Chin. Ann. of Math.
15B: 2(1994),225-234.

ON THE MULTIPLE TIME SET OF BROWNIAN MOTIONS**

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Abstract

Let S_d^p be the p-multiple time set of the Brownian motion in d dimensions. In this paper, the Hausdorff measure function for S_3^2 is proved to be $\varphi_3^{(2)} = t^{1/2}(\log|\log t|)^{3/2}$, and the Hausdorff measuure problem for S_2^p is also discussed. As a result, a conjecture suggested by J. Rosen is partially proved.

Keywords Hausdorff measure, Intersection local time, Multiple time set, Brownian motion.
1991 MR Subject Classification 60G17, 60j65.

§1. Introduction

Let $B^d = \{B_t^d\}_{t\geq 0}$ be the Brownian motion in R^d starting at the origin, whose probability measure is denoted by μ. It is well known that the sample path of B^2 has points of p-multiplicity for any $p \geq 2$, and that of B^2 has only double points. For convenience, we let

$$D_d^p = \{x \in R^d : \ B_{t_1}^d = \cdots = B_{t_p}^d = x \text{ for some } 0 \leq t_1 < \cdots < t_p < \infty\},$$

$$S_d^p = \{(t_1, \cdots, t_p) \in R_+^d : \ B_{t_1}^d = \cdots = B_{t_p}^d \text{ for some } 0 \leq t_1 < \cdots < t_p < \infty\}.$$

There are already a lot of works on the study of the Hausdorff measure problem or Hausdorff dimension problem for D_d^p and S_d^p. More precisely, Le Gall[2] proved that the Hausdorff measure function of D_d^p is $h_p(x)$ and $k_2(x)$ respectively for $d = 2, \ p \geq 2$ and $d = 3, \ p = 2$, where

$$h_p(x) = x^2(\log x^{-1} \log\log\log x^{-1})^p, \quad \forall p \geq 2, \ \forall x \in (0, 16^{-1}),$$

$$k_2(x) = x\big(\log\log x^{-1}\big)^2, \quad x \in (0, 1/4).$$

J. Rosen[6,7] proved that for $p \geq 2$

$$\dim S_3^2 = \frac{1}{2}; \quad \dim S_2^p = 1, \quad \text{a.e.} - \mu,$$

Manuscript received October 8, 1991. Revised July 9, 1993.
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**Project supported by the National Natural Science Foundation of China

where dim A denotes the Hausdorff dimension of the set A. However, the Hausdorff measure problem is still open for the set S_d^p. in [6], J. Rosen conjectured that the Hausdorff measure function of S_d^2 is $\varphi_d^{(2)}$, where

$$\varphi_d^{(p)} = t^{2-d/2}(\log|\log t|)^{d(p-1)/2}, \quad p \geq 2.$$

The aim of this paper is to investigate the Hausdorff measure problem for S_d^p. To state the main theorem, we let $\alpha_p^d(x, A)$ be the $p-$multiple intersection local time which satisfies

$$\int_{R^{d(p-1)}} f(x)\alpha_p^d(x, A)dx = \underset{A}{\int \cdots \int} f\big(B_{t_2}^d - B_{t_1}^d, \cdots, B_{t_p}^d - B_{t_{p-1}}^d\big)dt_1 \cdots dt_p$$

for any bounded Borel function f on $R^{d(p-1)}$, where

$$A \subset R_<^{+p} = \{(s_1, \cdots, s_p) : 0 \leq s_1 < \cdots < s_p < \infty\} \quad \text{and} \quad p \geq 2.$$

It is well known that $\alpha_p^d(x, A)$ exists only for $d = 2,\ p \geq 2$ and $d = 3,\ p = 2$. Usually, one denotes

$$\alpha_p(x, A) = \underset{A}{\int \cdots \int} \delta_{(x)}\big(B_{t_2}^d - B_{t_1}^d, \cdots, B_{t_p}^d - B_{t_{p-1}}^d\big)dt_1 \cdots dt_p$$

for $A \subset R_<^{+p},\ p \geq 2$, and $x \in R^{d(p-1)}$.

The main result of this paper is as follows.

Theorem 1.1. *There are constants $c_0, c_1 \in (0, \infty)$ such that*

$$c_0\alpha_2^3(0, A) \leq \varphi_3^{(2)} - m(S_3^2 \cap A) \leq c_1\alpha_2^3(0, A), \ \mu - \text{a.e.} \tag{1.1}$$

for any $A \subset R_<^{+2} \cap [0, t]^2$ and $t > 0$, where $\varphi - m(B)$ denotes the Hausdorff-φ-measure of the set $B \subset R^2$.

This paper is organized as follows. In Section 2, the lower bound in (1.1) is proved, and the upper bound in (1.1) is proved in Section 3. Our idea to prove Theorem 1.1 is basically from [2]. In Section 4, we make an argument for the Hausdorff measure problem for S_2^p with $p \geq 2$. Unfortunately, we are unable to solve completely this problem for $d = 2$.

§2. Lower Bound

The aim of this section is to show that for some constant $c_1 \in (0, \infty)$

$$c_1\alpha_2^3(0, S_3^2 \cap [a_1, b_1] \times [a_2, b_2]) \leq \varphi_3^{(2)} - m(S_3^2 \cap [a_1, b_1] \times [a_2, b_2]), \quad \mu - \text{a.e.} \tag{2.1}$$

where $a_1 < b_1 < a_2 < b_2$. We begin with a lemma.

Lemms 2.1. *There is a constant $c_2 \in (0, \infty)$ such that for any $s \in (0, \infty)$*

$$\overline{\lim_{a \to 0^+}} \big(\varphi_3^{(2)}(a)\big)^{-1}\alpha_2^3(0, [s-a, s] \times [s, s+a]) \leq c_2, \quad \text{a.e.} - \mu. \tag{2.2}$$

Proof. It is easy to see that for any $s \in (a, 1)$

$$\alpha_2^3(0, [s-a, s] \times [s, s+a]) \overset{(d)}{=} d^{1/2}\alpha_2^3(0, [0, 1] \times [1, 2]).$$

Thus, by [4, Lemma 2.2] we know that for ome constant $c_3 \in (0, \infty)$

$$E_\mu\Big[\alpha_2^3(0, [s-a, s] \times [s, s+a])\Big] \leq c_3^k a^{k/2}(k!)^{3/2}, \quad \forall k \geq 1, \quad \forall a \in (0, s),$$

where E_μ is the expectation with respect to μ. Hence, there are constant $c_4, c_5 \in (0,\infty)$ such that

$$E_\mu\Big[\Big(\exp\Big(c_4\big(\alpha_2^3(0,[s-a,s]\times[s,s+a])\big)\Big/a^{1/2}\Big)^{2/3}\Big)\Big] \le c_5, \quad \forall a\in(0,s). \tag{2.3}$$

Fix $s\in(0,\infty)$ and let

$$a_n = e^{-n/\log n}, \quad \forall n\ge 2,$$

and for $\varepsilon\in(0,\infty)$

$$F_n(\varepsilon)=\Big\{\alpha_2^3(0,[s-a_n,s]\times[s,s+a_n])\ge(c_4^{-1}+\varepsilon)^{3/2}a_{n+1}^{1/2}(\log|\log a_{n+1}|)^{3/2}\Big\}.$$

By (2.3) we have

$$\begin{aligned}\mu(F_n(\varepsilon)) \le&\mu\Big\{\alpha_2^3(0,[s-a_n,s]\times[s,s+a_n])\ge(c_4^{-1}+\varepsilon)^{3/2}a_n^{1/2}(\log|\log a_n|)^{3/2}\Big\}\\ \le& c_5\exp\big(-_4(c_4^{-1}+\varepsilon/2)\log|\log a_n|\big)\\ \le& c_5(n/\log n)^{-1-c_4\varepsilon/2},\end{aligned}$$

which implies

$$\sum_{n=2}^{\infty}\mu(F_n(\varepsilon))<\infty.$$

Note that $\alpha_2^3(0,[s-a_n,s]\times[s,s+a_n])$ is increasing as a function of $a\in(0,s)$, and $\varphi_3^{(2)}$ is also an increasing function. Then, by means of the Borel-Cantelli lemma we can easily show that (2.2) holds.

Now we can prove (2.1).

Proof of (2.1). For any process $\{X(t)\}_{t\ge0}$ and $0\le u\le v$, let

$${}_uX_v(t)=\begin{cases}X(u+t)-X(t), & t\le v-u,\\ X(v)-X(u), & t>v-u,\end{cases}$$

$${}_vX_u(t)=\begin{cases}X(v-t)-X(u), & t\le v-u,\\ X(u)-X(v), & t>v-u.\end{cases}$$

Let $\{X^1(t)\}_{t\ge0}$ and $\{X^2(t)\}_{t\ge0}$ be independent Brownian motion in R^3 starting at the origin, and $\tilde\alpha_{(s_1,s_2)}(0,A)$ be the intersection local time of ${}_0X^1_{b_1-s_1}$ and ${}_0X^2_{b_2-s_2}$ for any fixed $a_1<s_1<b_1<a_2<s_2<b_2$. It is clear that

$$\tilde\alpha_{s_1,s_2)}(0,[0,a]\times[0,a])\overset{(d)}{=}\alpha_2^3(0,[1-a,1]\times[1,1+a]), \quad \forall a\in(0,1).$$

Thus, by Lemma 2.1 we know that

$$\overline{\lim_{a\to0^+}}\big(\varphi_3^{(2)}(a)\big)^{-1}\tilde\alpha_{(s_1,s_2)}(0,[0,a]\times[0,a])\le c_2, \quad \mu-\text{a.e.} \tag{2.4}$$

By setting up a product measure in $[a_1,b_1]\times[a_2,b_2]$ and using Fubini's theorem, we can prove that (2.4) holds for a.e.-$m(s_1,s_2)\in[a_1,b_1]\times[a_2,b_2]$, where m is the Lebesgue measure in R^2. Thus, we can easily get the following from [5, Theorem 2.2]

$$\alpha_3^2\Big(0,\Big\{(s_1,s_2):\overline{\lim_{a\to0^+}}\big(\varphi_3^{(2)}(a)\big)^{-1}\alpha_2^3(0,[s_1,s_2+a]\times[s_2,s_2+a])\le c_2\Big\}\Big)=0, \quad \text{a.e.}-\mu. \tag{2.5}$$

By (2.5) and [2, Proposition 2.6] one can show that for some constant $c_6 \in (0,\infty)$

$$\begin{aligned}
&\varphi_3^{(2)} - m\Big(\Big\{(s_1,s_2)\in[a_1,b_1]\times[a_2,b_2] : B^3(s_1)=B^3(s_2)\Big\}\Big)\\
\geq&\varphi_3^{(2)} - m\Big(\Big\{(s_1,s_2)\in[a_1,b_1]\times[a_2,b_2] : B^3(s_1)=B^3(s_2),\\
&\varlimsup_{a\to 0^+}\big(\varphi_3^{(2)}(a)\big)^{-1}\alpha_2^3(0,[s_1,s_2+a]\times[s_2,s_2+a])\leq c_2\Big\}\Big)\\
\geq&c_6c_2^{-1}\alpha_2^3\Big(0,\Big\{(s_1,s_2)\in[a_1,b_1]\times[a_2,b_2] : B^3(s_1)=B^3(s_2),\\
&\varlimsup_{a\to 0^+}\big(\varphi_3^{(2)}(a)\big)^{-1}\alpha_2^3(0,[s_1,s_2+a]\times[s_2,s_2+a])\leq c_2\Big\}\Big)\\
=&c_6c_2^{-1}\alpha_2^3\Big(0,\Big\{(s_1,s_2)\in[a_1,b_1]\times[a_2,b_2] : B^3(s_1)=B^3(s_2)\Big\}\Big)\\
=&c_6c_2^{-1}\alpha_2^3(0,[a_1,b_1]\times[a_2,b_2]).
\end{aligned}\tag{2.6}$$

Note that the constants c_2 and c_6 do not depend on the choices of $a_1<b_1<a_2<b_2$. Hence (2.6) also holds if $[a_1,b_1]\times[a_2,b_2]$ is replaced by any set $A\subset R_<^{+2}\cap[0,t]^2$ with $t<\infty$. In other words,

$$\varphi_3^{(2)} - m(S_3^2\cap A)\geq c_6c_2^{-1}\alpha_2^3(0,A)$$

for any $A\subset R_<^{+2}\cap[0,t]^2$ with $t<\infty$, which proves (2.1).

§3. Upper Bound

The aim of this section is to prove that for some constant $c_1\in(0,\infty)$

$$\varphi_3^{(2)} - m(S_3^2\cap[a_1,b_1]\times[a_2,b_2])\leq c_1\alpha_2^3(0,S_3^2\cap[a_1,b_1]\times[a_2,b_2]),\ \text{a.e.}-\mu. \tag{3.1}$$

We also begin with a lemma

Lemms 3.1. *There is a constant $C_\gamma\in(0,\infty)$ for any $\gamma\in(0,1/2)$ such that*

$$\begin{aligned}
&\mu\Big(|\alpha_2^3(x,[1-t,1]\times[1,1+t])-\alpha_2^3(y,[1-t,1]\times[1,1+t])|\\
\geq&t^{1/2}|t^{-1/2}x-t^{-1/2}|^\gamma\Big)\leq\exp(-c_\gamma n^2),\quad\forall n\geq 1
\end{aligned}\tag{3.3}$$

for any $|x|\leq 1,\ |y|\leq 1$ and $t\in(0,1)$.

Proof. As in §2, let $\{X^1(t)\}_{t\geq 0}$ and $\{X^2(t)\}_{t\geq 0}$ be independent Brownian motions in R^3, starting at the origin, and $\beta_2(x,A)$ be the intersection local time of them, i.e.,

$$\beta_2(x,A)=\iint_A\delta(x)\Big(X^1(u)-X^2(v)\Big)dudv.$$

It is clear that

$$\alpha_2^3(x,[1-t,1]\times[1,1+t])\overset{(d)}{=}\beta_2(x,[0,t]\times[0,t]).$$

By the scaling property of Brownian motion, one easily shows that

$$\begin{aligned}
&\beta_2(x,[0,t]\times[0,t])-\beta_2(y,[0,t]\times[0,t])\\
\overset{(D)}{=}&[\beta_2(t^{1/2}x,[0,t]\times[0,t])-\beta_2(t^{-1/2}y,[0,t]\times[0,t])].
\end{aligned}$$

As in [1], we can show for any given $\gamma \in (0, \frac{1}{2})$

$$\begin{aligned}&E_\mu|\beta_2(t^{-1/2}x, [0,t] \times [0,t]) - \beta_2(t^{-1/2}y, [0,t] \times [0,t])|^k \\ \le& C_\gamma^k (k!)^2 |t^{-1/2}(x-y)|^{k\gamma}, \quad \forall k \ge 1,\end{aligned}$$

where $C_\gamma \in (0, \infty)$. Hence,

$$\begin{aligned}&E_\mu|\alpha_2^3(x, [1-t,1] \times [1,1+t]) - \alpha_2^3(y, [1-t,1] \times [1,1+t])|^k \\ =& t^{k/2} E_\mu|\beta_2(t^{-1/2}x, [0,t] \times [0,t]) - \beta_2(t^{-1/2}y, [0,t] \times [0,t])|^k \\ \le& C_\gamma^k (k!)^2 |t^{-1/2}(x-y)|^{k\gamma} t^{k/2}. \ \forall k \ge 1.\end{aligned}$$

Thus, there is a constant $C-2 \in (0,\infty)$ such that

$$\begin{aligned}&E_\mu\Big\{\exp\Big[\frac{1}{2}C_\gamma^{-1}\big|\alpha_2^3(x, [1-t,1] \times [1,1+t]) - \alpha_2^3(y, [1-t,1] \times [1,1+t])\big| \\ &\cdot t^{-1/2}|t^{-1/2}x - t^{-1/2}y|^\gamma\Big]^{1/2}\Big\} \le C_2, \quad \forall t \in (0,1), \quad |x|, \ |y| \le 1.\end{aligned}$$

By the Chebyshev inequality, one can easily get the desired result from the above estimate.

We are now in a position to prove (3.1).

Proof of (3.1). Let $a_n = 2^{-n^{1+\delta}}$ for some $\delta \in (0,1)$, $\forall n \ge 1$, and Ω_n denote the collection of the following type set in $[0,1] \times [1,2]$

$$E = \prod_{i=1}^{2} \big[k_i a_n, (k_i+1)a_n\big],$$

where k_1 and k_2 are integers satisfying

$$0 \le k_1 \le 2^{n^{1+\delta}}, \quad 2^{n^{1+\delta}} \le k_2 \le 2 \cdot 2^{n^{1+\delta}}.$$

Let N_n denote the number of the set E belonging to Ω_n which intersects with S_3^2, and such that for large enought $n_0 \ge 1$

$$\alpha_2^3(0, [s_E - a_k, s_E + a_k] \times [t_E - a_k, t_E + a_k]) \le r\varphi_3^{(2)}(a_k), \quad \forall k \in [n_0, n],$$

where r is a sufficient small positive constant, and (s_E, t_E) is the center of E. From the argument in [8, §6] one can see that it suffices to prove the following for proving (3.1)

$$\varliminf_{n\to\infty} \varphi_3^{(2)}(a_n) N_n = 0, \quad \text{a.e.} - \mu. \tag{3.3}$$

Now we let $\overline{\Omega}_n$ be the collection of the following type set in R^3

$$F = \prod_{i=1}^{3} \Big[l_i 2^{-\frac{1}{2}n^{1+\delta}}, (l_i+1)2^{-\frac{1}{2}n^{1+\delta}}\Big],$$

where l_i is an integer number and satisfies

$$-n2^{\frac{1}{2}n^{1+\delta}} \le l_i \le n2^{\frac{1}{2}n^{1+\delta}}, \quad i = 1,2,3.$$

Let N_n' be the number of the cubes in $\overline{\omega}_n$ to which the point $B^3(s) = B^3(t)$ belongs for some $(s,t) \in S_3^2$. Denote

$$B^{-1}(I) = \{s \in [0,2] : \ B^3(s) \in I\}, \quad I \subset R^3,$$

and let S be the number of k's such that $k \le 2[2^{n^{1+\delta}}]$ and there is a cubed I of length

$2^{-\frac{1}{2}n^{1+\delta}}$ satisfying

$$B^{-1}(I) \cap \left[k2^{-n^{1+\delta}}, (k+1)2^{-n^{1+\delta}}\right] \neq \emptyset.$$

As in the proof of [9, Lemma 2.4], one can prove that

$$\mu(S \geq n^4) \leq 2^{-3n^{1+\delta}}, \quad \forall n \gg 1. \tag{3.4}$$

Let $\widetilde{N}_n$ denote the number of E belonging to Ω_n which intersects with S_3^2. Then

$$N_n \leq 2^{-2n^{1+\delta}}, \quad \forall n \geq 1.$$

Note that for some constant $c_3 \in (0, \infty)$

$$\mu\Big(\max_{0 \leq t \leq 2} |B^3(t)| \geq n\Big) \leq e^{-c_3 n} \ll 2^{-3n^{1+\delta}}, \quad \forall n \gg 1.$$

Thus, by (3.4) we can show that

$$E_\mu \widetilde{N}_n \leq n^4 E_\mu N_n'.$$

By [3, Corollary 1.2] we know that for some constant $c_4 \in (0, \infty)$

$$E_\mu N_n' \leq c_4 2^{\frac{1}{2}n^{1+\delta}}.$$

Therefore,

$$E_\mu \widetilde{N}_n \leq c_4 n^4 2^{\frac{1}{2}n^{1+\delta}}, \quad \forall n \leq 1. \tag{3.5}$$

For any $E \in \Omega$, let $J_E = \{E \cap S_3^2 \neq \emptyset\}$. Then

$$\begin{aligned} E_\mu N_n = \sum_{E \in \Omega_n} & \mu(J_E) \cdot \mu\Big((\varphi_3^{(2)}(a_k))^{-1} \alpha_2^3(0, [s_E - a_k, s_E + a_k] \\ & \times [t_E - a_k, t_E + a_k]) \leq r, \quad \forall k \in [n_0, n] / J_E\Big). \end{aligned} \tag{3.6}$$

Note that there is a point $(\tau, \sigma) \in E$ such that

$$B^3(\tau) = B^3(\sigma),$$

and

$$\begin{aligned} & \alpha_2^3(0, [s_E - a_k, s_E + a_k] \times [t_E - a_k, t_E + a_k]) \\ = & \int_{s_E - a_k}^{s_E + a_k} \int_{t_E - a_k}^{t_E + a_k} \delta\big(B_s^3 - B_\tau^3 - (B_t^3 - B_\sigma^3)\big) ds dt, \end{aligned} \tag{3.7}$$

where we have assumed $s_E < t_E$. Since $(\tau, \sigma) \in E$, we have

$$\mu\Big(|B_\tau^3 - B_{s_E}^3| \geq 2^{-\frac{1}{2}n^{1+\delta}} n\Big) \leq \mu\Big(\sup_{0 \leq s \leq a_n} |B_s^3| \geq a_n^{1/2} n\Big) \ll 2^{-2n^{1+\delta}}, \quad \forall n \gg 1, \tag{3.8$_1$}$$

$$\mu\Big(|B_\sigma^3 - B_{t_E}^3| \geq 2^{-\frac{1}{2}n^{1+\delta}} n\Big) \ll 2^{-2n^{1+\delta}}, \quad \forall n \gg 1. \tag{3.8$_2$}$$

Without loss of the generality, we may assume $\tau \in [s_E - a_k, s_E]$ and $\sigma \in [t_E - a_k, t_E]$. Then

(3.7) and Lemma 3.1 can imply that

$$
\begin{aligned}
&\mu\Big(\Big|\int_{s_E}^{s_E+a_k}\int_{t_E}^{t_E+a_k}\delta\big(B_s^3-B_\tau^3-(B_t^3-B_\sigma^3)\big)dsdt\\
&-\int_{s_E}^{s_E+a_k}\int_{t_E}^{t_E+a_k}\delta\big(B_s^3-B_{s_E}^3-(B_t^3-B_{t_E}^3)\big)dsdt\Big|\ge\frac{1}{2}a_k^{1/2},\\
&|B_\tau^3-B_{s_E}^3|\le k^4a_k^{1/2},\ \ |B_\sigma^3-B_{t_E}^3|\le k^4a_k^{1/2}\Big)\\
\le&\mu\Big(\Big|\int_{s_E}^{s_E+a_k}\int_{t_E}^{t_E+a_k}\delta\big(B_s^3-B_{s_E}^3+(B_{s_E}^3-B_\tau^3)-(B_t^3-B_{t_E}^3)-(B_{t_E}^3-B_\sigma^3)\big)dsdt\\
&-\int_{s_E}^{s_E+a_k}\int_{t_E}^{t_E+a_k}\delta\big(B_s^3-B_{s_E}^3-(B_t^3-B_{t_E}^3)\big)dsdt\Big|\ge\frac{1}{2}a_k^{1/2},\\
&|B_\tau^3-B_{s_E}^3|\le k^4a_k^{1/2},\ \ |B_\sigma^3-B_{t_E}^3|\le k^4a_k^{1/2}\Big)\\
\le&2^{2k^{1+\delta}},\ \ \forall k\in[n_0,n-1],
\end{aligned}\tag{3.9}
$$

where we have used the strong Markov property of B^3 and the following fact

$$
\frac{a_n}{a_k}\le 2^{-n\delta},\ \ \forall k\in[n_0,n-1],\ \ n\ge n_0+1.\tag{3.10}
$$

By (3.7), (3.8) and (3.9) we can show that

$$
\begin{aligned}
&\text{r.h.s. of (3.6)}\le 2^{-n^{1+\delta}}+\sum_{E\in\Omega_n}\mu(J_n)\cdot\Big((\varphi_3^{(2)}(a_k))^{-1}\\
&\cdot\int_{s_E}^{s_E+a_k}\int_{t_E}^{t_E+a_k}\delta\Big(B^3(s)-B^3(s_E)-(B^3(t)-B^3(t_E))\Big)dsdt\le r,\\
&\forall k\in[n_0,n-1]/J_E\Big)\\
=&2^{-n^{1+\delta}}+E_\mu\widetilde{N}_n\cdot\mu\Big((\varphi_3^{(2)}(a_k))^{-1}\int_0^{a_k}\int_0^{a_k}\delta(X^1(s)-X^2(t))dsdt\le r,\\
&\forall k\in[n_0,n-1]\Big).
\end{aligned}\tag{3.11}
$$

We now estimate the following quantity

$$
\begin{aligned}
&\mu\Big(\int_0^{a_{n-1}}\int_0^{a_{n-1}}\delta(X^1(s)-X^2(t))dsdt\le r\varphi_3^{(2)}(a_{n-1})\Big)\\
\le&1-\exp(2c_5r[(1+\delta)\log n+\log\log 2]).
\end{aligned}\tag{3.12}
$$

By (3.10), Lemma 3.12 and [4, Lemma 2.2] we can show that for some constant $c_6,\ c_7\in$

$(0,\infty)$

$$
\begin{aligned}
&\mu\Big((\varphi_3^{(2)}(a_k))^{-1}\int_{a_{k+1}}^{a_k}\int_{a_{k+1}}^{a_k}\delta(X^1(s)-X^2(t))dsdt\le r\Big)\\
\le&\exp(-c_6k^2)+\mu\Big(|X^1(a_{k+1})|\le ka_{k+1}^{1/2},\quad |X^2(a_{k+1})|\le ka_{k+1}^{1/2},\\
&(\varphi_3^{(2)}(a_k))^{-1}\int_{a_{k+1}}^{a_k}\int_{a_{k+1}}^{a_k}\delta(X^1(s)-X^1(a_{k+1})-(X^2(t)-x^2(a_{k+1}))\\
&+X^1(a_{k+1})-X^2(a_{k+1}))dsdt\le r\Big)\\
\le&\exp(-c_6k^2)+\mu\Big(\int_0^{a_k-a_{k+1}}\int_0^{a_k-a_{k+1}}\delta(X^1(s)-X^2(t))dsdt\le 2r\varphi_3^{(2)}(a_k)\Big)\\
\le&1-\exp(-c_7r((1+\delta)\log k+\log\log 2)).
\end{aligned}
$$

By the Markov property we can show that

$$
\begin{aligned}
(3.12)=&\prod_{k=n_0}^{n-1}\Big[1-\exp(-c_7r((1+\delta)\log k+\log\log 2))\Big]\\
\le&\prod_{k=n_0}^{n-1}(1-k^{-c_7r(1+\delta/2)}),
\end{aligned}
$$

if $n_0\ge 1$ is large enough. We now choose $r>0$ to be small ehough. Then, there is a constant $c_8\in(0,\infty)$ for any given $K\ge 1$ such that

$$(3.12)\le c_8n^{-K},\quad \forall n\gg 1.$$

By (3.5) and (3.11) we know that for some constant $c_9\in(0,\infty)$

$$E_\mu N_n\le c_9n^{-k}n^4 2^{\frac12 n^{1+\delta}},\quad \forall n\gg 1.$$

Thus we have, if $K>4$,

$$\varphi_3^{(2}(a_n)E_\mu N_n\le c_9n^{-(K_4)}|\log\log 2^{-n^{1+\delta}}|\to 0,\quad n\to\infty,$$

which implies (3.3).

To sum up, we complete the proof of Theorem 1.2.

§4. Remark

In this section, we give some remarks on the Hausdorff measure problem for the multiple time set of the Brownian motions in R^2. Let $\theta_1,\cdots,\theta_p$ be p independent Brownian motion in R^2 and β_p be the multiple intersection local time of them, which is defined formally by

$$\beta_p(y,A)=\int\cdots\int_A\delta_{(y)}\big(\theta_1(s_1)-\theta_2(s_2),\cdots,\theta_{p-1}(s_{p-1})-\theta_p(s_p)\big)ds_1\cdots ds_p$$

for any $y\in R^{2p}$, $A\subset(R^+_<)^p$, and $p\ge 2$. Denote the p multiple intersection local time of B^2 by $\bar\alpha_p(y,A)$, i.e.,

$$\bar\alpha_p(y,A)=\int\cdots\int_A\delta_{(y)}\big(B^2(s_1)-B^2(s_2),\cdots,B^2(s_{p-1})-B^2(s_)\big)ds_1\cdots ds_p$$

for any $A \subset (R^+_<)^p$, $y \in R^{2p}$, and $p \geq 2$.

By [4, Lemma 2.2] thee are constants $c_1, c_2 \in (0,\infty)$ such that

$$c_1^k (k!)^{p-1} \leq E_\mu \Big[\beta_p(0,[0,1]\times\cdots\times[0,1])\Big]^k \leq c_2^k (k!)^{p-1}(\log k)^p. \tag{4.1}$$

By a similar argument in §3, we know that there is a constant $c_3 \in (0,\infty)$ such that

$$c_3 \bar\alpha_2(0, S_2^2 \cap [a_1,b_1]\times[a_2,b_2]) \geq \varphi_2^{(2)} - m(S_2^2 \cap [a_1,b_1]\times[a_2,b_2]), \quad \text{a.e.} - \mu,$$

where S_2^2 and $\varphi_2^{(2)}$ were defined in §1. One can easily generalize the corresponding result to $d=2$ and $p \geq 3$. In order words, there is a constant $C_p \in (0,\infty)$ such that

$$\begin{aligned}&\bar\alpha_p(0, S_2^p \cap [a_1,b_1]\times\cdots\times[a_p,b_p])\\ \geq& C_p \varphi_2^{(p)} - m(S_2^p \cap [a_1,b_1]\times\cdots\times[a_p,b_p]).\end{aligned} \tag{4.2}$$

Note that (see [5]) the distribution of

$$\{B^2(a_1+s_1), B^2(a_2+s_2), \cdots, B^2(a_p+s_p) : (s_1,\cdots,s_p) \in [0,b_1-a_1]\times\cdots\times[0,b_p-a_p]\}$$

is absolutely continuous with respect to that of

$$\{\theta_1(s_1), \theta_2(s_2), \cdots, \theta_p(s_p) \in [0,b_1-a_1]\times\cdots\times[0,b_p-a_p]\}$$

and vice versa, where $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$. Then, by a similar argument in [2, §4] we can show that it suffices to prove the following for proving (4.2)

$$\begin{aligned}&\beta_p(0, \overline{S}_2^p \cap [0,a_1,b_1]\times\cdots\times[0,a_p,b_p])\\ \geq& C'_p \varphi_2^{(p)} - m(\overline{S}_2^p \cap [0,a_1,b_1]\times\cdots\times[0,a_p,b_p])\end{aligned} \tag{4.3}$$

for some constant $C'_p \in (0,\infty)$, where

$$\overline{S}_2^p = \{(s_1,\cdots,s_p) : \theta_1(s_1) = \cdots = \theta_p(s_p)\} \subset (R^+_<)^p.$$

By a similar argument in §3, one can also show that (4.3) is actually a result of the lower bound in (4.1). Hence (4.2) is correct.

We now consider the estimate for the lower bound of

$$\varphi_2^{(p)} - m(S_2^p \cap [a_1,b_1]\times\cdots\times[a_2 b_2]).$$

For any $\epsilon \in (0,1)$, let

$$\varphi_\epsilon^{(p)}(t) = t(\log|\log t|)^{p-1+\epsilon}, \quad \forall t \in (0,1/4).$$

By (4.1) one can show that for some constant $c_4(\epsilon) \in (0,\infty)$

$$E_\mu\Big[\exp(c_4(\epsilon)\beta_p(0,[0,a]\times[0,a])/a)^{\frac{1}{p-1+\epsilon}}\Big] < \infty, \quad \forall a \in (0,1),\ \epsilon \in (0,1).$$

By a similar argument in the proof of Lemma 2.1 we can show

$$\varlimsup_{a\to 0^+} \big(\varphi_\epsilon^{(p)}(a)\big)^{-1} \beta_p(0,[0,a]\times[0,a]) = 0, \quad \text{a.e.} - \mu.$$

Then, as in §2 one can show that for any $c_5 \in (0,\infty)$

$$\begin{aligned}\bar\alpha_p\Big(0, \Big\{&(s_1,\cdots,s_p) \in (R^+_<)^p : \varlimsup_{a\to 0^+} \big(\varphi_\epsilon^{(p)}(a)\big)^{-1}\\ &\cdot \bar\alpha_p(0,[s_1,s_1+a]\times\cdots\times[s_p,s_p+a]) > 5\Big\}\Big) = 0.\end{aligned}$$

By the density theorem (see [2, Proposition 5 and Proposition 6]) we know that

$$\begin{aligned}&\varphi_\epsilon^{(p)} - m(S - 2^p \cap [a_1, b_1] \times \cdots \times [a_p, b_p])\\ \geq & rc_5\bar{\alpha}_p(0, [a_1, b_1] \times \cdots \times [a_p, b_p]), \quad \text{a.e.} - \mu.\end{aligned}$$

From the above argument one can see that the estimate (4.1) plays an important role in studying the Hausdorff measure problem for the set S_2^p. Unfortunately, we are so far unable to improve the estimate (4.1).

Acknowledgements. Many thanks to Prof. S. James Taylor and Prof. Shi Jianyan for their help and encouragements.

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