k+1)$-periodic solution $x^*(t)$ of Equation (1.1), it holds that

$$x^*(t) = -x^*(t - \frac{2}{4k+1}) = -x^*(t - \frac{2}{4k+1} - \frac{8k}{4k+1}) = -x^*(t-2).$$

Then

$$\begin{aligned}\dot{x}^*(t) &= -f(x^*(t), x^*(t-1)) \\ &= \begin{cases} -F(x^*(t), x^*(t-1), \ldots, (-1)^{n/2}x^*(t)), & \text{for n= even} \\ -F(x^*(t), x^*(t-1), \ldots, (-1)^{\frac{n-1}{2}}x^*(t), (-1)^{\frac{n-1}{2}}x^*(t-1)), & \text{for n= odd} \end{cases} \\ &= -F(x^*(t), x^*(t-1), \ldots, x^*(t-n)).\end{aligned}$$

That is to say, $x^*(t)$ is a nontrivial $4/(4k+1)$-periodic solution of Equation (1.2) under the conditions of Theorem 3.1.

Remark. With our correction and improvement, the corollaries in [4] will hold. We do not list them here.

§5. Two Examples

After checking the conditions given in this paper, we know that the conclusions about those examples in paper [4] are true. We give other two examples here.

Example 5.1. If $f(x,y) = (ax^2+by^2)y, a, b > 0$, then Equation (1.1) has infinitely many nontrivial periodic solutions.

Proof. Since

$$\lim_{(x,y)\to(u,\infty)} \Big|\frac{f(y,x)}{f(x,y)}\Big| = \lim_{(x,y)\to(u,\infty)} \Big|\frac{ay^2+bx^2}{ax^2+by^2}\frac{x}{y}\Big| = 0 \text{ for } u \in \mathbb{R},$$

it is obvious that $f(x,y)$ satisfies hypothesis (H_2) with $\alpha = 0$ and $\beta = \infty$. Our conclusion comes from Corollary 3.1.

Example 5.2. If

$$f(x,y) = \frac{x^2+2y^2}{2x^2+y^2}\big(y^{1/3}+y^3\big),$$

then Equation (1.1) has infinitely many nontrivial periodic solutions.

Proof. It is easy to see that

$$\mid f(x,y) \mid \leq h(y) = 2 \mid y^{1/3} + y^3 \mid$$

and

$$\Big|\frac{f(x,y)}{y}\Big| \geq \frac{1}{2\mid y\mid}\big|y^{1/3}+y^3\big|.$$

Therefore $\mid f(x,y)/y \mid \to \infty$ both as $y \to 0$ and as $y \to \infty$. After verifying the conditions of (H_1), we reach the conclusion in view of Corollary 3.1.

Those theorems and corollaries in [4] can not give the above results even if they were correct.

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